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Optimal Bundling Strategies For Complements And Substitutes With Heavy-Tailed Valuations

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OPTIMAL BUNDLING STRATEGIES FOR COMPLEMENTS AND SUBSTITUTES WITH HEAVY-TAILED VALUATIONS [1]

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ABSTRACT

We develop a framework that allows one to model the optimal bundling problem of a multiproduct monopolist providing interrelated goods with an arbitrary degree of complementarity or substitutability. Characterizations of

[1]The results in this paper constitute a part of the author's dissertation "New majorization theory in economics and martingale convergence results in econometrics" presented to the faculty of the Graduate School of Yale University in candidacy for the degree of Doctor of Philosophy in Economics in March, 2005. Some of the results were originally contained in the work circulated under the title "On the robustness of economic models to heavy-tailedness assumptions"

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optimal bundling strategies are derived for the seller in the case of long-tailed valuations and tastes for the products. We show, in particular, that if goods provided in a Vickrey auction or any other revenue equivalent auction are substitutes and bidders' tastes for the objects are not extremely heavy-tailed, then the monopolist prefers separate provision of the products. However, if the goods are complements and consumers' tastes are extremely thick-tailed, then the seller prefers providing the products on a single auction. We also present results on consumers' preferences over bundled auctions in the case when their valuations exhibit heavy-tailedness. In addition, we obtain characterizations of optimal bundling strategies for a monopolist who provides complements or substitutes for profit-maximizing prices to buyers with long-tailed tastes.

JEL Classification: D42, D44, L12, L21

KEYWORDS: Optimal bundling strategies, multiproduct monopolist, Vickrey auction, substitutes, complements, heavy-tailed valuations, tastes, robustness

1. INTRODUCTION AND DISCUSSION OF THE RESULTS

1.1. Optimal bundling decisions for a multiproduct monopolist. The last quarter of a century has witnessed a surge in the interest in the analysis of optimal bundling strategies for a multiproduct monopolist. In particular, many studies in the literature emphasized that bundling decisions of a monopoly providing two goods depend on correlations between consumers' valuations for the products (see Adams and Yellen, 1976, McAfee, McMillan and Whinston, 1989, Schmalensee, 1984, and Salinger, 1995), the degrees of complementarity and substitutability between the goods (e.g., Lewbel, 1985, and Venkatesh and Kamakura, 2003) and the marginal costs for the products (see, among others, Salinger, 1995, and Venkatesh and Kamakura, 2003). Most of studies on bundling have focused, however, on prescribed distributions for reservation prices in the case of two products and their packages, such as bivariate uniform or Gaussian distributions, and only a few general results are available for larger bundles (e.g., Palfrey, 1983, Bakos and Brynjolfsson, 1999, 2000, and Fang and Norman, 2003a, b).[3] For instance, Palfrey (1983) obtained characterizations of the monopolist's and buyers' preferences over bundled Vickrey auctions of *independently priced goods*, that is, products such that consumers' valuations for their bundles are additive in those for the component goods, as opposed to the case of *interrelated goods*, e.g., substitutes or complements (see Dansby and Conrad, 1984, Lewbel, 1985, Section 3.2 in Bakos and Brynjolfsson, 1999, Venkatesh and Kamakura, 2003, and Section 4 in the present paper). Palfrey (1983) showed that, in the case of two bidders with bounded valuations satisfying certain distributional assumptions, the seller maximizes her profit by selling the goods in a single bundle; the two buyers, however, unanimously prefer separate provision of objects to any other bundling decision. Palfrey's (1983) results also imply that, if stand-alone valuations are concentrated on a finite interval, then consumers never unanimously prefer separate provision of items to a single Vickrey auction, *ex ante*, if there are more than two buyers (see Theorems 5-7 in Palfrey, 1983). In a related paper, Chakraborty (1999) obtained characterizations of optimal bundling strategies for a monopolist providing two independently priced goods on Vickrey auctions under a regularity condition on quantiles of reservation prices. As follows from Proschan's (1965) results given by Proposition 6.1 in Appendix A1 in this paper, this regularity condition is satisfied for symmetric valuations with log-concave densities. Bakos and Brynjolfsson (1999) investigated optimal bundling decisions for a multiproduct monopolist providing large bundles of independently priced goods with zero marginal costs (information goods) for profit-maximizing prices to consumers whose valuations belong to a class that includes, again by Proschan (1965), reservation prices with symmetric log-concave densities.[4] Among other results, Bakos and Brynjolfsson (1999) showed that, in the latter setting, if the seller prefers bundling a certain number of goods to selling them separately and if the optimal price per good for the bundle is less than the mean valuation, then bundling any greater number of goods will further increase the seller's profits, compared to the case where the additional goods are sold separately. According to the result, if consumers' valuations have the above regularity property, then a form of superadditivity for bundling decisions holds, that is, the benefits to the seller grow as the number of goods in the bundle increases.[5] Recently, Fang and Norman (2003b) showed that a multiproduct monopolist providing bundles of independently priced goods to

[3]The terms "reservation prices" and "valuations" are used as synonyms in this paper, in accordance with the well-established tradition in the bundling literature.

[4]More precisely, according to Proschan's (1965) results given by Proposition 6.1 in Appendix A1, the distributional assumptions in Bakos and Brynjolfsson (1999) are satisfied for valuations with log-concave densities symmetric about the mean reservation price. In particular, the assumptions are satisfied for valuations with a finite support $[\underline{v}, \overline{v}]$ distributed as the truncation $XI(|X-\mu|<h)$, $h>0$, of an arbitrary random variable X with a log-concave density symmetric about $\mu=(\underline{v}+\overline{v})/2$, where $h=(\overline{v}-\underline{v})/2$ and $I(\cdot)$ is the indicator function (see also Remark 2 in An, 1998).

[5]This property is similar to the case of Vickrey auctions with two buyers (see Remark 2 in Palfrey, 1983).

consumers with valuations with log-concave densities prefers selling them separately to any other bundling decision if the marginal costs of all the products are greater than the mean valuation; under some additional distributional assumptions, the seller prefers providing the goods as a single bundle to any other bundling decision if the marginal costs of the goods are identical and are less than the mean reservation price.

To our knowledge, all general results in the bundling literature available for an arbitrary number of goods are based on conditions which are satisfied only for valuations with *light-tailed* distributions, such as those with log-concave densities or with a bounded support, and also typically maintain the assumption that the goods provided are independently priced. In particular, as far as we know, there are no results in the literature on the optimal bundling problem in the case of *heavy-tailed* valuations for the products, even in the case of independently priced goods and bundles consisting of two items.

1.2. Heavy-tailedness paradigm. Much of the modern literature in economics and finance have focused on the study of the so-called "thick-tailedness" paradigm. This stream of literature goes back to Mandelbrot (1963) (see also the papers in Mandelbrot, 1997, and Fama, 1965), who pioneered the study of heavy-tailed distributions with tails declining as $x^{-\alpha}$, $\alpha > 0$, in these fields. If a model involves a r.v. X with such thick-tailed distribution, then

$$P(|X| > x) \sim x^{-\alpha}. \tag{1.1}$$

The r.v. X for which this is the case has finite moments $E|X|^p$ of order $p < \alpha$. However, the moments are infinite for $p \geq \alpha$.

It was documented in numerous studies that the time series encountered in many fields in economics and finance are heavy-tailed (see the discussion in Loretan and Phillips, 1994, Meerschaert and Scheffler, 2000, Gabaix, Gopikrishnan, Plerou and Stanley, 2003, and references therein). Motivated by these empirical findings, a number of studies in financial economics have focused on portfolio and value-at-risk modelling with heavy-tailed returns (see, e.g., the reviews in Duffie and Pan, 1997, Uchaikin and Zolotarev, 1999, Ch. 17, and Glasserman, Heidelberger and Shahabuddin, 2002). Several authors considered problems of statistical inference for data from thick-tailed populations (see Loretan and Phillips, 1994, the papers in Adler, Feldman and Taqqu, 1998, and references therein). Mandelbrot (1963) presented evidence that historical daily changes of cotton prices have the tail index $\alpha \approx 1.7$, and thus have infinite variances. Using different models and statistical techniques, subsequent research reported the following estimates of the tail parameters α for returns on various stocks and stock indices: $3 < \alpha < 5$ (Jansen and de Vries, 1991); $2 < \alpha < 4$ (Loretan and Phillips, 1994); $1.5 < \alpha < 2$ (McCulloch, 1996, 1997); $0.9 < \alpha < 2$ (Rachev and Mittnik, 2000). Recent studies (see Gabaix et. al., 2003, and references therein) have found that the returns on many stocks and stock indices have the tail exponent $\alpha \approx 3$, while the distributions of trading volume and the number of trades on financial markets obey power laws (1.1) with $\alpha \approx 1.5$ and $\alpha \approx 3.4$, respectively. As discussed in Gabaix et. al. (2003), these estimates of the tail indices α are robust to different types and sizes of financial markets, market trends and are similar for different countries. Motivated by these empirical findings, Gabaix et. al. (2003) proposed a model that demonstrates that the above power laws for stock returns, trading volume and the number of trades are explained by trading of large market participants, namely, the largest mutual funds whose sizes have the tail exponent $\alpha \approx 1$. Power laws (1.1) with $\alpha \approx 1$ (Zipf laws) have also been found to hold for firm sizes (see

Axtell, 2001) and city sizes (see Gabaix, 1999a, b for discussion and explanations of the Zipf law for cities). One should also note that some studies also report the tail exponent to be close to one or even slightly less than one for such financial time series as Bulgarian lev/US dollar exchange spot rates and increments of the market time process for Deutsche Bank price record (see Rachev and Mittnik, 2000).

The fact that a number of economic and financial time series have the tail exponents of approximately one is very important in the context of the results in this paper: as we demonstrate, optimal bundling strategies for substitutes and consumers' valuations and tastes with the tail exponents $\alpha < 1$ and infinite means are the opposites of those for complements and buyers' reservation prices and tastes with $\alpha > 1$ for which the first moment is finite.

It is important to emphasize here that heavy-tailedness concepts provide a natural framework for modelling marketing strategies for goods with extreme valuations in the real world. For example, strategies involving exclusion of goods with extreme valuations and selling them separately are often employed on the market, in particular, by cable and direct satellite broadcast television firms. The latter firms typically offer a "basic" bundle and use such strategies as pay-per-view approach for unusual special events such as, for instance, boxing matches (see Bakos and Brynjolfsson, 1999). The high valuations for the special events are concentrated among a small fraction of consumers and thus are likely to be very heavy-tailed. Season tickets for entertainment performances offered by sporting and cultural organizations might illustrate the dual pattern in bundling. It seems plausible that most of the demand for season tickets is concentrated around a relative small fraction of consumers that have high (and very thick-tailed) valuations for performances offered by the entertainment organizations. However, in contrast to television firms, the companies offering the tickets often choose providing them in bundles to consumers with extreme tastes for the performances.[6]

One should also note here that distributions with log-concave densities for which several general results in the optimal bundling literature exist (see the previous subsection) *cannot* be used to model heavy-tailedness. This is because any such density has at most an exponential tail and thus *all* its moments are finite (see An, 1998, and Section 2 in this paper).

