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ABSTRACT

This paper develops a new unified approach to copula-based modeling and characterizations for time series and stochastic processes. We obtain complete characterizations of many time series dependence structures in terms of copulas corresponding to their finite-dimensional distributions. In particular, we focus on copula-based representations for Markov chains of arbitrary order, $m-$dependent and $r-$independent time series as well as martingales and conditionally symmetric processes. Our results provide new methods for modeling time series that have prescribed dependence structures such as, for instance, higher order Markov processes as well as non-Markovian processes that nevertheless satisfy Chapman-Kolmogorov stochastic equations. We also focus on the construction and analysis of new classes of copulas that have flexibility to combine many different dependence properties for time series. Among other results, we present a study of new classes of copulas based on expansions by linear functions (Eyraud-Farlie-Gumbel-Mongenstern copulas), power functions (power copulas) and Fourier polynomials (Fourier copulas) and introduce methods for modeling time series using these classes of dependence functions. We also focus on the study of weak convergence of empirical copula processes in the time series context and obtain new results on asymptotic gaussianity of such processes for a wide class of $\beta-$mixing sequences.

Key words and phrases: copulas, dependence, characterization, time series, Markov processes, $m-$dependence, $r-$independence, stochastic differential equations, Fourier copulas

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1 Introduction

1.1 Objectives and key results

The present paper develops a new unified approach to copula-based modeling and characterizations for time series and stochastic processes. Among other results, we obtain complete characterizations of a number of time series dependence structures in terms of copulas corresponding to their finite-dimensional distributions. In particular, we focus on copula-based representations for Markov chains of arbitrary order as well as $m-$dependent and $r-$independent time series. The results presented in the paper provide new methods for modeling time series having prescribed dependence structures such as, for instance, higher order Markov processes as well as non-Markovian processes that nevertheless satisfy Chapman-Kolmogorov stochastic equations. We also focus on the construction and analysis of new classes of copulas that have flexibility to combine many different dependence properties for time series. In particular, we present a study of new classes of copulas based on expansions by linear functions (Eyraud-Farlie-Gumbel-Mongenstern copulas), power functions (power copulas) and Fourier polynomials (Fourier copulas) and introduce methods for modeling time series using these classes of dependence functions. The paper also considers the problem of weak convergence of empirical copula processes in the time series context and presents new results on asymptotic gaussianity of such processes for a wide class of $\beta-$mixing sequences.

1.2 Discussion and relation to the literature

In recent years, a number of studies in economics, finance and risk management have focused on dependence measuring and modeling as well as on testing for serial dependence in time series. It was observed in several studies that the use of the most widely applied dependence measure, the correlation, is problematic in many setups. For example, Boyer, Gibson and Loretan (1999) reported that correlations can provide little information about the underlying dependence structure in the case of asymmetric dependence. Naturally (see Blyth (1996) and Shaw (1997)), the linear correlation fails to capture nonlinear dependencies in data on risk factors. Embrechts, McNeil and Straumann (2002) presented a rigorous study of the problems related to the use of correlation as measure of dependence in risk management and finance. As discussed in Embrechts, McNeil and Straumann (2002) (see also Hu (2001)), one of the cases when the use of correlation as measure of dependence becomes problematic is the departure from multivariate normal and, more generally, elliptic distributions. As reported in Shaw (1997), Ang and Chen (2002) and Longin and Solnik (2001), the departure from Gaussianity and elliptical distributions occurs in real world risks and financial market data. Some of other problems with using correlation is that it is a bivariate measure of dependence and even using its time varying versions, at best, leads to only capturing the pairwise dependence in data sets, failing to measure more complicated dependence structures. Also, the correlation is defined only in the case of data with finite second moments and its reliable estimation is problematic in the case of infinite higher moments. However, as reported in a number of studies (see, e.g., the discussion in Loretan and Phillips (1994), Cont (2001) and Ibragimov (2004, 2005) and references therein), many financial and commodity market data sets exhibit heavy-tailed behavior with higher moments failing to exist and even variances being infinite for certain time series in finance and economics.[1] Several approaches have been proposed recently to deal with the above problems. A number of papers have focused on statistical and econometric applications of mutual information and other dependence measures (e.g., Golan (2002), Golan and Perloff (2002), Massoumi and Racine (2002), Miller and Liu (2002), Soofi and Retzer (2002) and Ullah (2002) and references therein). Several recent papers in econometrics (Robinson (1991), Granger and Lin (1994) and Hong and White (2000)) considered the problems of estimating entropy measures of serial dependence in time series. In a study of multifractals and generalizations of Boltzmann-Gibbs statistics, Tsalis (1988) proposed a class of generalized entropy measures that include, as a particular case, the Hellinger distance and the mutual information measure. The latter measures were used by Fernandes and Flôres (2001) in testing for conditional independence and noncausality. Another approach, which is becoming increasingly popular in econometrics and dependence modeling in finance and risk management is the one based on copulas. Copulas are functions that allow one, by a celebrated theorem due to Sklar (1959), to represent a joint distribution of random variables (r.v.'s) as a function of marginal distributions. Copulas, therefore, capture all the dependence properties of the data generating process. In recent year, copulas and related concepts in dependence modeling and measuring have been applied to a wide range of problems in economics, finance and risk management (e.g., Taylor (1990), Fackler (1991), Frees, Carriere and Valdez (1996), Klugman and Parsa (1999), Patton (2000), Richardson, Klose and Gray (2000), Cherubini and Luciano (2001), Hu (2001), Reiss and Thomas (2001), Cherubini and Luciano (2002), Granger, Teräsvirta and Patton (2002), Miller and Liu (2002), Patton (2002), Embrechts, Lindskog and McNeil (2003) and Rosenberg (2003)). In particular, Patton (2000) studied modeling time-varying dependence in financial markets using the concept of conditional copula. Patton (2002) applied copulas to model asymmetric dependence in the joint distribution of stock returns. Hu (2001) used copulas to study the structure of dependence across financial markets. Miller and Liu (2002) proposed methods for recovery of multivariate joint distributions and copulas from limited information using entropy and other information theoretic concepts. Hennessy and Lapan (2002) used Archimedian copulas for modeling portfolio allocations. Cherubini and Luciano (2002) and Rosenberg (2003) applied bivariate copulas in the analysis of option pricing problems. Williamson and Downs (1990), Denuit, Genest and Marceau (1999), Durrleman, Nikeghbali and Roncalli (2000), Cherubini and Luciano (2001), Taylor (2002) and Embrechts, Hoeing and Juri (2003), among others, used Fréchet-Hoeffding inequalities for copulas and their analogues for cdf's to obtain distributional bounds for functions of dependent risk

[1] A number of frameworks have been proposed to model heavy-tailedness phenomena, including stable distributions and their truncated versions, Pareto distributions, multivariate $t-$distributions, mixtures of normals, power exponential distributions, ARCH processes, mixed diffusion jump processes, variance gamma and normal inverse Gamma distributions (see Cont (2001) and Ibragimov (2004, 2005) and references therein). The debate concerning the values of the tail indices for different heavy-tailed financial data and on appropriateness of their modeling based on certain above distributions is, however, still under way in empirical literature. In particular, as discussed in Ibragimov (2004, 2005), a number of studies continue to find tail parameters less than two in different financial data sets and also argue that stable distributions are appropriate for their modeling.

and estimates for VaR quantities. Bouyé, Gaussel and Salmon (2002) used copulas in modeling nonlinear autoregressive time series. Gagliardini and Gourieroux (2002) considered copula-based time-series models with constrained nonparametric dependence. An approach based on concepts closely related to copulas was applied by Lee (1982, 1983) in the analysis of econometric models with selectivity and by Heckman and Honoré (1989) in the study of competing risks models.

Measures of dependence and copula-based approaches to dependence modeling are two interrelated parts of the study of joint distributions of r.v.'s in mathematical statistics and probability theory. A problem of fundamental importance in the field is to determine a relationship between a multivariate cumulative distribution function (cdf) and its lower dimensional margins and to measure degrees of dependence that correspond to particular classes of joint cdf's. The problem is closely related to the problem of characterizing the joint distribution by conditional distributions (see Gouriéroux and Monfort (1979)). Remarkable advances have been made in the latter research area in recent years in statistics and probability literature (see papers in Dall'Aglio, Kotz and Salinetti (1991), Beneš and Štěpán (1997) and the monographs by Joe (1997) and Nelsen (1999)).

One should note that, so far, most of the studies have focused on the analysis of copulas and dependence measures only in the bivariate case and only a few papers have considered the problems in copula theory in the time series context. A drawback of the approach based on bivariate copulas and dependence measures is that, similar to the case of linear correlation, it can capture, at best, only pairwise dependence patterns and can not be used in the case of more complicated dependence structures. It is evident, however, that the dependence characteristics of real data sets can be far more general than those determined by pairs of variables. For example, the behavior of financial indices across markets is interrelated and is affected by a number of factors common to all of the markets. Modeling dependence in financial markets on the base of bivariate copulas, without considering dependence patterns for more than just two of financial indices might be quite a simplification. Furthermore, estimation procedures for copulas developed in the context of independent observations of random vectors can not be used in the analysis of time series dependence characteristics.

The problems of copula theory and its applications in the multivariate and time series context have been considered, in particular, in the following papers. Joe (1987, 1989) proposed multivariate extensions of Pearson's coefficient and the Kullback-Leibler and Shannon mutual information. Nelsen (1996) considered measures of multivariate association generalizing bivariate Spearman's rho and Kendall's tau. Darsow, Nguyen and Olsen (1992) obtained characterizations of first-order Markov chains in terms of copula functions corresponding to their two-dimensional distributions. Chen and Fan (2004a, b) considered parametric copula estimation procedures for time-series based on bivariate copulas and applied the results in the problems of evaluating density forecasts. Recently, de la Peña, Ibragimov and Sharakhmetov (2003) obtained general $U-$statistics-based representations for joint distributions and copulas of arbitrary dependent r.v.'s. As a corollary of the results, de la Peña, Ibragimov and Sharakhmetov (2003) derived new representations for multivariate divergence measures as well as complete characterizations of important classes of dependent r.v.'s that give, in particular, methods for constructing new copulas and modeling different dependence structures.

In this paper, we develop complete characterizations of a number of time series dependence structures in terms of copulas corresponding to the finite-dimensional distributions of the processes in consideration. In particular, we focus on the problems of characterizations and modeling for Markov processes of an arbitrary order as well as for $m-$dependent and $r-$independent time series (Theorems 1-3 and Corollary 1). The results obtained in the present paper provide new methods for modeling time series that exhibit prescribed dependence structures including, for instance, higher order Markov processes as well as non-Markovian processes that nevertheless satisfy Chapman-Kolmogorov stochastic equations (see the discussion at the beginning of Section 3).