Several frameworks have been proposed to model heavy-tailedness phenomena, including stable distributions, Pareto distributions, multivariate $t-$distributions, mixtures of normals, power exponential distributions, ARCH processes, mixed diffusion jump processes, variance gamma and normal inverse Gamma distributions. However, the debate concerning the values of the tail indices for different heavy-tailed financial data and on appropriateness of their modelling based on certain above distributions is still under way in empirical literature. In particular, as indicated before, a number of studies continue to find tail parameters less than two in different financial data sets and also argue that stable distributions are appropriate for their modelling.

[6]see Remark 5.4 in this paper for explanation of the dual patterns in the bundling decisions in the above settings on the basis of our results.

1.3. The main results of the paper: optimal bundling decisions for a multiproduct monopolist in the case of heavy-tailed reservation prices and interrelated goods. The present paper contributes to the existing literature on bundling and thick-tailedness in economics in a number of ways. First, we develop a framework that allows one to model the optimal bundling problem of a multiproduct monopolist providing large bundles of interrelated goods with an arbitrary degree of complementarity or substitutability. Second, we derive characterizations of optimal bundling strategies for the seller in this setup in the case of long-tailed valuations and tastes for the products, where, as indicated in Subsection 1.1, no results in the literature are available even in the case of two independently priced goods, to our knowledge. Third, our analysis provides a unified approach to the study of optimal bundling problems in the case where the goods are provided on an auction as well as in the setting where the prices for the products are set by the monopolist (see Sections 3-5 that present our main results along the above aims of the paper). In particular, the approach developed in this paper reveals that the analysis of optimal bundling strategies in both of the above cases is based on the same probabilistic concepts and results (see the next subsection for details).

Moreover, our study shows that patterns in the optimal bundling strategies are the opposites of one another, depending on the degrees of thick-tailedness of consumers' valuations and the degrees of complementarity and substitutability among the goods provided (e.g., Theorems 4.1 and 4.2 and Theorems 5.1 and 5.2). In that, for instance, the solutions to the optimal bundling problem with not too long-tailed valuations are the opposites of those in the case of very thick-tailed reservation prices, even in the case of independently priced goods. In particular, our analysis shows that many of the results available in the literature for independently priced goods with very light-tailed valuations (such as reservation prices with log-concave densities or those with a bounded support) are reversed in the case of valuations with very thick-tailed distributions. However, the results for very light-tailed reservation prices continue to hold under the assumption that distributions of the reservation prices are not too thick-tailed. In other words, the optimal bundling strategies analyzed in the literature in the case of very light-tailed valuations are robust to thick-tailedness assumptions for consumers' valuations as long as the distributions entering the assumptions are not too heavy-tailed.[7] However, they are reversed in the case of assumptions that involve very thick-tailed distributions.

In addition, the approach developed in this paper allows one to study, in a unified way, both the seller's and consumers' preferences over bundling decisions under heavy-tailed valuations. According to our results, the seller's and the buyers' preferences over bundles continue to be the opposites of one another in the case of very thick-tailed valuations, similar to the results with very light-tailed reservation prices available in the literature. However, in the framework with very long-tailed valuations, consumers' surplus is maximized under separate provision of independently priced goods, regardless of the number of buyers. This conclusion established in the present paper is in contrast with the results available in the literature for very light-tailed case.

More precisely, we show that if the goods provided on a Vickrey auction are independently priced or are substitutes (or complements with not very high degree of complementarity) and bidders' tastes for the objects are not very heavy-tailed, then the risk-neutral monopolist strictly prefers separate provision of the products to any

[7]According to well-established parlance in the many scientific literatures, robustness is understood to mean sensitivity to distributional assumptions. In the paper, the use of the terms "robust" and "robustness" accords with this tradition.

other bundling decision (Theorem 4.1). The results are reversed, however, in the case of a risk-averse auctioneer providing independently priced goods or complements (or substitutes with not very high degree of substitutability) to consumers with very long-tailed tastes for the products (Theorem 4.2).[8] According to our analysis, in the latter case, regardless of the number of consumers, the seller always strictly prefers providing the goods on a single Vickrey auction to any other bundling decision, as in the setting with two buyers in Palfrey (1983). This conclusion provides, in particular, a reversal of the results in Chakraborty (1999) from which it follows that, in the case of symmetric valuations satisfying comparisons that hold for distributions with log-concave densities, provision of independently priced goods through separate Vickrey auctions generates larger expected profits to the seller than any other bundling decision if the number of buyers is sufficiently large. We also obtain a characterization of consumers' preferences over the monopolist's bundling decision in a Vickrey auction in the case of heavy-tailed valuations for the products. We show, for instance, that if bidders' reservation prices for independently priced goods are very heavy-tailed, as modelled by positive stable distributions (see Section 2), then they unanimously prefer separate Vickrey auctions to any other bundling decision (Theorem 4.3 and Remark 4.2). These results are at odds with a setting where valuations have a finite distributional support in which, according to Palfrey (1983), consumers never unanimously prefer separate provision of the products, as indicated in Subsection 1.1.

We also obtain characterizations of optimal bundling strategies for a monopolist who provides goods with an arbitrary degree of complementarity or substitutability to consumers with heavy-tailed tastes for profit-maximizing prices (Theorems 5.1-5.4). We show, in particular, that, for products with high marginal costs, the seller's optimal strategy is to provide complements with very heavy-tailed consumers' tastes for them separately and those with sufficiently light-tailed valuations as a single bundle. For relatively low marginal costs, these conclusions are reversed (Theorems 5.1 and 5.2). In addition, contrary to the case of very light-tailed valuations and independently priced products considered in Bakos and Brynjolfsson (1999) and Fang and Norman (2003b), if consumers' tastes for the products are very long-tailed, then the monopolist's optimal strategy is to provide independently priced goods or complements with relatively high marginal costs as a single bundle and those with sufficiently low marginal costs separately (Theorem 5.4). Our results also imply, for instance, that for positive stable distributions of tastes, irrespective of the marginal costs of producing the goods in question, the optimal strategy is to provide the goods as a single bundle if the goods are independently priced or are complements (Remark 5.3). Such distributions are of particular importance since bundling models based on them satisfy the free disposal condition often imposed in the case of information goods and in the economics of the Internet.

1.4. Probabilistic foundations for the main results. The proof of the results in this paper is based on general results on peakedness properties of convolutions of distributions and majorization phenomena for tail probabilities of linear combinations of random variables (r.v.'s) presented in Appendix A1. These properties and phenomena were first analyzed, under the assumptions of log-concavity of distributions, in the seminal paper by Proschan (1965) that found applications in the study of many problems in statistics, econometrics and economic theory and other fields (see the discussion in Ibragimov, 2005). The proof of the main results in this paper is based on analogues of the results in Proschan (1965) in the case of heavy-tailed distributions and majorization comparisons between powers of coefficients of linear combinations of r.v.'s recently obtained by Ibragimov (2004)

[8]The assumption of seller's risk aversion is necessary in the case of very heavy-tailed tastes and valuations since otherwise the monopolist's expected profit is infinite for any bundling decision.

and also presented in Ibragimov (2005). To our knowledge, the results in Ibragimov (2004, 2005) are the first ones in the literature that give extensions of those in Proschan (1965) for comparisons between arbitrary powers of components of linear combinations of r.v.'s and their reversals for general classes of distributions. These results provide the key to the analysis of bundling problems for complements and substitutes and to reversals of the optimal bundling strategies in the case of very thick-tailed valuations in this paper. Besides the analysis of optimal bundling strategies for complements and substitutes considered in this paper, the majorization results obtained in Ibragimov (2004, 2005) have many other applications. These applications include the study of efficiency of linear estimators and monotone consistency of the sample mean, robustness of the model of demand-driven innovation and spatial competition over time, value at risk analysis as well that of inheritance models in mathematical evolutionary theory.[9]

The main intuition behind the analysis of optimal bundling decisions under very light-tailed valuations in the literature (see the discussion in Palfrey, 1983, Schmalensee, 1984, Salinger, 1995, Bakos and Brynjolfsson, 1999, and Fang and Norman, 2003b) is that, for such reservation prices, consumers' valuations per good for a bundle typically have a lower variance relative to the valuations for individual goods.[10] The underlying intuition that drives our results on bundling under heavy-tailed valuations is closely related to that above. Namely, the results on peakedness and majorization obtained in Ibragimov (2004) imply, essentially, that, in the case of not very heavy-tailed reservation prices, the consumers' valuations per good for bundles *always* have less spread relative to the valuations for component products, as measured by their peakedness (see Appendix A1 for details on the concept of peakedness and related results). On the other hand, in the case of very heavy-tailed valuations, the spread of reservation prices per product for bundles, as measured by peakedness, is *always* greater than that of valuations for components (the reader is referred to Sections 4 and 5 for more on the intuition).

[9]The following list summarizes some of other applications of the main majorization results in Ibragimov (2004) presented in the author's Ph.D. dissertation Ibragimov (2005).

(i) From the majorization results it follows that the sample mean is the best linear unbiased estimator of the population mean for not extremely heavy-tailed populations in the sense of its peakedness properties. Moreover, in such a case, the sample mean exhibits the important property of monotone consistency and, thus, an increase in the sample size always improves its performance. However, efficiency of the sample mean in the sense of its peakedness decreases with the sample size if the sample mean is used to estimate the population center under extreme thick-tailedness. The main majorization results in Ibragimov (2004) also provide sharp concentration inequalities for linear estimators as well as their extensions to the case of wide classes of dependent data.

(ii) Using the general majorization results, we show, for the first time in the literature, that the stylized fact that portfolio diversification is always preferable is reversed for a wide class of distributions of risks. The class of distributions for which this is the case is the class of extremely heavy-tailed distributions. The encouraging message of the results is that the stylized facts on diversification are nevertheless robust to thick-tailedness of risks or returns as long as their distributions are not extremely long-tailed.

Moreover, we demonstrate that, in the world of not extremely heavy-tailed risks, VaR satisfies the important condition of coherency, which is a natural requirement to be imposed on a measure of risk from the points of view of exchange, regulators and society. However, coherency of the value at risk is always violated if distributions of risks are extremely thick-tailed. We also obtain sharp bounds on the VaR of the returns on portfolios of risks with long-tailed returns.

(iv) Another application of the main majorization results explored in depth in Ibragimov (2005) concerns the analysis of growth of firms that invest into learning about the next period's optimal product. We present a study of robustness of the model of demand-driven innovation and spatial competition over time with log-concavely distributed signals developed by Jovanovic and Rob (1987) to heavy-tailedness assumptions. The implications of the model remain valid for not extremely long-tailed distributions of consumers' signals. However, again these properties are reversed for signals with extremely thick-tailed densities.

(v) We study transmission of traits through generations in multifactorial inheritance models with sex- and time-dependent heritability. We further analyze the implications of these models under heavy-tailedness of traits' distributions. Among other results, we show that in the case of a trait (for instance, a medical or behavioral disorder or a phenotype with significant heritability affecting human capital in an economy) with not very thick-tailed initial density, the trait distribution becomes increasingly more peaked, that is, increasingly more concentrated and unequally spread, with time. But these patterns are reversed for traits with sufficiently heavy-tailed initial distributions (e.g., a medical or behavioral disorder for which there is no strongly expressed risk group or a relatively equally distributed ability with significant genetic influence). Such traits' distributions become less peaked over time and increasingly more spread in the population.

[10]Further intuition behind the power of bundling is that, for light-tailed distributions, it reduces uncertainty about consumers' valuations and leads to a decrease in extreme values of the distribution of valuations per good, thereby reducing buyer diversity and increasing the predictive power of the selling strategy (see Schmalensee, 1984, and Bakos and Brynjolfsson, 1999).

1.2. Thick tails and extremely thick tails and extensions to the case of dependence. To illustrate the main ideas of the proof and in order to simplify the presentation of the main results in this paper, we model heavy-tailedness using the framework of independent stable distributions and their convolutions. More precisely, the class of not extremely thick-tailed distributions is modelled using convolutions of stable distributions with (different) indices of stability greater than one. Similarly, the results of the paper for extremely heavy-tailed case are first presented and proven using the framework of convolutions of stable distributions with characteristic exponents less than one. The former class has tail exponents $\alpha > 1$ and for the latter class one has $\alpha < 1$.

However, as follows from the extensions of the majorization results in Appendix A1 in this paper to the dependent case obtained in Ibragimov (2004), all the results obtained in the paper continue to hold for a wide class of multivariate distributions for which marginals are dependent and can be non-identical and, in addition to that, can have finite variances, unlike stable distributions and their convolutions. Namely, all the results in the paper continue to hold for convolutions of dependent r.v.'s with joint $\alpha-$symmetric distributions and their analogues with non-identical marginals.[11] The class of $\alpha-$symmetric distributions is very wide and includes, in particular, spherical distributions corresponding to $\alpha = 2$. Important examples of spherical distributions, in turn, are given by Kotz type, multinormal and logistic distributions and multivariate stable laws. In addition, they include a subclass of mixtures of normal distributions as well as multivariate $t-$distributions that were used in a number of papers to model heavy-tailedness phenomena with dependence and finite moments up to a certain order (see, among others, Praetz, 1972, Blattberg and Gonedes, 1974, and Glasserman et. al., 2002). Moreover, the class of $\alpha-$symmetric distributions includes a wide class of convolutions of models with common shocks affecting all consumers' valuations (such as macroeconomic or political ones, see Andrews, 2003) which are of great importance in economics and finance. Similar to the framework based on stable distributions, optimal bundling strategies for substitutes and consumers' tastes with joint $\alpha-$symmetric distributions with relatively large α's are the opposites of those for complements and buyers' tastes with $\alpha-$symmetric distributions with relatively small α's. The proof of these generalizations is completely similar to the proof of the main results in this paper.

One should also emphasize here that the results in the paper do not require the distributions entering their assumptions to be extremely heavy-tailed with $\alpha < 1$ in order to exhibit reversals of optimal bundling strategies for interrelated goods. Namely, as we demonstrate, optimal bundling strategies of a multiproduct monopolist depend crucially on both thick-tailedness of the tastes (characterized by the tail parameters α) as well as on the degree r of complementarity or substitutability among the products provided, with $r > 1$ corresponding to the case of complements and $r < 1$ modelling the case of substitutes. For instance, the characterizations of the optimal bundling strategies for the seller of baskets of complements or substitutes derived in the paper depend on comparisons between α and r. Therefore, the strategies exhibit reversal patterns even in the case when consumers have tastes with $\alpha > 1$. Namely, the optimal bundling strategies of a multiproduct monopolist providing complements with $r > \alpha$ on auctions or for profit-maximizing prices are reversals of her optimal strategies in the case of substitutes (for which $r < 1$) and of those in the case of complements with $1 < r < \alpha$.[12]

[11] An $n-$dimensional distribution is called $\alpha-$symmetric if its characteristic function can be written as $\phi((\sum_{i=1}^n |t_i|^\alpha)^{1/\alpha})$, where ϕ is a continuous function and $\alpha > 0$. Such distributions should not be confused with multivariate spherically symmetric stable distributions, which have characteristic functions $\exp\big[-\lambda\big(\sum_{i=1}^n t_i^2\big)^{\beta/2}\big]$, $0 < \beta \leq 2$. Obviously, spherically symmetric stable distributions are particular examples of $\alpha-$symmetric distributions with $\alpha = 2$ (that is, of spherical distributions) and $\phi(x) = \exp(-x^\beta)$.

[12] The case $r > \alpha$ corresponds to complements with relatively high degree of complementarity. Similarly, the case $1 < r < \alpha$ represents

We also note that all the results in the paper are available for the case of skewed distributions, including skewed stable distributions (such as, for instance, extremely heavy-tailed Lévy distributions with $\alpha = 1/2$ concentrated on the positive semi-axis) and, according to the extensions discussed above, $\alpha-$symmetric distributions with skewed marginals. Therefore, this paper, in fact, succeeds in the unification of the analysis of the effects of all the main distributional properties of consumers' valuations, including heavy-tailedness, dependence, skewness and the case of non-identical one-dimensional distributions, on optimal bundling strategies for a multiproduct monopolist.

1.6. Organization of the paper. The paper is organized as follows: Section 2 contains notations and definitions of classes of distributions used throughout the paper and reviews their basic properties. Section 3 describes our framework for modelling optimal bundling with interrelated goods. Sections 4 and 5 present the main results of the paper on optimal bundling strategies for complements and substitutes with heavy-tailed tastes. Section 4 deals with the setting where the products and their bundles are provided on an auction. Section 5 considers the case where the prices for goods and their bundles are set by the monopolist. Appendix A1 reviews peakedness properties of log-concavely distributed r.v.'s and presents their analogues in the case of heavy-tailed distributions needed for the proof of the main results in the paper. In particular, the appendix reviews peakedness properties of r.v.'s with log-concave densities established by Proschan (1965) and presents their analogues in the case of heavy-tailed distributions obtained in Ibragimov (2004). Finally, Appendix A2 contains proofs of the results obtained in the paper.

2. NOTATIONS AND DEFINITIONS

In this section, we introduce classes of distributions we will be dealing with throughout the paper.

We say that a r.v. X with density $f : \mathbf{R} \to \mathbf{R}$ and the convex distribution support $\Omega = \{x \in \mathbf{R} : f(x) > 0\}$ is log-concavely distributed if $log\, f(x)$ is concave in $x \in \Omega$, that is, if for all $x_1, x_2 \in \Omega$, and any $\lambda \in [0, 1]$,

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq (f(x_1))^{\lambda}(f(x_2))^{1-\lambda}. \tag{2.1}$$

(see An, 1998). A distribution is said to be log-concave if its density f satisfies (2.1).

If a r.v. X is log-concavely distributed, then its density has at most an exponential tail, that is, $f(x) = o(exp(-\lambda x))$ for some $\lambda > 0$, as $x \to \infty$ and all the power moments $E|X|^{\gamma}$, $\gamma > 0$, of the r.v. exist (see Corollary 1 in An, 1998). The reader is referred to Karlin (1968), Marshall and Olkin (1979) and An (1998) for a survey of many other properties of log-concave distributions.[13]

Throughout the paper, $\mathcal{LC}$ denotes the class of symmetric log-concave distributions.[14]

complements with relatively low degree of complementarity.

[13]Some of these properties are the following:

Any log-concave density is unimodal. Moreover, it has the property of strong unimodality, that is, its convolution with any other unimodal density is again unimodal;

The class of log-concave distributions is closed under convolutions;

The survivor and distribution functions of log-concave densities are both log-concave and, thus, a log-concavely distributed r.v. has the new-better-than-used property;

A log-concave density is of Pólya frequency of order 2 (PF-2);

The hazard function of a log-concave density is monotonically increasing.

Examples of log-concave distributions include the normal distribution, the uniform density, the exponential density, the Gamma distribution $\Gamma(\alpha, \beta)$ with the shape parameter $\alpha \geq 1$, the Beta distribution $\mathcal{B}(a, b)$ with $a \geq 1$ and $b \geq 1$; the Weibull distribution $\mathcal{W}(\gamma, \alpha)$ with the shape parameter $\alpha \geq 1$.

[14]$\mathcal{LC}$ stands for "log-concave".

For $0 < \alpha \leq 2$, $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbf{R}$, we denote by $S_\alpha(\sigma, \beta, \mu)$ the stable distribution with the characteristic exponent (index of stability) α, the scale parameter σ, the symmetry index (skewness parameter) β and the location parameter μ. That is, $S_\alpha(\sigma, \beta, \mu)$ is the distribution of a r.v. X with the characteristic function

$$E(e^{ixX}) = \begin{cases} exp\left\{i\mu x - \sigma^\alpha |x|^\alpha (1 - i\beta sign(x) tan(\pi\alpha/2))\right\}, & \alpha \neq 1, \\ exp\left\{i\mu x - \sigma|x|(1 + (2/\pi) i\beta sign(x) ln|x|\right\}, & \alpha = 1, \end{cases}$$

$x \in \mathbf{R}$, where $i^2 = -1$ and $sign(x)$ is the sign of x defined by $sign(x) = 1$ if $x > 0$, $sign(0) = 0$ and $sign(x) = -1$ otherwise. In what follows, we write $X \sim S_\alpha(\sigma, \beta, \mu)$, if the r.v. X has the stable distribution $S_\alpha(\sigma, \beta, \mu)$.

A closed form expression for the density $f(x)$ of the distribution $S_\alpha(\sigma, \beta, \mu)$ is available in the following cases (and only in those cases): $\alpha = 2$ (Gaussian distributions); $\alpha = 1$ and $\beta = 0$ (Cauchy distributions); $\alpha = 1/2$ and $\beta \pm 1$ (Lévy distributions).[15] Degenerate distributions correspond to the limiting case $\alpha = 0$.