Our results give solutions to a number of problems of combining different dependence structures. In particular, we consider the problems of characterization and modeling for Markov processes that satisfy additional dependence assumptions such as the problems of combining higher order Markovness with $m-$dependence, $r-$independence or martingaleness (Theorems 2-4). We also focus on the construction and analysis of new classes of copulas that have flexibility to generate different dependence structures for time series. In particular, in Sections 4 and 5, we present a study of different classes of copulas based on expansions by orthogonal functions, such as linear functions (Eyraud-Farlie-Gumbel-Mongenstern copulas), power functions (power copulas) and Fourier polynomials (Fourier copulas). These results give new methods for modeling time series having prescribed general dependence structures (such as, e.g., higher order Markov and diffusion processes) and arbitrary one-dimensional margins via inversion of finite-dimensional cdf's of known examples of dependent time series (see the discussion at the end of Section 2). They also allow one to model and study time series with flexible dependence properties, such as, for instance, non-Markovian processes that nevertheless satisfy Chapman-Kolmogorov stochastic equations or Markov processes of higher order exhibiting $r-$independence or $m-$dependence properties.

We also present an analysis of applicability and limitations of different classes of copulas in time series modeling. We study dependence properties of time series based on bivariate and multivariate Eyraud-Farlie-Gumbel-Mongenstern copulas as well as their more general analogues, including power copulas and copulas based on products of nonlinear functions of the arguments. In particular, we focus on conditions on copulas under which copula-based time series with prescribed dependence properties reduce to jointly independent processes. Among others, we obtain *impossibility/reduction*-type results that show that time series based on such copulas that simultaneously exhibit Markovness and $m-$dependence or $r-$independence properties are, in fact, sequences of independent r.v.'s (Theorems 5 and 6 and Corollaries 2-4).

Finally, in Section 6, we focus on the study of weak convergence of empirical copula processes in the time series context and obtain new results on asymptotic gaussianity of such processes for a wide class of $\beta-$mixing sequences.

1.3 Organization of the paper

The paper is organized as follows. Section 2 presents the main results of the paper on copula-based time series characterizations and modeling. Among other results, it provides representation results for Markov processes of arbitrary order as well as discusses copula inversion methods for their construction. Section 3 focuses on applications of the main results obtained in Section 2 to the problems of combining different dependence properties in time series and stochastic processes. Section 4 presents new reduction and impossibility results for Markov processes that provide conditions under which copula-based dependent time series reduce to sequences of independent r.v.'s. Section 5 introduces new classes of copulas that provide a flexible framework for modeling time series exhibiting prescribed dependence patterns. In Section 6, we focus on the study of weak convergence of empirical copula processes in the time series framework and present new results on their asymptotic gaussianity for a wide class of $\beta-$mixing sequences. Appendix A1 reviews the definition and discusses the main properties of copula functions, together with their examples. In particular, the appendix discusses the $U-$statistics-based copula characterizations and representations obtained recently by de la Peña, Ibragimov and Sharakhmetov (2003). These representations provide the key to obtaining copula-based time series characterizations developed in the present paper. Appendix A2 contains the proofs of the results derived in the paper.

2 Main results: Copula-based time series characterizations

This section of the paper presents our main results on copula-based modeling and characterizations for time series models and stochastic processes. We obtain complete characterizations of a number of time series dependence structures in terms of copulas corresponding to their finite-dimensional distributions. In particular, we focus on copula-based representations for Markov chains of arbitrary order, $m-$dependent and $r-$independent time series.

For the definition of copulas and a review of their basic properties the reader is referred to Appendix A1.

Throughout the paper, we focus on processes that have absolutely continuous finite-dimensional copulas and, in particular, have continuous one-dimensional cdf's. However, all the results obtained in the paper can be easily generalized to the case of copulas which are not necessarily absolutely continuous and to the case of processes consisting of discrete r.v.'s.

Let $m, n \geq k \geq 1$. Let A and B be, respectively, $m-$ and $n-$dimensional copulas such that

$$A(u_1, ..., u_{m-k}, \xi_1, ..., \xi_k)\Big|_{u_i=1, i=1,...,m-k} = B(\xi_1, ..., \xi_k, v_{m+1}, ..., v_n)\Big|_{u_i=1, i=k+1,...,n} = C(\xi_1, ..., \xi_k), \qquad (1)$$

$\xi_i \in [0,1]$, $i = 1, ..., k$, where C is a $k-$dimensional copula (relation (1) means that copulas A and B are compatible in the sense that a $k-$dimensional margin of A is the same as a $k-$dimensional margin of B, see Joe (1997) and Nelsen (1999) for more on compatibility of copulas).

Let $V_1, ..., V_k$ and $W_1, ..., W_n$ be r.v.'s with joint cdf's A and B (see Definition 5 in Appendix A1). Denote by $A_{1,...,m|m-k+1,...,m}(u_1, ..., u_{m-k}, \xi_1, ..., \xi_k) = P(V_1 \leq u_1, ..., V_{m-k} \leq u_{m-k}|V_{m-k+1} = \xi_1, ..., V_k = \xi_k)$ and $B_{1,...,n|1,...,k}(\xi_1, ..., \xi_k, u_{m+1}, ..., u_{m+n-k}) = P(W_{m+1} \leq u_{m+1}, ..., W_{m+n-k} \leq u_{m+n-k} \leq u_{m+n-k}|W_1 = \xi_1, ..., W_k = \xi_k)$ the conditional analogues of the copulas A and B. Further, define the $\star^k-$product of the copulas A and B, $D = A \star^k B : [0,1]^{m+n-k} \to [0,1]$ via the relation

$$D(u_1, ..., u_{m+n-k}) = \int_0^{u_{m-k+1}} ... \int_0^{u_m} A_{1,...,m|m-k+1,...,m}(u_1, ..., u_{m-k}, \xi_1, ..., \xi_k) \times B_{1,...,n|1,...,k}(\xi_1, ..., \xi_k, u_{m+1}, ..., u_{m+n-k}) C(d\xi_1, ..., d\xi_k). \quad (2)$$

The $\star^k-$operator is a generalization of the star $\star-$operator considered in Darsow, Nguyen and Olsen (1992, hereafter DNO); the $\star-$operator in DNO is a particular case of its $\star^k-$analogue with $k = 1$ (see Appendix A1). Similar to the case of $k = 1$ in DNO, one can show that the operator $\star^k$ is associative, distributive over convex combinations and continuous in each place (but not jointly continuous).

In the case when $A(v_1, ..., v_m)$ and $B(v_1, ..., v_n)$ are absolutely continuous copulas with densities $\frac{\partial^m A(v_1,...,v_m)}{\partial v_1...\partial v_m}$ and $\frac{\partial^n B(v_1,...,v_n)}{\partial v_1...\partial v_n}$, relation (2) is equivalent to the following (here, $\frac{\partial^{m+n-k} D(v_1,...,v_{m+n-k})}{\partial v_1...\partial v_{m+n-k}}$ denotes the density of the copula $D = A \star^k B$):

$$\frac{\partial^{m+n-k} D(v_1, ..., v_{m+n-k})}{\partial v_1...\partial v_{m+n-k}} = \frac{\partial^m A(u_1, ..., u_{m-k}, u_{m-k+1}, ..., u_m)}{\partial v_1...\partial v_m} \times \frac{\partial^n B(u_{m-k+1}, ..., u_m, u_{m+1}, ..., u_{m+n-k})}{\partial v_1...\partial v_n} \Big/ \frac{\partial^k C(u_{m-k+1}, ..., u_m)}{\partial v_1...\partial v_k},$$

or, equivalently,

$$\frac{\partial^{m+n-k} D(v_1, ..., v_{m+n-k})}{\partial v_1...\partial v_{m+n-k}} \cdot \frac{\partial^k C(u_{m-k+1}, ..., u_m)}{\partial v_1...\partial v_k} = \frac{\partial^m A(u_1, ..., u_{m-k}, u_{m-k+1}, ..., u_m)}{\partial v_1...\partial v_m} \cdot \frac{\partial^n B(u_{m-k+1}, ..., u_m, u_{m+1}, ..., u_{m+n-k})}{\partial v_1...\partial v_n}.$$

Let $T \subseteq \mathbf{R}$. The processes considered throughout the paper are assumed to be real-valued and continuous and to be defined on the same probability space $(\Omega, \Im, P)$.

Definition 1 *A process* $\{X_t\}_{t \in T}$ *is called a* Markov process of order $k \geq 1$ *if, for all* $t_1 < ... < t_{n-k} < t_{n-k+1} < ... < t_n$,

$$P\big(X_t < x_t \big| X_{t_1}, ..., X_{t_{n-k}}, X_{t_{n-k+1}}, ..., X_{t_n}\big) = P\big(X_t < x_t \big| X_{t_{n-k+1}}, ..., X_{t_n}\big). \quad (3)$$

The following theorem provides a complete characterization of Markov processes of an arbitrary order in terms of their $(k+1)-$dimensional copulas.

Throughout the rest of the section, $C_{t_1,...,t_k}$, $t_i \in T$, $i = 1, ..., k$, $t_1 < ... < t_k$, stand for copulas corresponding to the joint distribution of the r.v.'s $X_{t_1}, ..., X_{t_k}$ in the process $\{X_t\}_{t \in T}$ in consideration. In addition, throughout the paper, formulated equalities and inequalities for two functions f and g defined on $[a,b]^n \subseteq \mathbf{R}^n$ are understood to hold almost everywhere on $[a,b]^n$. That is, we write $f = g$ (or $f(u) = g(u)$) if f and g coincide almost everywhere on $[a,b]^n$: $f(u) = g(u)$ for all $u \in [a,b]^n \setminus \mathcal{A}$, where $\mathcal{A}$ is a subset of $[a,b]^n$ with the Lebesgue measure zero. The meaning of the inequalities $f \geq g$ and $f \leq g$ (or $f(u) \geq g(u)$ and $f(u) \leq g(u)$) is similar.

Theorem 1 *A stochastic process $\{X_t\}_{t\in T}$, is a Markov process of order k, $k \geq 1$, if and only if for all $t_i \in T$, $i = 1, ..., n$, $n \geq k+1$, such that $t_1 < ... < t_n$,*

$$C_{t_1,\ldots,t_n} = C_{t_1,\ldots,t_{k+1}} \star^k C_{t_2,\ldots,t_{k+2}} \star^k \ldots \star^k C_{t_{n-k},\ldots,t_n}. \tag{4}$$

Let $n \geq k+1$ and $s \geq 1$. For an $n-$dimensional copula C denote by C^s the $s-$fold product $\star^k$ of C with itself.

Corollary 1 *A sequence of r.v.'s X_t, $t = 1, 2, ...$ is a stationary Markov chain of order k, $k \geq 1$, if and only if for all $n \geq k+1$,*

$$C_{1,\ldots,n}(u_1, ..., u_n) = C \star^k C \star^k \ldots \star^k C(u_1, ..., u_n) = C^{n-k+1}(u_1, ..., u_n), \tag{5}$$

where C is a $k+1-$dimensional copula such that $C_{i_1+h,\ldots,i_l+h} = C_{i_1,\ldots,i_l}$, $1 \leq h \leq k+1-i_l$, $1 \leq i_1 < ... < i_l \leq k+1$, $l = 2, ..., k$, where $C_{j_1,\ldots,j_l}$, $1 \leq j_1 < ... < j_l \leq k+1$, denote the corresponding marginals of C: $C_{j_1,\ldots,j_l} = C|_{u_i=1, i\neq j_1,\ldots,j_l}$.

Theorem 1 and Corollary 1 provide an approach to modeling Markov processes of higher order alternative to that based on their transition probability matrices. Instead of specifying the initial distribution and a family of transition probabilities, one can specify a Markov process of order k by prescribing all of the marginal distributions and a family of $(k+1)-$dimensional copulas satisfying and then generating the copulas of higher order and, thus, finite-dimensional cdf's using (4). The advantage of the approach based on copulas is that it allows one to separate in the dependence modeling the properties determined by marginal distributions such as fat-tailedness of time series and its dependence properties such as conditional symmetry, $m-$dependence, $r-$independence or mixing properties.

Corollary 1, together with the inversion method for constructing copulas described in Appendix A1, provide a device for constructing new Markov chains of an arbitrary order that exhibit dependence properties completely similar to those of a given Markov process of the same order but have other marginals. Namely, let X_t, $t = 1, 2, ...$ be a stationary Markov chain of order $k \geq 1$ with $(k+1)-$dimensional absolutely continuous cdf $\tilde{F}(x_1, ..., x_{k+1})$ and the one-dimensional cdf F. Then the $(k+1)-$dimensional copula generating the process $\{X_t\}$ is, via formula (27) in Appendix A1, $C(u_1, ..., u_{k+1}) = \tilde{F}(F^{-1}(u_1), ..., F^{-1}(u_{k+1}))$. Given an arbitrary one-dimensional cdf G, the stationary $k-$th order Markov chain that has completely the same dependence structure as that of $\{X_t\}$ but a different one-dimensional marginal cdf G can be constructed via

(5) by generating its copulas of an arbitrary order and substituting the new one-dimensional cdf to obtain its finite-dimensional cdf's.

In what follows, we refer to the processes $\{X_t\}$, $t = 1, 2, ...$, constructed via (5) as stationary $k-$th order Markov chains based on the ($(k+1)-$dimensional) copula C or as $C-$based $k-$th order stationary Markov chains for short.

3 Main results: Applications to combining higher-order Markovness with other dependence properties

In this section of the paper, we focus on the problems of combining higher-order Markovness in time series with other dependence properties. In particular, we develop complete characterizations of Markov processes of an arbitrary order that satisfy additional assumptions of $r-$independence or $m-$dependence defined as follows.

Definition 2 *R.v.'s $X_1, ..., X_n$ are called $r-$***independent** *($2 \leq r < n$) if any r of them are jointly independent.*

Definition 3 *R.v.'s $X_1, ..., X_n$ are called $m-$dependent ($1 \leq m \leq n$) if any two vectors $(X_{j_1}, X_{j_2}, ..., X_{j_{a-1}}, X_{j_a})$ and $(X_{j_{a+1}}, X_{j_{a+2}}, ..., X_{j_{l-1}}, X_{j_l})$, where $1 \leq j_1 < ... < j_a < ... < j_l \leq n$, $a = 1, 2, ..., l-1$, $l = 2, ..., n$, $j_{a+1} - j_a \geq m$, are independent.*

A number of studies in dependence modeling have focused on problems of combining Markovian structures with other types of dependence. E.g., Lévy (1949) constructed a 2nd order Markov chain consisting of pairwise independent uniformly distributed r.v.'s (a 2nd order pairwise independent Markov chain). Motivated by applications in the study of the mechanism of human vision, Rosenblatt and Slepian (1962) constructed Nth order stationary Markov chains consisting of discrete r.v.'s such that every N variables of the process are independent while $N + 1$ adjacent variables of the process are not independent (Nth order $N-$independent stationary Markov chain). Rosenblatt and Slepian (1962) also obtained a result that is naturally to refer to as an *impossibility* or a *reduction* property for Markov chains that shows that the only $N-$th order $N-$independent Markov processes for which X_n is concentrated on two points are the jointly independent ones. The $r-$independent Markov chains of higher order are important in testing empirically the sensitivity of commonly used statistical procedures developed on the independence assumption to weak dependence in the data generating process (see Rosenblatt and Slepian (1962)). In addition to that, such processes are of interest since they provide examples of processes which are not Markovian of first order but whose first order transition probabilities $P(s, x, t, A) = P(X_t \in A | X_s = x)$ nevertheless satisfy the Chapman-Kolmogorov stochastic equation

$$P(s, x, t, A) = \int_{-\infty}^{\infty} P(u, \xi, t, A) P(s, x, u, d\xi) \tag{6}$$

for all Borel sets A, all $s < t$ in T, $u \in (s,t) \cap T$ and for almost all $x \in \mathbf{R}$. [2]

Markov chains with 1-dependence appeared for the first time in Aaronson, Gilat and Keane (1992) and were considered, e.g., by Burton, Goulet and Meester (1993) and Matúš (1996), where the focus was on $1-$dependent Markov shifts and on the structure of block-factors. Matúš (1998) studied $m-$dependent Markov sequences consisting of discrete r.v.'s and showed, in particular, that generally no stationary sequence of r.v.'s which is Markov of order n but not of order $n-1$ exists if the state space of the sequence has small cardinality (another type of an *impossibility/reduction* result for Markov chains). Matúš (1998) also showed that to ensure the existence for the Markov processes of order $n = 1$ the number of attainable states must be at least $m + 2$ and that this bound is tight.

The following result gives a complete characterization of $k-$independent $k-$th order stationary Markov chains.

Theorem 2 *A sequence of r.v.'s $\{X_t\}_{t=1}^{\infty}$, is a $C-$based $k-$th order $k-$independent stationary Markov chain if and only if the density of C has the form*

$$\frac{\partial^{k+1} C(u_1, ..., u_{k+1})}{\partial u_1 ... \partial u_{k+1}} = 1 + g(u_1, ..., u_{k+1}), \tag{7}$$

where $g : [0,1]^{k+1} \to [0,1]$ is a function satisfying the conditions

$$\int_0^1 ... \int_0^1 |g(u_1, ..., u_{k+1})| du_1 ... du_{k+1} < \infty, \tag{8}$$

$$\int_0^1 ... \int_0^1 \prod_{j=1}^{s} g(u_j, ..., u_{k+j}) du_{i_1} ... du_{i_s} =$$

$$\int_0^1 ... \int_0^1 g(u_1, ..., u_{k+1}) g(u_2, ..., u_{k+2}) ... g(u_s, ..., u_{k+s}) du_{i_1} ... du_{i_s} = 0 \tag{9}$$

for all $s \leq u_{i_1} < ... < u_{i_s} \leq k+1$, $s = 1, 2, ..., \left[\frac{k+1}{2}\right]$, and

$$g(u_1, ..., u_{k+1}) \geq -1. \tag{10}$$

Remark 1 *Integration in condition (9) is with respect to all combinations of s variables among the arguments $u_s, u_{s+1}, ..., u_{k+1}$ that are common to all the functions $g(u_1, ..., u_{k+1})$, $g(u_2, ..., u_{k+2})$, ..., $g(u_s, ..., u_{k+s})$ appearing in the integrand. These conditions ensures that all the $k-$dimensional marginals of the copula of the r.v.'s $X_1, ..., X_{k+s}$ are product copulas (30) corresponding to independence and, thus, the $k-$independence property is satisfied for the stationary $k-$th order Markov process in consideration.*

The following theorem provides a characterization of Markov chains satisfying $m-$dependence properties.

[2] Examples of non-Markovian processes for which Chapman-Kolmogorov equation is satisfied were given, e.g., by Feller (1959) and Rosenblatt (1960).

Theorem 3 *A sequence* $\{X_t\}_{t=1}^{\infty}$ *is a* $C-$*based* $m-$*dependent first order stationary Markov chain if and only if the density of* C *satisfies*

$$\frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} = 1 + g(u_1, u_2), \tag{11}$$

where $g : [0, 1]^2 \to [0, 1]$ *is a function satisfying the conditions*

$$\int_0^1 \int_0^1 |g(u_1, u_2)| du_1 du_2 < \infty, \tag{12}$$

$$\int_0^1 g(u_1, u_2) du_i = 0, \ i = 1, 2, \tag{13}$$

$$g(u_1, u_2) \geq -1 \tag{14}$$

and such that

$$\int_0^1 \prod_{i=1}^{m} g(u_i, u_{i+1}) du_2 du_3 ... du_m = \int_0^1 g(u_1, u_2) g(u_2, u_3) ... g(u_m, u_{m+1}) du_2 du_3 ... du_m = 0. \tag{15}$$

Remark 2 *Similar to (9), integration in condition (15) is with respect to the variables* $u_2, u_3, ..., u_m$ *that appear more than once among the arguments of the functions* $g(u_1, u_2)$, $g(u_2, u_3)$, ..., $g(u_m, u_{m+1})$. *This condition ensures that the r.v.'s* X_1 *and* X_{m+1} *are independent. This, in turn, guarantees that the stationary Markov process in consideration is* $m-$*dependent.*

In a number of applications, e.g., in finance, it is important to have a simultaneous satisfaction of the Markov and martingale property. The martingale property, in contrast to the Markov (first and higher order) properties is evidently not determined by finite-dimensional copulas of a process only and can be affected by changes in one-dimensional marginal distributions. The property, however is determined by copulas alone for a wide subclass of martingale differences, namely for sequences satisfying conditional symmetry assumptions.