The index of stability α characterizes the heaviness (the rate of decay) of the tails of stable distributions $S_\alpha(\sigma, \beta, \mu)$. The distribution of a stable r.v. $X \sim S_\alpha(\sigma, \beta, \mu)$ with $\alpha \in (0, 2)$ obeys power law (1.1) and thus the $p-$th absolute moments $E|X|^p$ of X are finite if $p < \alpha$ and are infinite otherwise. The symmetry index β characterizes the skewness of the distribution. The stable distributions with $\beta = 0$ are symmetric about the location parameter μ. The stable distributions with $\beta = \pm 1$ and $\alpha \in (0, 1)$ (and only they) are one-sided, the support of these distributions is the semi-axis $[\mu, \infty)$ for $\beta = 1$ and is $(-\infty, \mu]$ (in particular, the Lévy distribution with $\mu = 0$ is concentrated on the positive semi-axis for $\beta = 1$ and on the negative semi-axis for $\beta = -1$). In the case $\alpha > 1$ the location parameter μ is the mean of the distribution $S_\alpha(\sigma, \beta, \mu)$. The scale parameter σ is a generalization of the concept of standard deviation; it coincides with the latter in the special case of Gaussian distributions ($\alpha = 2$).

Distributions $S_\alpha(\sigma, \beta, \mu)$ with $\mu = 0$ for $\alpha \neq 1$ and $\beta \neq 0$ for $\alpha = 1$ are called strictly stable. If $X_i \sim S_\alpha(\sigma, \beta, \mu)$, $\alpha \in (0, 2]$, are i.i.d. strictly stable r.v.'s, then, for all $a_i \geq 0$, $i = 1, ..., n$, $\sum_{i=1}^n a_i X_i / \Big(\sum_{i=1}^n a_i^\alpha\Big)^{1/\alpha} \sim S_\alpha(\sigma, \beta, \mu)$.

For a detailed review of properties of stable distributions the reader is referred to, e.g., the monographs by Zolotarev (1986) and Uchaikin and Zolotarev (1999).

For $0 < r < 2$, we denote by $\overline{\mathcal{CS}}(r)$ the class of distributions which are convolutions of symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with characteristic exponents $\alpha \in (r, 2]$ and $\sigma > 0$.[16] That is, $\overline{\mathcal{CS}}(r)$ consists of distributions of r.v.'s X such that, for some $k \geq 1$, $X = Y_1 + ... + Y_k$, where Y_i, $i = 1, ..., k$, are independent r.v.'s such that $Y_i \sim S_{\alpha_i}(\sigma_i, 0, 0)$, $\alpha_i \in (r, 2]$, $\sigma_i > 0$, $i = 1, ..., k$.

Further, for $0 < r \leq 2$, $\underline{\mathcal{CS}}(r)$ stands for the class of distributions which are convolutions of symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with indices of stability $\alpha \in (0, r)$ and $\sigma > 0$.[17] That is, $\underline{\mathcal{CS}}(r)$ consists of distributions of r.v.'s X such that, for some $k \geq 1$, $X = Y_1 + ... + Y_k$, where Y_i, $i = 1, ..., k$, are independent r.v.'s such that $Y_i \sim S_{\alpha_i}(\sigma_i, 0, 0)$, $\alpha_i \in (0, r)$, $\sigma_i > 0$, $i = 1, ..., k$.

Finally, we denote by $\overline{\mathcal{CSLC}}$ the class of convolutions of distributions from the classes $\mathcal{LC}$ and $\overline{\mathcal{CS}}(1)$. That is,

[15] The densities of Cauchy distributions are $f(x) = \sigma/(\pi(\sigma^2 + (x - \mu)^2))$. Lévy distributions have densities $f(x) = (\sigma/(2\pi))^{1/2} exp(-\sigma/(2x)) x^{-3/2}$, $x \geq 0$; $f(x) = 0$, $x < 0$, where $\sigma > 0$, and their shifted versions.

[16] Here and below, $\mathcal{CS}$ stands for "convolutions of stable"; the overline indicates that convolutions of stable distributions with indices of stability *greater* than the threshold value r are taken.

[17] The underline indicates considering stable distributions with indices of stability *less* than the threshold value r.

$\overline{\mathcal{CSLC}}$ is the class of convolutions of symmetric distributions which are either log-concave or stable with characteristic exponents greater than one.[18] In other words, $\overline{\mathcal{CSLC}}$ consists of distributions of r.v.'s X such that $X = Y_1 + Y_2$, where Y_1 and Y_2 are independent r.v.'s with distributions belonging to $\mathcal{LC}$ or $\overline{\mathcal{CS}}(1)$.

All the classes $\mathcal{LC}$, $\overline{\mathcal{CSLC}}$, $\overline{\mathcal{CS}}(r)$ and $\underline{\mathcal{CS}}(r)$ are closed under convolutions. In particular, the class $\overline{\mathcal{CSLC}}$ coincides with the class of distributions of r.v.'s X such that, for some $k \geq 1$, $X = Y_1 + ... + Y_k$, where Y_i, $i = 1, ..., k$, are independent r.v.'s with distributions belonging to $\mathcal{LC}$ or $\overline{\mathcal{CS}}(1)$.

A linear combination of independent stable r.v.'s with the *same* characteristic exponent α also has a stable distribution with the same α. However, in general, this does not hold true in the case of convolutions of stable distributions with *different* indices of stability. Therefore, the class $\overline{\mathcal{CS}}(r)$ of *convolutions* of symmetric stable distributions with *different* indices of stability $\alpha \in (r, 2]$ is wider than the class of *all* symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with $\alpha \in (r, 2]$ and $\sigma > 0$. Similarly, the class $\underline{\mathcal{CS}}(r)$ is wider than the class of *all* symmetric stable distributions $S_\alpha(\sigma, 0, 0)$ with $\alpha \in (0, r)$ and $\sigma > 0$.

Clearly, $\overline{\mathcal{CS}}(1) \subset \overline{\mathcal{CSLC}}$ and $\mathcal{LC} \subset \overline{\mathcal{CSLC}}$. It should also be noted that the class $\overline{\mathcal{CSLC}}$ is wider than the class of (two-fold) convolutions of log-concave distributions with stable distributions $S_\alpha(\sigma, 0, 0)$ with $\alpha \in (1, 2]$ and $\sigma > 0$.

By definition, for $0 < r_1 < r_2 \leq 2$, the following inclusions hold: $\overline{\mathcal{CS}}(r_2) \subset \overline{\mathcal{CS}}(r_1)$ and $\underline{\mathcal{CS}}(r_1) \subset \underline{\mathcal{CS}}(r_2)$.

In some sense, symmetric (about $\mu = 0$) Cauchy distributions $S_1(\sigma, 0, 0)$ are at the dividing boundary between the classes $\underline{\mathcal{CS}}(1)$ and $\overline{\mathcal{CS}}(1)$ (and between the classes $\underline{\mathcal{CS}}(1)$ and $\overline{\mathcal{CSLC}}$). Similarly, for $r \in (0, 2)$, symmetric stable distributions $S_r(\sigma, 0, 0)$ with the characteristic exponent $\alpha = r$ are at the dividing boundary between the classes $\underline{\mathcal{CS}}(r)$ and $\overline{\mathcal{CS}}(r)$. Further, symmetric normal distributions $S_2(\sigma, 0, 0)$ are at the dividing boundary between the class $\mathcal{LC}$ of log-concave distributions and the class $\underline{\mathcal{CS}}(2)$ of convolutions of symmetric stable distributions with indices of stability $\alpha < 2$.[19]

In what follows, we write $X \sim \mathcal{LC}$ (resp., $X \sim \overline{\mathcal{CSLC}}$, $X \sim \overline{\mathcal{CS}}(r)$ or $X \sim \underline{\mathcal{CS}}(r)$) if the distribution of the r.v. X belongs to the class $\mathcal{LC}$ (resp., $\overline{\mathcal{CSLC}}$, $\overline{\mathcal{CS}}(r)$ or $\underline{\mathcal{CS}}(r)$). We also denote $\mathbf{R}_+ = [0, \infty)$.

It is natural to refer to distributions from the class $\underline{\mathcal{CS}}(1)$ (and, more generally, to those from the classes $\underline{\mathcal{CS}}(r)$ with relatively small r) as very heavy-tailed. Similarly, it is natural to refer to distributions from the classes $\overline{\mathcal{CSLC}}$ and $\overline{\mathcal{CS}}(1)$ (or the classes $\overline{\mathcal{CS}}(r)$ with relatively high r) as not too thick-tailed ones. We will follow these conventions in the rest of the paper.

3. A FRAMEWORK FOR OPTIMAL BUNDLING MODELS WITH INTERRELATED GOODS

Throughout the paper, we consider a setting with a single seller providing m goods to n consumers. Let $M = \{1, 2, ..., m\}$ be the set of goods sold on the market and let $J = \{1, 2, ..., n\}$ denote the set of buyers. Let 2^M stand for the set of all subsets of M. As in Palfrey (1983), the seller's bundling decisions $\mathcal{B}$ are defined as partitions

[18] $\overline{\mathcal{CSLC}}$ is the abbreviation of "convolutions of stable and log-concave".

[19] More precisely, the symmetric Cauchy distributions are the only ones that belong to all the classes $\underline{\mathcal{CS}}(r)$ with $r > 1$ and all the classes $\overline{\mathcal{CS}}(r)$ with $r < 1$. Symmetric stable distributions $S_r(\sigma, 0, 0)$ are the only ones that belong to all the classes $\underline{\mathcal{CS}}(r')$ with $r' > r$ and all the classes $\overline{\mathcal{CS}}(r')$ with $r' < r$. Symmetric normal distributions are the only distributions belonging to the class $\mathcal{LC}$ and all the classes $\overline{\mathcal{CS}}(r)$ with $r \in (0, 2)$.

of the set of items M into a set of subsets, $\{B_1, ..., B_l\} = \mathcal{B}$, where l is the cardinality of $\mathcal{B}$; the subsets $B_s \in 2^M$, $s = 1, ..., l$, are referred to as bundles. That is, $B_s \neq \emptyset$ for $s = 1, ..., l$; $B_s \cap B_t =$ for $s \neq t$, $s, t = 1, ..., l$; and $\cup_{s=1}^{l} B_s = M$ (see Palfrey, 1983, Bakos and Brynjolfsson, 1999, and Fang and Norman, 2003b). It is assumed that the seller can offer one (and only one) partition $\mathcal{B}$ for sale on the market (this referred to as pure bundling, see Adams and Yellen, 1976).[20] We denote by $\underline{\mathcal{B}} = \{\{1\}, \{2\}, ..., \{m\}\}$ and $\overline{\mathcal{B}} = \{1, 2, ..., m\}$ the bundling decisions corresponding, respectively, to the cases where the goods are sold separately (that is, on separate auctions or using unbundled sales) and as a single bundle M.

For a bundle $B \in 2^M$, we write $card(B)$ for a number of elements in B and denote by π_B the seller's profit resulting from selling the bundle. For a bundling decision $\mathcal{B} = \{B_1, ..., B_l\}$, we write $\Pi_{\mathcal{B}}$ for the seller's total profit resulting from following $\mathcal{B}$, that is, $\Pi_{\mathcal{B}} = \sum_{s=1}^{l} \pi_{B_s}$.