Definition 4 *A sequence* $\{X_t\}_{t=1}^{\infty}$ *on a probability space* $(\Omega, \Im, P)$ *is a conditionally symmetric martingale difference with respect to an increasing sequence of* $\sigma-$*algebras* $\Im_0 = (\Omega, \emptyset) \subset \Im_1 \subseteq \Im_2 \subseteq ... \subseteq \Im_n \subseteq \Im$ *if r.v.'s* X_t *are conditionally symmetric on the* $\sigma-$*algebras* $\Im_{t-1}$, *that is, if* $P(X_t > x | \Im_{t-1}) = P(X_t < -x | \Im_{t-1})$, $x \geq 0$, $t = 1, 2, ...$

The following theorem characterizes the stationary Markov chains of the first order satisfying martingale property.

Theorem 4 *A* $C-$*based Markov chain* $\{X_t\}_{t=1}^{\infty}$ *consisting of symmetric r.v.'s with a continuous one-dimensional cdf* F *is a conditionally symmetric martingale difference with respect to the natural filtration* $\Im_0 = (\Omega, \emptyset)$, $\Im_t = \sigma(X_1, ..., X_t)$, $t \geq 1$, *if and only if*

$$\frac{\partial C(u_1, \ 1/2 - u)}{\partial u_1} + \frac{\partial C(u_1, \ 1/2 + u)}{\partial u_1} = 1, \tag{16}$$

or, equivalently, if the density of C satisfies

$$\frac{\partial C(u_1,\ 1/2-u)}{\partial u_1 \partial u_2} = \frac{\partial C(u_1,\ 1/2+u)}{\partial u_1 \partial u_2}, \tag{17}$$

$u \in [0, 1/2)$.

Given a particular $k-$th order $k-$independent Markov chain or an $m-$dependent Markov chain of the first order (say, those in the works by Lévy (1949), Rosenblatt and Slepian (1962), Aaronson, Gilat and Keane (1992) or Matúš (1998) discussed in the beginning of this section, one can use the inversion procedure described following Corollary 1 in Section 2 to construct Markov processes that exhibit the same dependence properties but have one-dimensional marginals different from those in the examples in the above papers.

4 Main results: reduction and impossibility theorems for Markov chains of an arbitrary order

Theorems 2 and 3 imply several *reduction* and *impossibility* results for Markov processes satisfying $m-$dependence and $r-$independence conditions. According to the results in the theorems, a number of copula based time series that simultaneously exhibit Markovness and $m-$dependence or $r-$independence properties are, in fact, sequences of independent r.v.'s.

The results in the following theorem, for instance, show that a construction of non-trivial Markov chains of higher order that exhibit $r-$independence properties is impossible on the base of copulas whose densities C in Theorem 2 (that provides a characterization of time series combining $r-$independence with Markovness) have functions g with a separable product form.

Theorem 5 *Suppose that $C : [0,1]^{k+1} \to [0,1]$ is a $(k+1)-$dimensional copula that has density (7), where $g(u_1, u_2, ..., u_{k+1}) = \alpha f(u_1) f(u_2) ... f(u_{k+1})$ for some $\alpha \in \mathbf{R}$ and some function $f : [0,1] \to [0,1]$. A sequence of r.v.'s $\{X_t\}$, $t = 1, 2, ...,$ is a $C-$based $k-$th order $k-$independent Markov chain if and only if $\{X_t\}$ is a sequence of i.i.d. r.v.'s.*

A particular case of copulas C in the form of Theorem 5 is given by a special case of $(k+1)-$dimensional Eyraud-Farlie-Gumbel-Mongenstern copulas (38) that have the form

$$C(u_1, u_2, ..., u_{k+1}) = \prod_{i=1}^{k+1} u_i \Big(1 + \alpha(1-u_1)(1-u_2)...(1-u_{k+1})\Big). \tag{18}$$

These copulas have densities (7) with

$$g(u_1, u_2, ..., u_{k+1}) = \alpha(1-2u_1)(1-2u_2)...(1-2u_{k+1}). \tag{19}$$

Corollary 2 *Let* $C : [0,1]^{k+1} \to [0,1]$ *be a* $(k+1)-$*dimensional Eyraud-Farlie-Gumbel-Mongenstern copula (18) with density (7) with* g *given by (19). A sequence of r.v.'s* $\{X_t\}$, $t = 1, 2, ...,$ *is a* $C-$*based* $k-$*th order* $k-$*independent Markov chain if and only if* $\{X_t\}$ *is a sequence of i.i.d. r.v.'s.*

The following results is a generalization of Corollary (2) to the case of a special case of $(k+1)-$dimensional power copulas (39), namely, for the copulas

$$C(u_1, u_2, ..., u_{k+1}) = \prod_{i=1}^{k+1} u_i \Big(1 + \alpha(u_1^l - u_1^{l+1})(u_2^l - u_2^{l+1})...(u_{k+1}^l - u_{k+1}^{l+1})\Big), \tag{20}$$

where $l \in \{0, 1, 2, ...\}$ (copulas (20) reduce to those in (18) for $l = 0$). These copulas have density (7) in which

$$g(u_1, u_2, ..., u_{k+1}) = \alpha\Big((l+1)u_1^l - (l+2)u_1^{l+1}\Big)\Big((l+1)u_2^l - (l+2)u_2^{l+1}\Big)...\Big((l+1)u_{k+1}^l - (l+2)u_{k+1}^{l+1}\Big). \tag{21}$$

Corollary 3 *Let* $C : [0,1]^{k+1} \to [0,1]$ *be a* $(k+1)-$*dimensional power copula (20) with density (7) with* g *given by (21). A sequence of r.v.'s* $\{X_t\}$, $t = 1, 2, ...,$ *is a* $C-$*based* $k-$*th order* $k-$*independent Markov chain if and only if* $\{X_t\}$ *is a sequence of i.i.d. r.v.'s.*

The following theorem is an analogue of Theorem 5 in the case of $m-$dependence. It concerns impossibility/reduction properties of $m-$dependent Markov chains. According to the theorem, a construction of non-trivial examples (that is, those more general than sequences of independent sequences) of Markov chains exhibiting $m-$dependence is impossible on the base of bivariate copulas that have, similar, to Theorem 5, the function g in a separable product form.

Theorem 6 *Suppose that* $C : [0,1]^2 \to [0,1]$ *is a bivariate copula that has the density* $\frac{\partial^2 C(u_1,u_2)}{\partial u_1 \partial u_2} = 1 + \alpha f(u_1) f(u_2)$ *for some* $\alpha \in \mathbf{R}$ *and some function* $f : [0,1] \to [0,1]$*. A sequence of r.v.'s* $\{X_t\}$, $t = 1, 2, ...,$ *is a* $C-$*based* $m-$*dependent Markov chain (of the first order) if and only if* $\{X_t\}$ *is a sequence of i.i.d. r.v.'s.*

The following corollary is a specialization of Theorem 6 to the special case of bivariate Eyraud-Farlie-Gumbel-Mongenstern copulas (18) with $k = 1$:

$$C(u_1, u_2) = u_1 u_2 \Big(1 + \alpha(1 - u_1)(1 - u_2)\Big) \tag{22}$$

that have density (11) with

$$g(u_1, u_2) = \alpha(1 - 2u_1)(1 - 2u_2). \tag{23}$$

According to the corollary, stationary Markov chains that exhibit $m-$dependence and are based on such copulas are, in fact, sequences of i.i.d. r.v.'s.

Corollary 4 *Let* $C : [0,1]^2 \to [0,1]$ *be a bivariate copula Eyraud-Farlie-Gumbel-Mongenstern copula (22) with density (11), where* g *is given by (23). A sequence of r.v.'s* $\{X_t\}$, $t = 1, 2, ...,$ *is a* $C-$*based* $m-$*dependent Markov chain (of the first order) if and only if* $\{X_t\}$ *is a sequence of i.i.d. r.v.'s.*

The results in the present section that demonstrate that Markov chains with $m-$dependence and $r-$independent Markov chains of higher order cannot be constructed from Eyraud-Farlie-Gumbel-Mongenstern copulas and other separable copulas complement and generalize substantially the results of Cambanis (1991). Cambanis (1991) showed that the most common dependence structures such as constant, exponential and $m-$dependence cannot be exhibited by stationary processes $\{X_n\}$ whose finite-dimensional copulas are the following multivariate analogues of bivariate Eyraud-Farlie-Gumbel-Mongenstern copulas (38):

$$C_{j_1,...,j_n}(u_{j_1},...,u_{j_n}) = \prod_{s=1}^{n} u_{j_k}\Big(1 + \sum_{1\le l<m\le n} \alpha_{lm}(1-u_{j_l})(1-u_{j_m})\Big).$$

The results also complement the above-mentioned results by Rosenblatt and Slepian (1962) on non-existence of non-trivial Nth order $N-$independent Markov chains consisting of two-valued r.v.'s since, according to Sharakhmetov and Ibragimov (2002), the finite-dimensional copulas of sequences of r.v.'s concentrated on two points have multivariate Eyraud-Farlie-Gumbel-Mongenstern structure (38).

5 Main results: flexible classes of copulas

The results on limitations of the separable copulas presented in the previous section emphasize the substantial technical difficulty in modeling copula-based time series with flexible dependence structures. According to the results in this section of the paper, a class of copulas based on expansions by Fourier polynomials we introduce in the present paper allows one to encompass this difficulty.

It is not difficult to check that the conditions of Theorem 2 are satisfied for the following functions g :

$$g(u_1,...,u_{k+1}) = \sum_{j=1}^{N}\big[\alpha_j sin(2\pi\sum_{i=1}^{k+1}\beta_i^j u_i) + \gamma_j cos(2\pi\sum_{i=1}^{k+1}\beta_i^j u_i)\big], \tag{24}$$

where $N \ge 1$, and $\alpha_j, \gamma_j \in \mathbf{R}$, and $\beta_i^j \in \mathbf{Z}$, $i = 1,...,k+1$, $j = 1,...,N$, are arbitrary numbers such that

$$\beta_1^{j_1} + \sum_{l=2}^{s} \epsilon_{l-1}\beta_l^{j_l} \ne 0,$$

for $j_1,...,j_s \in \{1,...,N\}$, $\epsilon_1,...,\epsilon_{s-1} \in \{-1,1\}$, $s = 2,...,k+1$, and

$$1 + \sum_{j=1}^{N}\big[\alpha_j\epsilon_j + \gamma_j\epsilon_{j+N}\big] \ge 0$$

for $\epsilon_1,...,\epsilon_{2N} \in \{-1,1\}$. The above functions g satisfy the conditions of Proposition 2 and, thus, define copulas via (28). We refer to the copulas C corresponding to the functions g in such a way,

$$C(u_1,...,u_{k+1}) = \int_0^{u_1} ... \int_0^{u_{k+1}} (1+g(u_1,...,u_{k+1}))du_1...du_{k+1},$$

as $(k+1)-$dimensional *Fourier* copulas. Each such copulas can thus be used to construct a $k-$independent $k-$th order stationary Markov chain via (5).