A risk-neutral seller prefers (strictly prefers) a bundling decision $\mathcal{B}_1$ to a bundling decision $\mathcal{B}_2$ *ex ante* if $E\Pi_{\mathcal{B}_1} \geq E\Pi_{\mathcal{B}_2}$ (resp., if $E\Pi_{\mathcal{B}_1} > E\Pi_{\mathcal{B}_2}$), where E denotes the expectation operator. The seller prefers a bundling decision $\mathcal{B}_1$ to a bundling decision $\mathcal{B}_2$ *ex post* if $\Pi_{\mathcal{B}_1} \geq \Pi_{\mathcal{B}_2}$ (a.s.), that is, if $P(\Pi_{\mathcal{B}_1} \geq \Pi_{\mathcal{B}_2}) = 1$.

More generally, if the seller has an increasing utility of wealth function $U : \mathbf{R}_+ \to \mathbf{R}$ with $U(0) = 0$, then she prefers (strictly prefers) a bundling decision $\mathcal{B}_1$ to a bundling decision $\mathcal{B}_2$ if $EU(\Pi_{\mathcal{B}_1}) \geq EU(\Pi_{\mathcal{B}_2})$ (resp., if $EU(\Pi_{\mathcal{B}_1}) > EU(\Pi_{\mathcal{B}_2})$). The setting with a concave function U represents the case of a risk-averse seller with the utility of wealth satisfying the property of diminishing returns. The case where U is convex models the framework with a risk-loving seller.

A representative consumer's preferences over the bundles $B \in 2^M$, on the other hand, are determined by her reservation prices (valuations) $v(B)$ for the bundles and, in particular, by her valuations $v(\{i\})$ for goods $i \in M$ (when the goods are sold separately) which are referred to as stand-alone reservation prices. In the case where the reservation prices for bundles are nonnegative: $v(B) \geq 0$, $B \in 2^M$, it is said that the goods in M and their bundles satisfy the *free disposal* condition.[21] The free disposal assumption is particularly important in the case of information goods and in the economics of the Internet (see Bakos and Brynjolfsson, 1999, 2000). If consumers' valuations for a bundle of goods are additive in those of component goods: $v(B) = \sum_{i \in B} v(\{i\})$, then the products provided by the monopolist are said to be *independently priced* (see Venkatesh and Kamakura, 2003). Under free disposal, the natural analogues of this property for interrelated goods are subadditivity $v(B) \leq \sum_{i \in B} v(\{i\})$ in the case of substitutes and superadditivity $\sum_{i \in B} v(\{i\}) \leq v(B)$ in the case of complements (see Dansby and Conrad, 1984, Lewbel, 1985, and Venkatesh and Kamakura, 2003).

In our main results presented in the next two sections, X_i, $i \in M$, denote i.i.d. r.v.'s representing the distribution of consumers' tastes for goods $i \in M$ that determine their reservation prices for the goods and their bundles. We suppose that a representative consumer's reservation price $v(B)$ for a bundle B of goods produced by the monopolist is a function of her tastes for the component goods in the bundle. More precisely, we model the

[20]The analysis of *mixed* bundling, in which consumers can choose among *all* bundling decisions available (see Adams and Yellen, 1976, and McAfee et. al., 1989) is beyond the scope of this paper.

[21]The case where the support of the valuations $v(B)$ intersects with $(-\infty, 0)$ corresponds to the situation where the goods have negative value to some consumers (e.g., articles exposing certain political views, advertisements or pornography in the case of information goods, see Bakos and Brynjolfsson, 1999).

setting with interrelated goods by assuming that a representative consumer's valuations for bundles $B \in 2^M$ are given by $v(g_r, B) = g_r\big(\sum_{i\in B} X_i\big)$ or by $v(h_r, B) = h_r\big(\sum_{i\in B} X_i\big)$, where, for $r \in (0, 2]$, $g_r(x) = x^r I(x \geq 0)$, $h_r(x) = x|x|^{r-1}$, $x \in \mathbf{R}$, and $I(\cdot)$ stands for the indicator function. The valuations for goods $i \in M$ in the case where they are sold separately are thus $v(g_r, \{i\}) = g_r(X_i)$ or $v(h_r, \{i\}) = h_r(X_i)$, $i \in M$. Clearly, in the case $r = 1$, one has $v(h_1, \{i\}) = h_1(X_i) = X_i$, $i \in M$. Also, the reservation prices $v(g_r, B)$ satisfy the free-disposal condition: $v(g_r, B) \geq 0$ for all $B \in 2^M$. It is easy to see that, for all $B \in 2^M$, $v(g_r, B) \leq \sum_{i\in B} v(g_r, \{i\})$, if $r \leq 1$, and $\sum_{i\in B} v(g_r, \{i\}) \leq v(g_r, B)$, if $r \geq 1$, and $X_i \geq 0$, $i \in B$. That is, consumers' reservation price $v(g_r, B)$ for a bundle is subadditive in those for the component products if $r \leq 1$, as it is typically required for substitutes, and is superadditive in the rectangle of non-negative tastes if $r \geq 1$, as it is usually assumed in the case of complements. Similarly, for $r \leq 1$, the reservation prices $v(h_r, B)$ are subadditive in those for component products in the rectangle of non-negative stand-alone valuation $v(h_r, \{i\})$ $i \in M$, and are superadditive in the components' valuations in the case where all the stand-alone valuations are non-positive. For $r \geq 1$, the valuations for bundles $v(h_r, B)$ are superadditive in those for the components if all the stand-alone reservation prices are non-negative and are subadditive if the valuations for all component products are non-positive. More precisely, if $v(h_r, \{i\}) \geq 0$, $i \in B$, then $\sum_{i\in B} v(h_r, \{i\}) \leq v(h_r, B)$ for $r \geq 1$, and $v(h_r, B) \leq \sum_{i\in B} v(h_r, \{i\})$ for $r \leq 1$. If $v(h_r, \{i\}) \leq 0$, $i \in B$, then $v(h_r, B) \leq \sum_{i\in B} v(h_r, \{i\})$ for $r \geq 1$, and $\sum_{i\in B} v(h_r, \{i\}) \leq v(h_r, B)$ for $r \leq 1$. The above super- and subadditivity properties of $v(h_r, B)$ for $r \geq 1$ are consistent with the assumption typically imposed on the value function of (complementary) gains and losses in mental accounting and prospect theory (e.g., Kahneman and Tversky, 1979, and Thaler, 1985). The case $r = 1$ with reservation prices for bundles $v(h_1, B) = \sum_{i\in B} X_i$ models the case of independently priced goods.

For $j \in J$, the jth consumer's tastes for goods in M are assumed to be $\tilde{X}_{ij}$, $i \in M$, where $\tilde{X}^{(j)} = (\tilde{X}_{1j}, ..., \tilde{X}_{nj})$, $j \in M$, are independent copies of the vector $(X_1, ..., X_n)$, and her reservation prices $v_j(B)$ for bundles $B \in 2^M$ of goods in M are given by $v_j(g_r, B) = g_r\big(\sum_{i\in B} \tilde{X}_{ij}\big)$ or by $v_j(h_r, B) = h_r\big(\sum_{i\in B} \tilde{X}_{ij}\big)$. The seller is assumed to know only the distribution of consumers' reservation prices for goods in M and their bundles. The valuations $v_j(g_r, B)$ $(v_j(h_r, B))$ for bundles $B \in 2^M$, are known to buyer j, however, the buyer has only the same incomplete information about the other consumers' reservation prices as does the seller (see Palfrey, 1983).

4. MAIN RESULTS: OPTIMAL BUNDLED AUCTIONS FOR COMPLEMENTS AND SUBSTITUTES WITH HEAVY-TAILED VALUATIONS

We first consider the case in which the goods in M sold by the monopolist and their bundles are provided through Vickrey auctions (see Palfrey, 1983). In this setting, the buyers submit simultaneous sealed bids for bundles of goods sold by the seller. The bidder with the highest bid wins the auction and pays the seller the second highest bid. It is well-known that, in such a setup, a dominant strategy for each bidder is to bid her true reservation prices. In accordance with the assumption of nonnegativity of bids and valuations usually imposed in the auction theory, we suppose that, for $j \in J$, the jth consumer's reservation price for a bundle $B \in 2^M$ of goods sold is given by $v_j(g_r, B) = g_r(\sum_{i\in B} \tilde{X}_{ij}) \geq 0$. The seller's profit from following a bundling decision $\mathcal{B} = \{B_1, ..., B_l\}$ is, evidently, $\sum_{s=1}^{l} v_{(n-1)}(g_r, B_s)$, where, for $s = 1, ..., l$, $v_{(n-1)}(g_r, B_s)$ denotes the second highest of consumers' reservation prices for the bundle B_s (that is, the second highest order statistic of the reservation prices for the bundle).

The following Theorem 4.1 extends the results in Palfrey (1983) and Chakraborty (1999) to the case of interrelated goods (with an arbitrary degree of complementarity or substitutability) and consumers with long-tailed valuations. According to the theorem, if consumers' tastes are not very heavy-tailed and the goods are independently priced or are substitutes (or are complements with not very high degree of complementarity) then the risk-neutral auctioneer strictly prefers separate provision of goods to any other bundling decision.

Theorem 4.1 *Let $r \in (0,2)$, and let the reservation prices for bundles $B \in 2^M$ of goods from M be given by $v(g_r, B)$. Suppose that the tastes X_i, $i \in M$, are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1,1]$ and $\alpha \in (r,2]$, where $\beta = 0$ for $\alpha = 1$, or $X_i \sim \overline{\mathcal{CS}}(r)$, $i \in M$. Then, for all $m \geq 2$, the risk-neutral seller strictly prefers (ex ante) $\underline{\mathcal{B}}$ (that is, m separate Vickrey auctions) to any other bundling decision.*

Remark 4.1 *From the proof of Theorem 4.1 it follows that, under its assumptions, for any bundle $B \in 2^M$ with the number of elements $card(B) = k \geq 2$, the seller's profit π_B from selling B on a Vickrey auction is strictly (first-order) stochastically dominated by the profit from selling one of goods in B, say good $i \in B$, separately k times, that is, by the r.v. $k\pi_i$, where $\pi_i = \pi_{B_i}$ with $B_i = \{i\}$. Namely, for all $x > 0$, one has $P(\pi_B > x) < P(k\pi_i > x)$ that means that selling one of goods in B k times separately is always likely to generate higher profits to the seller than selling the bundle B. We get, therefore, by Shaked and Shanthikumar (1994, pp. 3-4), that $EU(\pi_B) \leq EU(k\pi_i)$ for all increasing functions $U : \mathbf{R}_+ \to \mathbf{R}$ for which the expectations exist. Similar to the proof of Theorem 4.2 below, this, in turn, implies that Theorem 4.1 holds as well in the case of a risk-loving seller with any increasing convex utility of wealth function U such that $U(0) = 0$.*

There are no counterparts of Theorem 4.1 for very heavy-tailed distributions of consumers' valuations (such as $\underline{\mathcal{CS}}(r)$) if the seller is risk-neutral since, as it is not difficult to see, in this case, the seller's expected profits from following any bundling decision are infinite. However, in the case of a *risk-averse* seller with a concave utility of wealth function, the following reversal of Theorem 4.1 holds.

According to the following Theorem 4.2, in the latter case, the auctioneer strictly prefers providing all the items through one Vickrey auction to any other bundling decision, if consumers' tastes are very heavy-tailed and the goods are independently priced or are complements (or are substitutes with not very high degree of substitutability).

Theorem 4.2 *Let $r \in (0,2]$, and let the reservation prices for bundles $B \in 2^M$ of goods from M be given by $v(g_r, B)$. Suppose that the tastes X_i, $i \in M$, are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1,1]$ and $\alpha \in (0,r)$, where $\beta = 0$ for $\alpha = 1$, or $X_i \sim \underline{\mathcal{CS}}(r)$, $i \in M$. If the seller's utility of wealth is concave, then, for all $m \geq 2$, the seller strictly prefers (ex ante) $\overline{\mathcal{B}}$ (that is, a single Vickrey auction) to any other bundling decision.*

The underlying intuition behind the results given by Theorems 4.1 and 4.2 is the following. As follows from the results in Appendix A1, if consumers' tastes are not very heavy-tailed, then distributions of their valuations per good in bundles become increasingly more concentrated (more peaked) as the size of bundles becomes larger. Therefore, it is increasingly more likely that consumers with not too thick-tailed tastes are willing to pay more for a set of goods if they are provided separately. Thus, the monopolist's total profit is likely to be maximized if goods in the bundles are sold on separate auctions to such buyers. However, as the results in Appendix A1 imply, if buyers' tastes are very long-tailed, then concentration (peakedness) of their valuations per good in bundles decreases with the size of the bundles. Consequently, it is increasingly more likely that the seller's total profit derived from a set of products sold as a single bundle is greater than her profit under separate provision of the goods to such consumers.

Using the general majorization properties of long-tailed distributions presented in Appendix A1, one can also obtain the following Theorem 4.3 that characterizes buyers' preferences over the bundled auctions in the case of independently priced goods and very heavy-tailed reservation prices.

Let $j \in J$ and let $\tilde{x}^{(j)} = (\tilde{x}_{1j}, ..., \tilde{x}_{nj}) \in \mathbf{R}^n_+$. If a bundle B consisting of independently priced goods is offered for sale on a Vickrey auction then the expectation of the surplus $S_j(B, \tilde{x}^{(j)})$ to consumer j with the values of stand-alone reservation prices $\tilde{X}^{(j)} = \tilde{x}^{(j)}$ and induced valuations for bundles $v_j(B) = \sum_{i \in B} \tilde{x}_{ij}$, $B \in 2^M$, is (see Palfrey, 1983)

$$ES_j(B, \tilde{x}^{(j)}) = P\Big(\max_{s \in J, s \neq j} v_s(B) < v_j(B)\Big)\Big(v_j(B) - E\big(\max_{s \in J, s \neq j} v_s(B) | \max_{s \in J, s \neq j} v_s(B) < v_j(B)\big)\Big),$$

where $v_t(B) = \sum_{i \in B} \tilde{X}_{it}$, $B \in 2^M$, $t \in J$, $t \neq j$. If the seller follows a bundling decision $\mathcal{B} = \{B_1, ..., B_l\}$, then the expectation of the surplus $S_j(\mathcal{B}, \tilde{x}^{(j)})$ to the jth buyer with the vector of stand-alone valuations $\tilde{X}^{(j)} = \tilde{x}^{(j)}$ is $ES_j(\mathcal{B}, \tilde{x}^{(j)}) = \sum_{s=1}^{l} ES_j(B_s, \tilde{x}^{(j)})$. The jth buyer with $\tilde{X}^{(j)} = \tilde{x}^{(j)}$ is said to (strictly) prefer a bundling decision $\mathcal{B}_1$ to a bundling decision $\mathcal{B}_2$, *ex ante*, if $ES_j(\mathcal{B}_1, \tilde{x}^{(j)}) \geq ES_j(\mathcal{B}_2, \tilde{x}^{(j)})$ (resp., if $ES_j(\mathcal{B}_1, \tilde{x}^{(j)}) > ES_j(\mathcal{B}_2, \tilde{x}^{(j)})$). If all buyers $j \in J$ (strictly) prefer a bundling decision $\mathcal{B}_1$ to a bundling decision $\mathcal{B}_2$ *ex ante* for *almost all* realizations of their reservation prices $\tilde{X}^{(j)}$, it is said that buyers *unanimously* (strictly) prefer $\mathcal{B}_1$ to $\mathcal{B}_2$ *ex ante*. More precisely, buyers unanimously prefer (strictly prefer) a partition $\mathcal{B}_1$ to a partition $\mathcal{B}_2$ if, for all $j \in J$, $P[E(S_j(\mathcal{B}_1, \tilde{X}^{(j)})|\tilde{X}^{(j)}) \geq E(S_j(\mathcal{B}_2, \tilde{X}^{(j)})|\tilde{X}^{(j)})] = 1$ (resp., $P[E(S_j(\mathcal{B}_1, \tilde{X}^{(j)})|\tilde{X}^{(j)}) > ES_j(\mathcal{B}_2, \tilde{X}^{(j)})|\tilde{X}^{(j)})] = 1$), where, as usual, $E(\cdot|\tilde{X}^{(j)})$ stands for the expectation conditional on $\tilde{X}^{(j)}$. Clearly, in the case of absolutely continuous reservation prices X_i, $i \in M$, consumers unanimously prefer $\mathcal{B}_1$ to $\mathcal{B}_2$ *ex ante* if each of them prefers $\mathcal{B}_1$ to $\mathcal{B}_2$ for all but a finite number of realizations of their stand-alone valuations.

According to Theorem 4.3, consumers unanimously prefer (*ex ante*) separate provision of goods on Vickrey auctions to any other bundling decision in the case of an arbitrary number of buyers, if their valuations are very heavy-tailed, as modelled by positive stable distributions.[22] These results are reversals of those given by Theorem 6 in Palfrey (1983) from which it follows that if consumers' valuations are concentrated on a finite interval, then buyers never unanimously prefer separate provision auctions if there are more than two buyers on the market (Theorem 4.3 does not contradict Theorem 6 in Palfrey, 1983, since the support of heavy-tailed distributions in Theorem 4.3 is the infinite positive semi-axis $\mathbf{R}_+$).

Theorem 4.3 *Let the reservation prices for bundles $B \in 2^M$ be given by $v(B) = \sum_{i \in B} X_i$. Suppose that the stand-alone reservation prices X_i, $i \in M$, for goods in M are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, 1, 0)$ for some $\sigma > 0$ and $\alpha \in (0, 1)$. Then buyers unanimously strictly prefer (ex ante) $\underline{\mathcal{B}}$ (that is, n separate auctions) to any other bundling decision.*

The intuition behind the results given by Theorem 4.3 is similar to that behind Theorem 4.2 and is a reversal of

[22] As we indicated in Section 2, positive stable distributions are distributions $S_\alpha(\sigma, 1, 0)$ with any $\sigma > 0$ and $\alpha \in (0, 1)$. Any such distribution is very thick-tailed.

the intuition for the results in Palfrey (1983) and that for Theorem 4.1 above. Again, according to Appendix A1, in the case of very heavy-tailed tastes, consumers' valuations per good for bundles become less concentrated about the mean as the size of bundles increases. Buyers who are on the upper tail of the distributions for the goods are more likely to win separate auctions and the next highest bidder is likely to have relatively lower valuations than in the case of a bundled auction, as follows from the above. Therefore, contrary to the case of very light-tailed valuations (see the discussion preceding Theorem 5 in Palfrey, 1983) the winner of the auction is likely to prefer separate provision of the products.

Remark 4.2 *As shown by Palfrey (1983), in Vickrey auctions with independently priced goods and an arbitrary number of bidders, the total surplus (that is, the sum of the seller's profit and buyers' surplus) is always maximized in the case where the goods are provided on separate auctions. Palfrey (1983) also proves that, with two buyers, the bidders unanimously prefer separate provision of items ex post and thus ex ante and the seller, on the other hand, prefers a single auction. Since the above results are, essentially, deterministic, all they are robust with respect to risk attitudes of the seller and the buyer. However, as discussed in Palfrey (1983), the ex post results on the seller's and the buyers' preferences available in the two-buyer setup cannot be extended in any way to the case when there are more than two buyers. On the other hand, from Theorem 4.2 with* $r = 1$ *and Theorem 4.3 it follows that, in the case of an arbitrary number of buyers with (very heavy-tailed) positive stable reservation prices, the market participants' ex ante preferences over the bundling decisions are the same as in the case of the ex post analysis for two-buyer setting in Palfrey (1983). Namely, the seller's expected utility of wealth is maximized in the case of a single auction and the buyers unanimously prefer separate provision of goods to any other bundling decision. Thus, the effects of bundling on the seller's expected utility of wealth and the buyers' expected surplus continue to be the opposites of one another, although (by Palfrey, 1983) the expected total surplus is still maximized under the separate provision.*

5. MAIN RESULTS: OPTIMAL BUNDLING FOR COMPLEMENTS AND SUBSTITUTES WITH THICK-TAILED VALUATIONS AND PRICES SET BY THE SELLER

We turn now to the case in which the prices for goods on the market and their bundles are set by the monopolist. Let c_i, $i \in M$, be the marginal costs of goods in M. Suppose that the seller can provide bundles B of goods in M for prices per good $p \in [0, p_{max}]$, where p_{max} is the (regulatory) maximum price, with the convention that p_{max} can be infinite. For a bundle of goods $B \in 2^M$, denote by p_B the profit-maximizing price per good for the bundle, so that the seller's expected profit from selling B (at the price p_B per good) is $\pi_B = J(kp_B - \sum_{i \in B} c_i)P(v(B) \geq kp_B)$, where $k = card(B)$. Clearly, in the case where the marginal costs are identical for goods produced by the seller, that is, $c_i = c$ for all $i \in M$, the values of p_B are the same for all bundles B that consist of the same number $card(B)$ of goods. That is, $p_B = p_{B'}$, if $card(B) = card(B')$. With identical marginal costs, we denote by $\overline{p}$ the profit maximizing price per good in the case where all the goods in M are sold as a single bundle and by $\underline{p}$ the profit maximizing price of each good $i \in M$ under unbundled sales. That is, in the case where $c_i = c$ for all $i \in M$, $\overline{p} = p_B$ with $B = M$, and $\underline{p} = p_B$ with $B = \{i\}$, $i \in M$.