Similarly, conditions of Theorem 3 are satisfied with $m = 1$ for the bivariate Fourier copulas corresponding to the functions g defined in (24) with $k = 1$, that is, for the Fourier copulas

$$C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} (1 + g(u_1, u_2))du_1 du_2, \qquad (25)$$

where

$$g(u_1, u_2) = \sum_{j=1}^{N} \big[\alpha_j sin(2\pi(\beta_1^j u_1 + \beta_2^j u_2)) + \gamma_j cos(2\pi(\beta_1^j u_1 + \beta_2^j u_2))\big], \qquad (26)$$

$N \geq 1$, $\alpha_j, \gamma_j \in \mathbf{R}$, and $\beta_1^j, \beta_2^j \in \mathbf{Z}$, $j = 1, ..., N$, are arbitrary numbers such that

$$\beta_1^{j_1} + \beta_2^{j_2} \neq 0,$$

$$\beta_1^{j_1} - \beta_2^{j_2} \neq 0,$$

$$1 + \sum_{j=1}^{N} \big[\alpha_j \epsilon_j + \gamma_j \epsilon_{j+N}\big] \geq 0$$

for $\epsilon_1, ..., \epsilon_{2N} \in \{-1, 1\}$. The processes constructed from copulas (25) via (5) thus give examples of stationary first-order $1-$dependent Markov chains.

6 Non-parametric estimation procedures for copula-based processes

The present section focuses on copula function estimation procedures in the time series context. We focus on developing copula process convergence for $\beta-$mixing sequences.

Let $X_1, X_2, ..., X_n, ...$ be a stationary sequence of r.v.'s with the one-dimensional continuous marginal cdf's $F(x) = P(X_i \leq x)$, $i = 1, 2, ..., n$. Denote by $H(x_1, ..., x_{k+1}) = P(X_1 \leq x_1, ..., X_{k+1} \leq x_{k+1})$, $x_i \in \mathbf{R}$, $i = 1, ..., k+1$, the cdf of the $k+1$ successive r.v.'s $X_1, ..., X_n$.

The copula of the $k+1$ successive r.v.'s $X_1, ..., X_n$ is

$$C(u_1, ..., u_{k+1}) = H(F^{-1}(u_1), ..., F^{-1}(u_{k+1})),$$

where the inverse $F^{-1}(u)$ is defined as $F^{-1}(u) = \inf\{t : F(t) \geq u, \}$, $u \in [0, 1]$. Denote by $F_n(x) = 1/n \sum_{i=1}^{n} I_{X_i \leq x}$, $x \in \mathbf{R}$, the one-dimensional empirical cdf of $X_1, ..., X_n$ and by

$$H_n(x_1, ..., x_{k+1}) = (1/(n-k)) \sum_{i=k+1}^{n} \prod_{j=1}^{k+1} I_{X_{i+j-k-1} \leq x_j}$$

the empirical cdf of $k+1$ successive r.v.'s in the sample $X_1, ..., X_n$. Further, denote by

$$C_n(u_1, ..., u_{k+1}) = H_n(F_n^{-1}(u_1), ..., F_n^{-1}(u_{k+1}))$$

the empirical copula function of the $k+1$ successive r.v.'s in the sample $X_1, ..., X_n$.

Theorem 7 *If $\{X_t\}$, $t = 1, 2, ...$ is a stationary $\beta-$mixing sequence with the coefficients β_k satisfying $k^r \beta_k \to 0$ as $k \to \infty$ for some $r > 1$, then the empirical copula process*

$$\sqrt{n-k}(C_n(u_1, ..., u_{k+1}) - C(u_1, ..., u_{k+1}))$$

converges weakly in $l^\infty([0,1]^{k+1})$ to a tight Gaussian process $\{G_C(u_1, ..., u_{k+1}), u_i \in [0, 1], i = 1, ..., k+1\}$.

Similar to Fermanian, Radulovic and Wegcamp (1994), Theorem 7 implies weak convergence of the measures of dependence for $\beta-$mixing sequences based on the multivariate rank order statistics of the form

$$1/(n-k) \sum_{i=1}^{n-k} J(F_n(X_i), F_n(X_{i+1}), ..., F_n(X_{i+k})$$

(the convergence of the statistics in the case of bivariate i.i.d. vectors (X_i, Y_i) was studied, e.g., by Ruymgaart, Shorack and Van Zwet (1972), Ruymgaart (1974), Rüscehndorf (1976), Genest, Ghoudi and Rivest (1995) and Fermanian, Radulovic and Wegcamp (1994)). Using the functional delta method, from Theorem 7 we get that

$$1/\sqrt{n-k} \sum_{i=1}^{n-k} (J(F_n(X_i), F_n(X_{i+1}), ..., F_n(X_{i+k})) - EJ(F_n(X_i), F_n(X_{i+1}), ..., F_n(X_{i+k}))) \to$$

$$-\int_{[0,1]^{k+1}} G_C(u_1, ..., u_{k+1}) dJ(u_1, ..., u_{k+1})$$

for all functions J of bounded variation. It also implies weak convergence of the nonparametric estimates of the time series analogues of the multivariate orthant dependence coefficients introduced by Nelsen (1996):

$$\rho^+_{k+1} = \frac{k+2}{2^{k+1} - (k+2)} \left(2^{k+1} \int_{[0,1]^{k+1}} u_1 ... u_{k+1} dC_{k+1}(u) - 1 \right)$$

and

$$\rho^-_{k+1} = \frac{k+2}{2^{k+1} - (k+2)} \left(2^{k+1} \int_{[0,1]^{k+1}} C_{k+1}(u) du - 1 \right),$$

as well as of the multivariate time series analogues of the bivariate distance measures studied by Schweizer and Wolff (1981):

$$\delta_1 = \int_{[0,1]^{k+1}} |C_{k+1}(u) - u_1 ... u_{k+1}| du_1 ... du_{k+1}$$

and

$$\delta_2 = \left(\int_{[0,1]^{k+1}} (C_{k+1}(u) - u_1 ... u_{k+1})^2 du_1 ... du_{k+1} \right)^{1/2}.$$

7 Appendix A1. Copula functions and their properties. $U-$statistics based copula characterizations

We begin with the definition of copulas and formulation of Sklar's theorem mentioned in the introduction (see e.g., Nelsen (1999) and Embrechts, McNeil and Straumann (2002)).

Definition 5 *A function* $C : [0,1]^n \to [0,1]$ *is called a* $n-$*dimensional* **copula** *if it satisfies the following conditions:*

1. $C(u_1, ..., u_n)$ *is increasing in each component* u_i.
2. $C(u_1, ..., u_{k-1}, 0, u_{k+1}, ..., u_n) = 0$ *for all* $u_i \in [0,1]$, $i \neq k$, $k = 1, ..., n$.
3. $C(1, ..., 1, u_i, 1, ..., 1) = u_i$ *for all* $u_i \in [0,1]$, $i = 1, ..., n$.
4. *For all* $(a_1, ..., a_n)$, $(b_1, ..., b_n) \in [0,1]^n$ *with* $a_i \leq b_i$,

$$\sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+...+i_n} C(x_{1i_1}, ..., x_{ni_n}) \geq 0,$$

where $x_{j1} = a_j$ *and* $x_{j2} = b_j$ *for all* $j \in \{1, ..., n\}$. *Equivalently,* C *is a* $n-$*dimensional copula if it is a joint cdf of* n *r.v.'s* $U_1, ..., U_n$ *each of which is uniformly distributed on* $[0,1]$.

Definition 6 *A copula* $C : [0,1]^n \to [0,1]$ *is called* **absolutely continuous** *if, when considered as a joint cdf, it has a joint density given by* $\partial C^n(u_1, ..., u_n)/\partial u_1 ... \partial u_n$.

Proposition 1 *(Sklar (1959)). If* $X_1, ..., X_n$ *are r.v.'s defined on a common probability space, with the one-dimensional cdf's* $F_{X_k}(x_k) = P(X_k \leq x_k)$ *and the joint cdf* $F_{X_1,...,X_n}(x_1, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n)$, *then there exists an* $n-$*dimensional copula* $C_{X_1,...,X_n}(u_1, ..., u_n)$ *such that*
$F_{X_1,...,X_n}(x_1, ..., x_n) = C_{X_1,...,X_n}(F_{X_1}(x_1), ..., F_{X_n}(x_n))$ *for all* $x_k \in \mathbf{R}$, $k = 1, ..., n$.

If the univariate marginal cdf's $F_{X_1}, ..., F_{X_n}$ *are all continuous, then the copula is unique and can be obtained via* inversion *method:*

$$C_{X_1,...,X_n}(u_1, ..., u_n) = F_{X_1,...,X_n}(F_{X_1}^{-1}(u_1), ..., F_{X_n}^{-1}(u_n)), \tag{27}$$

where $F_{X_k}^{-1}(u_k) = \inf\{x : F_{X_k}(x) \geq u_k\}$.

The following representation results obtained by de la Peña, Ibragimov and Sharakhmetov (2003) provide $U-$statistics-based copula representations and characterizations of dependence structures for multivariate vectors.