The following Theorems 5.1 and 5.2 characterize the optimal bundling strategies for a multiproduct monopolist in the above setting with an arbitrary degree of complementarity or substitutability for goods in M (the cases of valuations $v(g_r, B)$ and $v(h_r, B)$ with an arbitrary $r \in (0, 2]$). From Theorem 5.1 it follows that if consumers' tastes are not very heavy-tailed and the goods are independently priced or are substitutes (or are complements with not very high degree of complementarity), then the patterns in seller's optimal bundling strategies are the same as in the case of independently priced goods with log-concavely distributed valuations (see Bakos and Brynjolfsson, 1999, and Fang and Norman, 2003b, and the discussion in Subsection 5.1 in the introduction to this paper).

Theorem 5.1 *Let $\mu \in \mathbf{R}$, $r \in (0,2)$, and let the reservation prices for bundles $B \in 2^M$ of goods from M be given by $v(g_r, B)$ or by $v(h_r, B)$. Suppose that the tastes X_i, $i \in M$, are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, \mu)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1,1]$ and $\alpha \in (r, 2]$, where $\beta = 0$ for $\alpha = 1$, or $X_i - \mu \sim \overline{\mathcal{CS}}(r)$, $i \in M$. The risk-neutral seller strictly prefers $\overline{\mathcal{B}}$ to any other bundling decision (that is, the goods are sold as a single bundle), if $c_i = c$, $i \in M$, and $\underline{p} < \mu$. The risk-neutral seller strictly prefers $\underline{\mathcal{B}}$ to any other bundling decision (that is, the goods are sold separately), if $c_i \geq \mu$, $i \in M$, or if $c_i = c$, $i \in M$, and $\overline{p} > \mu$.*

According to Theorem 5.2, the patterns in the solutions to the seller's optimal bundling problem in Theorem 5.1 are reversed if consumers' tastes are very heavy-tailed and the goods are independently priced or are complements (or are substitutes with not very high degree of substitutability).

Theorem 5.2 *Let $\mu \in \mathbf{R}$, $r \in (0,2]$, $p_{max} < \infty$, and let the reservation prices for bundles $B \in 2^M$ of goods from M be given by $v(g_r, B)$ or by $v(h_r, B)$. Suppose that the tastes X_i, $i \in M$, are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, \mu)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1,1]$ and $\alpha \in (0, r)$, where $\beta = 0$ for $\alpha = 1$, or $X_i - \mu \sim \underline{\mathcal{CS}}(r)$, $i \in M$. The risk-neutral seller strictly prefers $\underline{\mathcal{B}}$ to any other bundling decision (that is, the goods are sold separately), if $c_i = c$, $i \in M$, and $\overline{p} < \mu$. The risk-neutral seller strictly prefers $\overline{\mathcal{B}}$ to any other bundling decision (that is, the goods are sold as a single bundle), if $c_i \geq \mu$, $i \in M$, or if $c_i = c$, $i \in M$, and $\underline{p} > \mu$.*

Theorem 5.3 and 5.4 below give analogues of the results in Theorems 5.1 and 5.2 in the case of independently priced goods ($r = 1$).

Theorem 5.3 *Let $\mu \in \mathbf{R}$, and let the reservation prices for bundles $B \in 2^M$ be given by $v(h_1, B) = \sum_{i \in B} X_i$. Suppose that the stand-alone reservation prices $v(h_1, \{i\}) = X_i$, $i \in M$, for goods in M are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, \mu)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1,1]$ and $\alpha \in (1, 2]$, or $X_i - \mu \sim \overline{\mathcal{CSLC}}$, $i \in M$. Then the conclusion of Theorem 5.1 holds.*

Theorem 5.4 *Let $\mu \in \mathbf{R}$, $p_{max} < \infty$, and let the reservation prices for bundles $B \in 2^M$ be given by $v(h_1, B) = \sum_{i \in B} X_i$. Suppose that the stand-alone reservation prices $v(h_1, \{i\}) = X_i$, $i \in M$, for goods in M are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, \mu)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1,1]$ and $\alpha \in (0, 1)$, or $X_i - \mu \sim \underline{\mathcal{CS}}(1)$, $i \in M$. Then the conclusion of Theorem 5.2 holds.*

Similar to the argument based on variance in Bakos and Brynjolfsson (1999), the underlying intuition for Theorems 5.1 and 5.3 is that for not very heavy-tailed distributions of reservation prices and the marginal costs of goods on the right of the mean valuation, bundling decreases profits since it reduces concentration (peakedness) of the valuation per good and thereby decreases the fraction of buyers with valuations for bundles greater than their total marginal costs (this is implied by the results in Appendix A1). For the identical marginal costs of goods less than the mean valuation, bundling is likely to have the opposite effect on the profit.

On the other hand, similar to Vickrey auctions in Section 4, the results in Theorems 5.2 and 5.4 are driven by the fact that, in the case of very heavy-tailed reservation prices, concentration and peakedness of the valuations per good in bundles decreases with their size (see Appendix A1). Therefore, bundling of goods in the case of very long-tailed valuations and marginal costs of goods higher than the mean reservation price increases the fraction of buyers with reservation prices for bundles greater than their total marginal costs and thereby leads to an increase in the monopolist's profit. This effect is reversed in the case of the identical marginal costs on the left of the mean valuation.

Remark 5.3 *The assumptions of Theorem 5.2 with $r \geq 1$ (and those of Theorem 5.4) are satisfied, in particular, for positive stable tastes (stand-alone reservation prices) $X_i \sim S_\alpha(\sigma, 1, \mu)$, $i \in M$, where $\sigma > 0$ and $\alpha \in (0, 1)$, for which thus the free disposal condition holds, including the Lévy distributions $S_{1/2}(\sigma, 1, \mu)$. Furthermore, from the proof of Theorems 5.1-5.4 it follows that the first parts (second parts) of conclusions in the theorems hold as well in the case of arbitrary marginal costs c_i if the price per good p_B in each bundle $B \in 2^M$ is less than (greater than) μ. One should also note here that the conditions $p_{max} < \infty$ in Theorems 5.2 and 5.4 are necessary since otherwise the monopolist would set an infinite price for each bundle of goods under very heavy-tailed distributions of consumers' tastes considered in the theorems.*

Remark 5.4 *It is important to note that Theorems 5.2 and 5.4 shed new light on marketing strategies involving exclusion of goods for which observations of extreme (both positive and negative) valuations are more likely from the bundle and selling them separately. Such strategies are often observed on the market, in particular, in the bundling decisions of cable and direct satellite broadcast television firms that have marginal costs of reproduction close to zero. The latter firms typically offer a "basic" bundle and use such strategies as pay-per-view approach for unusual special events such as boxing matches (see Bakos and Brynjolfsson, 1999). The high valuations for the special events are concentrated among a small fraction of consumers and thus are likely to be very heavy-tailed. Therefore, the optimal bundling strategies for the special events are likely to be the opposites of those for light-tailed distributions of valuations and thus, in contrast to the basic bundles, the events are likely to be provided on pay-per-view basis. Season tickets for entertainment performances offered by sporting and cultural organizations that have sufficiently high marginal costs of production might illustrate the dual pattern in bundling. It seems plausible that most of the demand for season tickets is concentrated around a relative small fraction of consumers that have high valuations for performances offered by the entertainment organization. The optimal strategy is to offer tickets to such consumers as a bundle, as predicted by our results for heavy-tailed tastes under the free disposal assumption or symmetric long-tailed valuations in the case of sufficiently large marginal costs. This strategy is the opposite of separate provision of the most of tickets to performances to consumers who are likely not to have very extreme valuations.*

6. APPENDIX A1. MAJORIZATION AND PEAKEDNESS PROPERTIES OF LOG-CONCAVE AND HEAVY-TAILED DISTRIBUTIONS

Definition 6.1 *(Marshall and Olkin, 1979). Let $a, b \in \mathbf{R}^n$. The vector a is said to be majorized by the vector b, written $a \prec b$, if $\sum_{i=1}^k a_{[i]} \le \sum_{i=1}^k b_{[i]}$, $k = 1, ..., n-1$, and $\sum_{i=1}^n a_{[i]} = \sum_{i=1}^n b_{[i]}$, where $a_{[1]} \ge \ldots \ge a_{[n]}$ and $b_{[1]} \ge \ldots \ge b_{[n]}$ denote components of a and b in decreasing order.*

The relation $a \prec b$ implies that the components of the vector b are more diverse than those of a (see Marshall and Olkin, 1979). In this context, it is easy to see that the following relations hold:

$$\Big(\sum_{i=1}^n a_i/n, ..., \sum_{i=1}^n a_i/n\Big) \prec (a_1, ..., a_n) \prec \Big(\sum_{i=1}^n a_i, 0, ..., 0\Big), \quad a \in \mathbf{R}_+^n, \tag{6.1}$$

In particular,

$$(1/(n+1), ..., 1/(n+1), 1/(n+1)) \prec (1/n, ..., 1/n, 0), \quad n \ge 1. \tag{6.2}$$

Definition 6.2 *(Marshall and Olkin, 1979). A function $\phi : A \to \mathbf{R}$ defined on $A \subseteq \mathbf{R}^n$ is called Schur-convex (resp., Schur-concave) on A if $(a \prec b) \Longrightarrow (\phi(a) \le \phi(b))$ (resp. $(a \prec b) \Longrightarrow (\phi(a) \ge \phi(b))$ for all $a, b \in A$. If, in addition, $\phi(a) < \phi(b)$ (resp., $\phi(a) > \phi(b)$) whenever $a \prec b$ and a is not a permutation of b, then ϕ is said to be strictly Schur-convex (resp., strictly Schur-concave) on A.*

The following concept of peakedness of r.v.'s was introduced by Birnbaum (1948).

Definition 6.3 *(Birnbaum, 1948, see also Proschan, 1965, and Marshall and Olkin, 1979, p. 372). A r.v. X is more peaked about $\mu \in \mathbf{R}$ than is Y if $P(|X-\mu| > x) \le P(|Y-\mu| > x)$ for all $x \ge 0$. If these inequalities are strict whenever the two probabilities are not both 0 or both 1, then the r.v. X is strictly more peaked about μ than is Y. A r.v. X is said to be (strictly) less peaked about μ than is Y if Y is (strictly) more peaked about μ than is X.*

In the case $\mu = 0$, we simply say that the r.v. X is (strictly) more peaked than Y.

Roughly speaking, a r.v. X is more peaked about $\mu \in \mathbf{R}$ than is Y, if the distribution of X is more concentrated about μ than is that of Y.

Proschan (1965) obtained the following well-known result concerning majorization and peakedness properties of tail probabilities of linear combinations of log-concavely distributed r.v.'s:

Proposition 6.1 *(Proschan, 1965).*[23] *If $X_1, ..., X_n$ are i.i.d. r.v.'s such that $X_i \sim \mathcal{LC}$, $i = 1, ..., n$, then the function $\psi(a, x) = P\big(\sum_{i=1}^n a_i X_i > x\big)$ is strictly Schur-convex in $a = (a_1, ..., a_n) \in \mathbf{R}_+^n$ for $x > 0$ and is strictly Schur-concave in $a = (a_1, ..., a_n) \in \mathbf{R}_+^n$ for $x < 0$.*

[23] This proposition is formulated as Theorem 12.J.1 in Marshall and Olkin (1979) and is the main result in Section 12.J in the book. Proschan's (1979) work is also presented, in a rearranged form, in Section 11 of Chapter 7 in Karlin (1968).