Proposition 2 *(de la Peña, Ibragimov and Sharakhmetov (2003)). A function* $C : [0,1]^n \to [0,1]$ *is an absolutely continuous* $n-$*dimensional copula if and only if there exist functions* $g_{i_1,...,i_c} : \mathbf{R}^c \to \mathbf{R}$, $1 \leq i_1 < ... < i_c \leq n$, $c = 2, ..., n$, *satisfying the conditions*

A1 (integrability):

$$\int_0^1 ... \int_0^1 |g_{i_1,...,i_c}(t_{i_1}, ..., t_{i_c})| dt_{i_1} ... dt_{i_c} < \infty,$$

A2 (degeneracy):

$$E(g_{i_1,...,i_c}(u_{i_1}, ..., u_{i_{k-1}}, u_{i_k}, u_{i_{k+1}}, ..., u_{i_c}) | u_{i_1}, ..., u_{i_{k-1}}, u_{i_{k+1}}, ..., u_{i_c}) =$$

$$\int_0^1 g_{i_1,...,i_c}(u_{i_1},...,u_{i_{k-1}},t_{i_k},u_{i_{k+1}},...,u_{i_c})dt_{i_k} = 0,$$

$1 \le i_1 < ... < i_c \le n,\ k = 1,2,...,c,\ c = 2,...,n,$

A3 (positive definiteness):

$$\sum_{c=2}^{n} \sum_{1\le i_1<...<i_c\le n} g_{i_1,...,i_c}(u_{i_1},...,u_{i_c}) \ge -1,$$

and such that

$$C(u_1,...,u_n) = \int_0^{u_1} ... \int_0^{u_n} \left(1 + \sum_{c=2}^{n} \sum_{1\le i_1<...<i_c\le n} g_{i_1,...,i_c}(t_{i_1},...,t_{i_c})\right) \prod_{i=1}^{n} dt_i, \tag{28}$$

or, equivalently, such that the density of C satisfies

$$\frac{\partial^n C(u_1,...,u_n)}{\partial u_1...\partial u_n} = 1 + \sum_{c=2}^{n} \sum_{1\le i_1<...<i_c\le n} g_{i_1,...,i_c}(u_{i_1},...,u_{i_c}). \tag{29}$$

R.v.'s $X_1,...,X_n$ with copula $C(u_1,...,u_n)$ are jointly independent if and only if C is the product copula:

$$C(u_1,...,u_n) = u_1...u_n. \tag{30}$$

Well-studied examples of copulas are given by (see, e.g., Joe (1997) and Nelsen (1999))

$$C(u_1,u_2) = (u_1^{-\theta} + u_2^{-\theta})^{-1/\theta},\ \ \theta > 0 \tag{31}$$

(Clayton copulas);

$$C(u_1,u_2) = exp\big(-\big[(-ln\ u_1)^\theta + (-ln\ u_2)^\theta\big]\big)^{-1/\theta},\ \ \theta > 0 \tag{32}$$

(Gumbel copulas);

$$C(u_1,u_2) = -\frac{1}{\theta} ln\Big(1 + \frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{e^{-\theta}-1}\Big),\ \ \theta > 0 \tag{33}$$

(Frank copulas).

Taking in (27 F to be the $n-$dimensional normal cdf with a covariance matrix Σ:

$$F = F_\Sigma(x)^N = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} \phi_{n,\Sigma}(x)dx, \tag{34}$$

where $\phi_{n,\Sigma}(x) = 1/((2\pi)^{n/2}|\Sigma|^{1/2})\exp(-\frac{1}{2}x/\Sigma^{-1}x)$ one obtains the well-known normal copula $C_\Sigma(u_1,...,u_n)$:

$$C_\Sigma^N(u_1,...,u_n) = F_\Sigma(\Phi^{-1}(u_1),...,\Phi^{-1}(u_n)), \tag{35}$$

where $\Phi(x)$ denotes the one-dimensional normal cdf.

It is natural to consider the following generalizations of normal copulas (35) that are natural to refer to as Gram-Charlier copulas:

$$C_{\Sigma}^{GC}(u_1,...,u_n) = F_{\Sigma}^{GC}(F_1^{-1}(u_1),...,F_n^{-1}(u_n)), \tag{36}$$

where F^{GC} is the $n-$dimensional Gram-Charlier approximation to the normal cdf (34):

$$F_{\Sigma}^{GC}(x) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} \phi_{n,\Sigma}(x)\Big(1 + \sum_{1\le j_1<...<j_n\le m} c_{j_1,...,j_n} H_{j_1,...,j_n}(x)\Big)dx \tag{37}$$

with $n-$variate Hermite polynomials $H_{j_1,...,j_n}(x)$ such that $\frac{\partial^{j_1+...+j_n}}{\partial x_1^{j_1}...\partial x_n^{j_n}} H_{j_1,...,j_n}(x) = (-1)^{j_1+...+j_n}\phi_{n,\Sigma}(x)$, and F_k are one-dimensional marginal cdf's of F^{GC}.

Up to our knowledge, the Gram-Charlier copulas (36) are for the first time proposed in the present paper.

As discussed in de la Peña, Ibragimov and Sharakhmetov (2003), Proposition 2 provides a general device for constructing multivariate copulas and joint distributions. E.g., taking in (28) $n = 2$, $g_{1,2}(t_1,t_2) = \alpha(1-2t_1)(1-2t_2)$, $\alpha \in [-1,1]$, we get the family of bivariate Eyraud-Farlie-Gumbel-Morgenstern copulas $C_\alpha(u_1,u_2) = u_1u_2\,(1+\alpha(1-u_1)(1-u_2))$. More generally, taking $g_{i_1,...,i_c}(t_{i_1},...,t_{i_c}) = 0$, $1 \le i_1 < ... < i_c \le n$, $c = 2,...,n-1$, $g_{1,2,...,n}(t_1,t_2,...,t_n) = \alpha(1-2t_1)(1-2t_2)...(1-2t_n)$, we obtain the following multivariate Eyraud-Farlie-Gumbel-Morgenstern copulas: $C_\alpha(u_1,u_2,...,u_n) = \prod_{i=1}^n u_i\,(1+\alpha\prod_{i=1}^n(1-u_i))$.

Let $\alpha_{i_1,...,i_c} \in \mathbf{R}$ be constants such that $\sum_{c=2}^n \sum_{1\le i_1<...<i_c\le n} \alpha_{i_1,...,i_c}\delta_{i_1}...\delta_{i_c} \ge -1$ for all $\delta_i \in \{0,1\}$, $i = 1,...,n$. The choice $g_{i_1,...,i_c}(t_{i_1},...,t_{i_c}) = \alpha_{i_1,...,i_c}(1-2t_{i_1})(1-2t_{i_2})...(1-2t_{i_c})$, $1 \le i_1 < ... < i_c \le n$, $c = 2,...,n$, gives the following generalized multivariate Eyraud-Farlie-Gumbel-Morgenstern copulas (see Johnson and Kotz (1975) and Cambanis (1977))

$$C(u_1,...,u_n) = \prod_{k=1}^n u_k \left(1 + \sum_{c=2}^n \sum_{1\le i_1<...<i_c\le n} \alpha_{i_1,...,i_c}(1-u_{i_k})\right). \tag{38}$$

The bivariate cases of these copulas have the form (22).

The importance of the generalized Eyraud-Farlie-Gumbel-Morgenstern copulas and cdf's stems, in particular, from the fact that, as shown in Sharakhmetov and Ibragimov (2002), they completely characterize joint distributions of two-valued r.v.'s.

Let now $\alpha_1,...,\alpha_n \in (-1,1)\setminus\{0\}$, $\sum_{i=1}^n |\alpha_i| \le 1$. Taking in Proposition 2

$$g_{i_1,...,i_c}(t_{i_1},...,t_{i_c}) = \frac{\alpha_1...\alpha_n}{\alpha_{i_1}...\alpha_{i_c}}\left((l+1)t_{i_1}^l - (l+2)t_{i_1}^{l+1}\right)...\left((l+1)t_{i_c}^l - (l+2)t_{i_c}^{l+1}\right),$$

where $l \in \{0,1,2,...\}$, we obtain the following extensions of Eyraud-Farlie-Gumbel-Morgenstern copulas (38) that are natural to call power copulas:

$$C(u_1,...,u_n) = \prod_{i=1}^n u_i \left(1 + \sum_{1\le i_1<...<i_c\le n} \frac{\alpha_1...\alpha_n}{\alpha_{i_1}...\alpha_{i_c}}(u_{i_1}^l - u_{i_1}^{l+1})...(u_{i_c}^l - u_{i_c}^{l+1})\right). \tag{39}$$

As discussed in Remark 5 in de la Peña, Ibragimov and Sharakhmetov (2003), power copulas (39) with $g_{i_1,...,i_c}(t_{i_1},...,t_{i_c}) = 0$, $1 \leq i_1 < ... < i_c \leq n$, $c = 2,...,n$, $c \neq r+1$, give examples of copulas of $r-$independent r.v.'s obtained by Wang (1990) whose r.v.'s have copulas (39) with $l = 0$.

Taking $n = 2$, $g_{1,2}(t_1, t_2) = \theta c(t_1, t_2)$, where c is a continuous function on the unit square $[0, 1]^2$ satisfying the properties $\int_0^1 c(t_1, t_2)dt_1 = \int_0^1 c(t_1, t_2)dt_2 = 0$, $1 + \theta c(t_1, t_2) \geq 0$ for all $0 \leq t_1, t_2 \leq 1$, one obtains the class of bivariate densities studied by Rüschendorf (1985) and Long and Krzysztofowicz (1995) (see also Mari and Kotz (2001), pp. 73-78) $f(x_1, x_2) = f_1(x_1)f_2(x_2)(1 + \theta c(F_1(x_1), F_2(x_2)))$ with the covariance characteristic c and the covariance scalar θ. Furthermore, from Proposition 2 it follows that this representation in fact holds for an arbitrary density function and the function $\theta c(t_1, t_2)$ is unique.

As discussed in de la Peña, Ibragimov and Sharakhmetov (2003), Proposition 2, in particular, provides a device for obtaining complete characterizations of copulas of r.v.'s exhibiting different dependence structures. For instance, the following characterizations of r.v.'s satisfying $m-$dependence or $r-$independence hold.

Proposition 3 *(de la Peña, Ibragimov and Sharakhmetov (2003, Theorem 13)) R.v.'s $X_1, ..., X_n$ are $r-$independent if and only if the functions $g_{i_1,...,i_c}$ in representation (28) satisfy the conditions $g_{i_1,...,i_c}(u_{i_1}, ..., u_{i_c}) = 0$, $1 \leq i_1 < ... < i_c \leq n$, $c = 2, ..., r$.*

Proposition 4 *(de la Peña, Ibragimov and Sharakhmetov (2003, Theorem 11)). R.v.'s $X_1, ..., X_n$ are $m-$dependent if and only if the functions g in representation (28) satisfy the conditions $g_{i_1,...,i_k,i_{k+1},...,i_c}(u_{i_1}, ..., u_{i_k}, u_{i_{k+1}}, ..., u_{i_c}) = g_{i_1,...,i_k}(u_{i_1}, ..., u_{i_k})g_{i_{k+1},...,i_c}(u_{i_{k+1}}, ..., u_{i_c})$ for all $1 \leq i_1 < ... < i_k < i_{k+1} ... < i_c \leq n$, $i_{k+1} - i_k \geq m$, $k = 1, ..., c-1$, $c = 2, ..., n$.*

DNO obtained the following necessary and sufficient conditions for a time series process based on bivariate copulas to be first-order Markov.

For copulas $A, B : [0, 1]^2 \to [0, 1]$, set

$$(A * B)(x, y) = \int_0^1 \frac{\partial A(x, t)}{\partial t} \cdot \frac{\partial B(t, y)}{\partial t} dt.$$

Further, for copulas $A : [0, 1]^m \to [0, 1]$ and $B : [0, 1] \to [0, 1]$, define their $\star-$product $A \star B : [0, 1]^{m+n-k} \to [0, 1]$ via

$$A \star B(x_1, ..., x_{m+n-1}) = \int_0^{x_m} \frac{\partial A(x_1, ..., x_{m-1}, \xi)}{\partial \xi} \cdot \frac{\partial B(\xi, x_{m+1}, ..., x_{m+n-1})}{\partial \xi} d\xi.$$

As shown in DNO, the operators $*$ and $\star$ on the class of copulas are distributive over convex combinations, associative and continuous in each place, but not jointly continuous.

DNO proved that the transition probabilities $P(s, x, t, A) = P(X_t \in A|X_s = x)$ of a real stochastic process X_t, $t \in T$, satisfy the Chapman-Kolmogorov equations (6) if and only if the copulas corresponding to bivariate distributions of X_t are such that

$$C_{st} = C_{su} * C_{ut} \tag{40}$$

for all $s < u < t$. DNO also showed that a real valued stochastic process X_t, $t \in T$, is a first-order Markov process if and only if the copulas corresponding to the finite-dimensional distributions of X_t satisfy the conditions

$$C_{t_1,...,t_n} = C_{t_1 t_2} \star C_{t_2 t_3} \star ... \star C_{t_{n-1} t_n}$$

for all $t_1, ..., t_n \in T$ such that $t_k < t_{k+1}$, $k = 1, ..., n-1$.

8 Appendix A2. Proofs

Throughout the rest of the paper, for a r.v. X and $x \in \mathbf{R}$, $I_{X<x}$ denotes the indicator of the event $\{X < x\}$. In addition, we denote by F_X the cdf of the r.v. X. As usual, for a Borel set $A \in \Im$, the notation $X^{-1}(A)$ will stand for the event $\{\omega \in \Omega : X(\omega) \in A\}$.

Proof of Theorem 1. Let $n \geq k+1$. Let us show that the Markovian (order k) property (3) holds for $t_1 = 1, ..., t_n = n$ and $t = n+1$ if and only if

$$P\big(X_1 < x_1, ..., X_{n-k} < x_{n-k}, X_{n+1} < x_{n+1} \big| X_{n-k+1}, ..., X_n\big) =$$
$$P\big(X_1 < x_1, ..., X_{n-k} < x_{n-k} \big| X_{n-k+1}, ..., X_n\big) P\big(X_{n+1} < x_{n+1} \big| X_{n-k+1}, ..., X_n\big) \ (a.s.). \quad (41)$$

Indeed, suppose that (3) holds for $t_1 = 1, ..., t_n = n$ and $t = n+1$. Then we have

$$\begin{aligned}
P\big(X_1 < x_1, ..., X_{n-k} < x_{n-k}, X_{n+1} < x_{n+1} \big| X_{n-k+1}, ..., X_n\big) &= \\
E\big(I_{X_1<x_1}, ..., I_{X_{n-k}<x_{n-k}}, I_{X_{n+1}<x_{n+1}} \big| X_{n-k+1}, ..., X_n\big) &= \\
E\Big\{E\big(I_{X_1<x_1}, ..., I_{X_{n-k}<x_{n-k}}, I_{X_{n+1}<x_{n+1}} \big| X_1, ..., X_n\big) \Big| X_{n-k+1}, ..., X_n\Big\} &= \\
E\Big\{I_{X_1<x_1}, ..., I_{X_{n-k}<x_{n-k}} E\big(I_{X_{n+1}<x_{n+1}} \big| X_1, ..., X_n\big) \Big| X_{n-k+1}, ..., X_n\Big\} &= \\
E\Big\{I_{X_1<x_1}, ..., I_{X_{n-k}<x_{n-k}} E\big(I_{X_{n+1}<x_{n+1}} \big| X_{n-k+1}, ..., X_n\big) \Big| X_{n-k+1}, ..., X_n\Big\} &= \\
E\big(I_{X_1<x_1}, ..., I_{X_{n-k}<x_{n-k}} \big| X_{n-k+1}, ..., X_n\big) P\big(I_{X_{n+1}<x_{n+1}} \big| X_{n-k+1}, ..., X_n\big) &= \\
P\big(X_1 < x_1, ..., X_{n-k} < x_{n-k} \big| X_{n-k+1}, ..., X_n\big) P\big(X_{n+1} < x_{n+1} \big| X_{n-k+1}, ..., X_n\big)&,
\end{aligned}$$

that is, (41) holds. Conversely, if (41) holds, then from the above chain of equalities read in the opposite order it follows that

$$\begin{aligned}
E\Big\{I_{X_1<x_1}, ..., I_{X_{n-k}<x_{n-k}} E\big(I_{X_{n+1}<x_{n+1}} \big| X_1, ..., X_n\big) \Big| X_{n-k+1}, ..., X_n\Big\} &= \\
E\Big\{I_{X_1<x_1}, ..., I_{X_{n-k}<x_{n-k}} E\big(I_{X_{n+1}<x_{n+1}} \big| X_{n-k+1}, ..., X_n\big) \Big| X_{n-k+1}, ..., X_n\Big\}&,
\end{aligned}$$

that is, for all Borel subsets $B_{n-k+1}, ..., B_n$ of $\mathbf{R}$,

$$\begin{aligned}
E\Big\{I_{X_1<x_1} ... I_{X_{n-k}<x_{n-k}} I_{X_{n-k+1}\in B_{n-k+1}} ... I_{X_n \in B_n} E\big(I_{X_{n+1}<x_{n+1}} \big| X_1, ..., X_n\big)\Big\} &= \\
E\Big\{I_{X_1<x_1} ... I_{X_{n-k}<x_{n-k}} I_{X_{n-k+1}\in B_{n-k+1}} ... I_{X_n \in B_n} E\big(I_{X_{n+1}<x_{n+1}} \big| X_n\big)\Big\}&.
\end{aligned}$$

This relation means that (3) holds for $t_1 = 1, ..., t_n = n$ and $t = n+1$.

Suppose now that X_t, $t \in T$, is a Markov process of order k. Integrating (41) over $X_{n-k+1}^{-1}((-\infty, x_{n-k+1})) \times ... \times X_n^{-1}((-\infty, x_n))$, we get

$$C_{1,2,...,n+1}(F_1(x_1), ..., F_{n+1}(x_{n+1})) = F_{1,2,...,n+1}(x_1, ..., x_{n+1}) =$$
$$\int_{-\infty}^{x_{n-k+1}} ... \int_{-\infty}^{x_n} C_{1,2,...,n|n-k+1,...,n}(F_1(x_1), ..., F_{n-k}(x_{n-k}), F_{n-k+1}(\eta_{n-k+1}), ..., F_n(\eta_n)) \times$$
$$C_{n-k+1,...,n,n+1|n-k+1,...,n}(F_{n-k+1}(\eta_{n-k+1}), ..., F_n(\eta_n), F_{n-k+1}(x_{n+1}))dF_{n-k+1,...,n}(\eta_{n-k+1}, ..., \eta_n) =$$
$$\int_0^{F_{n-k+1}(x_{n-k+1})} ... \int_0^{F_n(x_n)} C_{1,2,...,n|n-k+1,...,n}(F_1(x_1), ..., F_{n-k}(x_{n-k}), \xi_{n-k+1}, ..., \xi_n) \times$$
$$C_{n-k+1,...,n,n+1|n-k+1,...,n}(\xi_{n-k+1}, ..., \xi_n, F_{n+1}(x_{n+1}))C_{n-k+1,...,n}(d\xi_{n-k+1}, ..., d\xi_n) =$$
$$C_{1,2,...,n} \star^k C_{n-k+1,...,n+1}(F_1(x_1), ..., F_{n+1}(x_{n+1})).$$

By induction, this implies that (4) holds.

Conversely, suppose that (4) holds. Then we have

$$EI_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}I_{X_{n-k+1}<x_{n-k+1}}...I_{X_n<x_n}I_{X_{n+1}<x_{n+1}} =$$
$$C_{1,2,...,n+1}(F_1(x_1), ..., F_{n-k}(x_{n-k}), F_{n-k+1}(x_{n-k+1}), ..., F_n(x_n), F_{n+1}(x_{n+1})) =$$
$$\int_0^{F_{n-k+1}(x_{n-k+1})} ... \int_0^{F_n(x_n)} C_{1,2,...,n|n-k+1,...,n}(F_1(x_1), ..., F_{n-k}(x_{n-k}), \xi_{n-k+1}, ..., \xi_n) \times$$
$$C_{n-k+1,...,n,n+1|n-k+1,...,n}(\xi_{n-k+1}, ..., \xi_n, F_{n+1}(x_{n+1}))C_{n-k+1,...,n}(d\xi_{n-k+1}, ..., d\xi_n) =$$
$$E\big(E(I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}|X_{n-k+1}, ..., X_n)I_{X_{n-k+1}<x_{n-k+1}}...I_{X_n<x_n}E(X_{n+1}|X_{n-k+1}, ..., X_n)\big).$$

This implies that (41) holds. ■

Proof of Theorem 2. Let $\{X_t\}_{t=0}^{\infty}$ be a $C-$based kth order stationary Markov chain. Using Propositions 2 and 3, we obtain that if the sequence $\{X_t\}_{t=0}^{\infty}$ exhibits $k-$independence, then the density of the copula C has form (29) with $n = k+1$ and the functions g such that $g_{i_1,...,i_c}(u_{i_1}, ..., u_{i_c}) = 0$, $1 \leq i_1 < ... < i_c \leq n$, $c = 2, ..., k$, that is, (7) holds with $g(u_1, ..., u_{k+1}) = g_{1,...,k+1}(u_1, ..., u_{k+1})$. In addition, by the same propositions, the above function g satisfies conditions (8) and (10) and is such that

$$\int_0^1 g(u_1, ..., u_{k+1})du_j = 0, \quad j = 1, 2, ..., k+1. \tag{42}$$

Further, from Corollary 1 it follows that the density of the copula $C_{1,2,...,k+1,k+2}$ of r.v.'s $X_1, X_2, ..., X_{k+1}$, X_{k+2} is given by

$$\frac{\partial^{k+2}C_{1,2,...,k+1,k+2}}{\partial u_1 \partial u_2 ... \partial u_{k+1} \partial u_{k+2}} = \frac{\partial^{k+1}C}{\partial u_1 \partial u_2 ... \partial u_{k+1}} \frac{\partial^{k+1}C}{\partial u_2 \partial u_3 ... \partial u_{k+2}} =$$
$$(1 + g(u_1, ..., u_{k+1}))(1 + g(u_2, ..., u_{k+2})) =$$
$$1 + g(u_1, ..., u_{k+1}) + g(u_2, ..., u_{k+2}) + g(u_1, ..., u_{k+1})g(u_2, ..., u_{k+2}). \tag{43}$$

Using (42) and (43) we get that, for $2 \leq i_1 < i_2 \leq k+1$, the density of the copula of the r.v.'s X_j, $j \in \{1, 2, ..., k+2\} \setminus \{i_1, i_2\}$ is given by

$$1 + \int_0^1 \int_0^1 g(u_1, ..., u_{k+1}) g(u_2, ..., u_{k+2}) du_{i_1} du_{i_2}.$$

This and $k-$independence of $\{X_t\}$ imply that

$$\int_0^1 \int_0^1 g(u_1, ..., u_{k+1}) g(u_2, ..., u_{k+2}) du_{i_1} du_{i_2} = 0, \quad 2 \leq i_1 < i_2 \leq k+1. \tag{44}$$

In complete similarity, by considering the $k-$dimensional marginal copulas of the r.v.'s $X_1, X_2, ..., X_{k+2}, X_{k+3}$ and using (42), (43) and (44), we obtain

$$\int_0^1 \int_0^1 \int_0^1 g(u_1, ..., u_{k+1}) g(u_2, ..., u_{k+2}) g(u_3, ..., u_{k+3}) du_{i_1} du_{i_2} du_{i_3} = 0, \quad 3 \leq i_1 < i_2 < i_3 \leq k+1.$$

Continuing in the same fashion, we get that the property that $\{X_t\}_{t=0}^{\infty}$ is a $C-$based $k-$th order $k-$independent Markov chain implies that (9) holds for all $s \leq u_{i_1} < ... < u_{i_s} \leq k+1$, $1, 2, ..., \left[\frac{k+1}{2}\right]$.

It is not difficult to see that relations (9) guarantee that, for $m = \left[\frac{k+1}{2}\right]$, the copula $C_{1,2,...,k+m}$ of the first $k+m$ r.v.'s $X_1, X_2, ..., X_{k+m}$ in a $k-$th order Markov chain $\{X_t\}_{t=0}^{\infty}$, has $k-$dimensional marginal copulas in the product form (30) with $n = k$, that is, the Markov chain exhibits $k-$independence. By stationarity of the process $\{X_t\}_{t=0}^{\infty}$, this completes the proof. ■

Proof of Theorem 3. Let $\{X_t\}_{t=0}^{\infty}$ be a $C-$based first order stationary Markov chain. By Proposition 2, the density of the copula C is given by (11) with the function $g(u_1, u_2)$ satisfying conditions (12)-(14). In addition, according to Corollary 1, the density of the copula $C_{1,2,...,m,m+1}$ of the r.v.'s $X_1, X_2, ..., X_m, X_{m+1}$ has the form

$$\frac{\partial^{m+1} C_{1,2,...,m,m+1}}{\partial u_1 \partial u_2 ... \partial u_m \partial u_{m+1}} =$$

$$(1 + g(u_1, u_2))(1 + g(u_2, u_3))...(1 + g(u_m, u_{m+1})) = \prod_{s=1}^{m} (1 + g(u_s, u_{s+1})).$$

Using relations (13) we thus get that the copula $C_{1,m+1}$ of the r.v.'s X_1 and X_{m+1} is given by

$$C_{1,m+1}(u_1, u_{m+1}) = \int_0^1 ... \int_0^1 \prod_{s=1}^{m} (1 + g(u_s, u_{s+1})) du_2 ... du_m = 1 + \int_0^1 ... \int_0^1 \prod_{s=1}^{m} g(u_s, u_{s+1}) du_2 ... du_m.$$

Thus, the copula $C_{1,m+1}$ is the product copula: $C_{1,m+1}(u_1, u_{m+1}) = u_1 u_{m+1}$ if and only if condition (15) is satisfied. By stationarity of $\{X_t\}_{t=0}^{\infty}$ this implies that it is $m-$dependent $k-$th order $C-$based Markov chain if and only if C satisfies the conditions of Theorem 3. ■

Proof of Theorem 5. Using relations (9) in Theorem 2 with $s = 2$ and $i_1 = 2$, $i_2 = 3$, we get that if, under the conditions of Theorem (5), $\{X_t\}$ is a $C-$based $k-$independent $k-$th order Markov chain, then

$$\int_0^1 \int_0^1 g(u_1, u_2, ..., u_{k+1}) g(u_2, u_3, ..., u_{k+2}) du_2 du_3 =$$

$$\int_0^1 \int_0^1 \alpha^2 f(u_1) f^2(u_2) f^2(u_3) ... f^2(u_{k+1}) f(u_{k+2}) du_2 du_3 = 0,$$

that is,

$$\alpha^2 \Big[\int_0^1 f^2(u_2) du_2 \Big] \Big[\int_0^1 f^2(u_3) du_3 \Big] f(u_1) f^2(u_3) ... f^2(u_{k+1}) f(u_{k+2}) = 0.$$

This evidently implies that $g(u_1, u_2, ..., u_{k+1}) = \alpha f(u_1) ... f(u_{k+1}) = 0$ and, thus, $\{X_t\}$ is a sequence of i.i.d. r.v.'s. ■

Proof of Corollary 2. The corollary is a consequence of Theorem 5 applied to the function $f(u) = 1 - 2u$. ■

Proof of Corollary 3. The corollary is a consequence of Theorem 5 applied to the function $f(u) = (l+1)u^l - (l+2)u^{l+1}$. ■

Proof of Theorem 6. Using relation (15) in Theorem 3, we obtain that, if, under the conditions of Theorem 6, $\{X_t\}$ is an $m-$dependent $C-$based Markov chain, then

$$\int_0^1 ... \int_0^1 \alpha^m f(u_1) f^2(u_2) ... f^2(u_m) f(u_{m+1}) du_2 ... du_m = 0,$$

that is,

$$\alpha^m f(u_1) f(u_{m+1}) \Big[\int_0^1 f^2(u_2) du_2 \Big]^{m-1} = 0.$$

This evidently implies that $f(u) = 0$ and, thus, $\{X_t\}$ is a sequence of i.i.d. r.v.'s. ■

Proof of Corollary 4. The corollary is a consequence of Theorem 6 applied to the function $f(u) = 1 - 2u$. ■

Proof of Theorem 4. By Markov property, $P(X_n > x | \Im_{n-1}) = P(X_n < -x | \Im_{n-1})$, if and only if $P(X_n > x | X_{n-1}) = P(X_n < -x | X_{n-1})$. The latter inequality, in turn, is equivalent to $P(V_n > 1/2 + u | V_{n-1}) = P(V_n < 1/2 - u | V_{n-1})$, where $V_t = F(X_t)$ and, by stationarity, to $P(V_2 > 1/2 + u | V_1) = P(V_2 < 1/2 - u | V_1)$, $u \in [0, 1/2)$. We have therefore, that $\{X_n\}$ is a conditionally symmetric martingale difference if and only if $\frac{\partial C(V_1, 1/2-u)}{\partial u_1} = 1 - \frac{\partial C(V_1, 1/2+u)}{\partial u_1}$ (a.s.), or, equivalently, if and only if (16) and (17) hold. ■

Proof of Theorem 7. The argument for the theorem is based on the results obtained in Arcones and Yu (1994) concerning convergence of empirical processes for mixing processes under almost minimal conditions and the approach utilizing the empirical transformation of the original process.[3] Consider the process $\{Y_t\}$, $t = 1, 2, ...,$ obtained from the process $\{X_t\}$, $t = 1, 2, ...,$ via the empirical transformation $Y_t = F(X_t)$. It is well-known that under the assumption of continuity of the cdf F each of the r.v.'s Y_t is uniformly distributed on $[0, 1]$ and all the finite-dimensional copulas of the process $\{Y_t\}$, $t = 1, 2, ...,$ are the same as those of the original process $\{X_t\}$, $t = 1, 2, ...$ Furthermore, the process $\{Y_t\}$, $t = 1, 2, ...,$ is stationary and satisfies the same mixing conditions as the original process. By definition, the copula $C(u_1, ..., u_{k+1})$ is the joint cdf of the $k+1$ successive r.v.'s in the sequence $\{Y_t\}$. Denote by $F_n^*(u) = 1/n \sum_{i=1}^n I_{U_i \le u}$ the one-dimensional empirical cdf of the sample $U_1, ..., U_n$. Let

$$H_n^*(u_1, ..., u_{k+1}) = (1/(n-k)) \sum_{i=k+1}^{n} \prod_{j=1}^{k+1} I_{U_{i+j-k-1} \le x_j}$$

stand for the empirical cdf of the $k + 1$ successive r.v.'s in $\{Y_t\}$, $t = 1, 2, ...,$ and let

$$C_n^*(u_1, ..., u_{k+1}) = H_n^*((F_n^*)^{-1}(u_1), ..., (F_n^*)^{-1}(u_{k+1}))$$

denote the empirical copula of the successive $k + 1$ variables in $\{Y_t\}$. Similar to Lemma 3 in Fermanian, Radulovic and Wegkamp (2002) one can show that

$$C_n(i_1/n, ..., i_{k+1}/n) = C_n^*(i_1/n, ..., i_{k+1}/n) \tag{45}$$

for $i_1, ..., i_{k+1} = 0, 1, ..., n$. Indeed, let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ be the order statistics of the sample $X_1, X_2, ..., X_n$, $X_{(0)} = -\infty$ and $X_{(n+1)} = +\infty$, $i_{sn} = i_s/n$, $s = 1, ..., k + 1$. We have

$$C_n(i_{1n}, ..., i_{k+1,n}) = H_n(F_n^{-1}(i_{1n}), ..., F_n^{-1}(i_{k+1,n})) =$$

$$H_n(X_{(i_1)}, ..., X_{(i_k)}) = C_n^*(F(X_{(i_1)}), ..., F(X_{(i_k)})) =$$

$$C_n^*(Y_{(i_1)}, ..., Y_{(i_k)}) = C_n^*(i_{1n}, ..., i_{k+1,n}).$$

We have that for all $u_1, ..., u_{k+1} \in [0, 1]$ there exist $i_1/n, ..., i_{k+1}/n \in [0, 1]$ such that

$$C_n(u_1, ..., u_{k+1}) = C_n(i_1/n, ..., i_{k+1}/n).$$

By (45) we get therefore that

$$\sqrt{n-k}(C_n - C)(u_1, ..., u_{k+1}) = \sqrt{n-k}(C_n^* - C)(u_1, ..., u_{k+1}).$$

The statement of the theorem now follows from Corollary 2.1 in Arcones and Yu (1994).

[3] The approach based on the empirical transformations was applied by Fermanian, Radulovic and Wegcamp (1994) in the case of two independent samples.

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