Clearly, from Proposition 6.1 it follows that, under its assumptions, $\sum_{i=1}^n a_i X_i$ is strictly more peaked than $\sum_{i=1}^n b_i X_i$ if $a \prec b$ and a is not a permutation of b.

Proschan (1965) notes that Proposition 6.1 also holds for (two-fold) convolutions of log-concave distributions with symmetric Cauchy distributions and shows that comparisons implied by the proposition are reversed for $n = 2^k$, vectors $a = (1/n, 1/n, ..., 1/n) \in \mathbf{R}^n$ with identical components and certain transforms of symmetric Cauchy r.v.'s.

Theorems 6.1-6.4 in this appendix give analogues of Proposition 6.1 for heavy-tailed r.v.'s obtained in Ibragimov (2004) (see Theorems 2.1-2.4 in that paper). In particular, according to the following Theorem 6.1, the majorization properties of convex combinations of r.v.'s in the classes $\overline{\mathcal{CS}}(r)$ are of the same type as in Proposition 6.1 with respect to the comparisons between the powers of components of the vectors of weights of the combinations.

Theorem 6.1 *Let $r \in (0, 2)$. If $X_1, ..., X_n$ are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i = 1, ..., n$, for some $\sigma > 0$, $\beta \in [-1, 1]$ and $\alpha \in (r, 2]$, where $\beta = 0$ for $\alpha = 1$, or $X_i \sim \overline{\mathcal{CS}}(r)$, $i = 1, ..., n$, then the function $\psi(a, x)$, $a \in \mathbf{R}_+^n$ in Proposition 6.1 is strictly Schur-convex in $(a_1^r, ..., a_n^r)$ for $x > 0$ and is strictly Schur-concave in $(a_1^r, ..., a_n^r)$ for $x < 0$.*

As follows from Theorem 6.2 below, the majorization and peakedness properties of the tail probabilities $\psi(a, x)$ in Theorem 6.1 are reversed in the case of r.v.'s from the classes $\underline{\mathcal{CS}}(r)$.

Theorem 6.2 *Let $r \in (0, 2]$. If $X_1, ..., X_n$ are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i = 1, ..., n$, for some $\sigma > 0$, $\beta \in [-1, 1]$ and $\alpha \in (0, r)$, where $\beta = 0$ for $\alpha = 1$, or $X_i \sim \underline{\mathcal{CS}}(r)$, $i = 1, ..., n$, then the function $\psi(a, x)$, $a \in \mathbf{R}_+^n$ in Proposition 6.1 is strictly Schur-concave in $(a_1^r, ..., a_n^r)$ for $x > 0$ and is strictly Schur-convex in $(a_1^r, ..., a_n^r)$ for $x < 0$.*

According to Theorem 6.3 below, peakedness and majorization properties of linear combinations of r.v.'s with not too heavy-tailed distributions, as modelled, e.g., by convolutions of log-concave distributions and symmetric stable distributions with characteristic exponents greater than one, are the same as in the case of log-concave distributions in Proschan (1965).

Theorem 6.3 *Proposition 6.1 holds if $X_1, ..., X_n$ are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i = 1, ..., n$, for some $\sigma > 0$, $\beta \in [-1, 1]$ and $\alpha \in (1, 2]$, or $X_i \sim \overline{\mathcal{CSLC}}$, $i = 1, ..., n$.*

As follows from Theorem 6.4, peakedness properties given by Proposition 6.1 and Theorem 6.3 above are reversed in the case of r.v.'s with very heavy-tailed distributions, as modelled by convolutions of stable distributions with indices of stability less than one.

Theorem 6.4 *If $X_1, ..., X_n$ are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i = 1, ..., n$, for some $\sigma > 0$, $\beta \in [-1, 1]$ and $\alpha \in (0, 1)$, or $X_i \sim \underline{\mathcal{CS}}(1)$, $i = 1, ..., n$, then the function $\psi(a, x)$ in Proposition 6.1 is strictly Schur-concave in $(a_1, ..., a_n) \in \mathbf{R}_+^n$ for $x > 0$ and is strictly Schur-convex in $(a_1, ..., a_n) \in \mathbf{R}_+^n$ for $x < 0$.*

From Theorem 6.3 it follows that, similar to the class $\mathcal{LC}$ covered by Proposition 6.1, $\sum_{i=1}^n a_iX_i$ is strictly more peaked than $\sum_{i=1}^n b_iX_i$ for not too heavy-tailed $X_i's$, if $a \prec b$ and a is not a permutation of b. However, according to Theorem 6.4, if $a \prec b$ and a is not a permutation of b, then $\sum_{i=1}^n a_iX_i$ is strictly less peaked than $\sum_{i=1}^n b_iX_i$ for very thick-tailed $X_1, ..., X_n$.

7. APPENDIX A2. PROOFS

Proof of Theorem 4.1. Let $r \in (0, 2)$ and let X_i, $i \in M$, be i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1, 1]$, and $\alpha \in (r, 2]$, $\beta = 0$ for $\alpha = 1$, or $X_i \sim \overline{\mathcal{CS}}(r)$, $i \in M$. Consider any bundle $B \in 2^M$ with $card(B) = k \geq 2$. Denote $H_k(x) = P(\sum_{i=1}^k X_i \leq x)$, $x \in \mathbf{R}$. Clearly, the cdf of the r.v. $v(g_r, B) = g_r(\sum_{i\in B} X_i)$ is $P(v(g_r, B) \leq x) = H_k(x^{1/r})$ for $x \geq 0$, $P(v(g_r, B) \leq x) = 0$ otherwise. Therefore, we have that, for all $x > 0$, the cdf of the seller's profit π_B resulting from selling B is

$$P(\pi_B \leq x) = P(v_{(n-1)}(g_r, B) \leq x) = n(H_k(x^{1/r}))^{n-1} - (n-1)(H_k(x^{1/r}))^n \tag{7.1}$$

(this cdf is zero for $x < 0$). For $i \in M$, let π_i be the seller's profit resulting from selling good i separately, that is, $\pi_i = \pi_{B_i}$ with $B_i = \{i\}$. For $x > 0$, the cdf of the r.v. $k\pi_1$ (that represents the seller's profit resulting from selling good 1 k times) is

$$P(k\pi_1 \leq x) = P(v_{(n-1)}(g_r, \{1\}) \leq x/k) = n(H_1(x^{1/r}/k^{1/r}))^{n-1} - (n-1)(H_1(x^{1/r}/k^{1/r}))^n. \tag{7.2}$$

By Theorem 6.1 and comparisons (6.2), $H_k(xk^{1/r}) > H_1(x)$, $x > 0$, and, therefore, $H_k(x^{1/r}) > H_1(x^{1/r}/k^{1/r})$, $x > 0$. Since the function $ny^{n-1} - (n-1)y^n$ is increasing in $y \in (0, 1)$, this, together with (7.1) and (7.2) implies that $P(\pi_B \leq x) > P(k\pi_1 \leq x)$ for all $x > 0$, and, therefore (see Shaked and Shanthikumar, 1994, pp. 3-4, and Remark 4.1 in this paper), $E(\pi_B) < E(k\pi_1) = \sum_{i\in B} E(\pi_i)$. Consequently, we get that for any bundling decision $\mathcal{B} = \{B_1, ..., B_l\}$ such that $card(B_s) = k_s$, $s = 1, ..., l$, and $k_t \geq 2$ for at least one $t \in \{1, ..., l\}$,

$$E(\Pi_{\mathcal{B}}) = \sum_{s=1}^{l} E(\pi_{B_s}) < \sum_{s=1}^{l} \sum_{i\in B_s} E(\pi_i) = \sum_{i=1}^{m} E(\pi_i) = E(\Pi_{\underline{\mathcal{B}}}). \tag{7.3}$$

The proof is complete.

Proof of Theorem 4.2. Let $r \in (0, 2]$ and let X_i, $i \in M$, be i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, 0)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1, 1]$ and $\alpha \in (0, r)$, $\beta = 0$ for $\alpha = 1$, or $X_i \sim \underline{\mathcal{CS}}(r)$, $i \in M$. Consider any bundle $B \in 2^M$ with $card(B) = k \leq m - 1$. With the same notations as in the proof of Theorem 4.1, comparisons (6.2) and Theorem 6.2 imply that $H_k(xk^{1/r}) > H_m(xm^{1/r})$, $x > 0$, and, therefore, $H_k(x^{1/r}) > H_m(x^{1/r}m^{1/r}/k^{1/r})$, $x > 0$. Similar to the proof of Theorem 4.1, we get, therefore, that $P(\pi_B \leq x) > P((k/m)\Pi_{\overline{\mathcal{B}}} \leq x)$ for all $x > 0$. By Shaked and Shanthikumar (1994, pp. 3-4) and the property that U is an increasing concave function with $U(0) = 0$, we get, therefore, that $EU(\pi_B) < EU((k/m)\Pi_{\overline{\mathcal{B}}}) \leq (k/m)EU(\Pi_{\overline{\mathcal{B}}})$. Consequently, for any bundling decision $\mathcal{B} = \{B_1, ..., B_l\}$ such that $card(B_s) = k_s$, $s = 1, ..., l$, and $k_t \leq m - 1$ for at least one $t \in \{1, ..., l\}$,

$$EU(\Pi_{\mathcal{B}}) = EU(\sum_{s=1}^{l} \pi_{B_s}) \leq \sum_{s=1}^{l} EU(\pi_{B_s}) < \sum_{s=1}^{l} EU((k_s/m)\Pi_{\overline{\mathcal{B}}}) \leq \sum_{s=1}^{l} (k_s/m)EU(\Pi_{\overline{\mathcal{B}}}) = EU(\Pi_{\overline{\mathcal{B}}}).$$

The proof is complete.

Proof of Theorem 4.3. Let $j \in J$. Let the vector $\tilde{X}^{(j)}$ of the jth buyer's reservation prices for goods in M take a value $\tilde{x}^{(j)} = (\tilde{x}_{1j}, ..., \tilde{x}_{nj}) \in \mathbf{R}^n_+$, $(\tilde{x}_{1j}, ..., \tilde{x}_{nj}) \neq (0, 0, ..., 0)$. Consider any bundle $B \in 2^M$ with $card(B) = k \geq 2$. The $j-$th buyer's reservation price for the bundle is $v_j(B) = \sum_{i \in B} \tilde{x}_{ij}$. Using the same notations as in the proof of Theorem 4.1, we get, similar to Palfrey (1983), that the expected surplus to the buyer when B is offered for sale is

$$ES_j(B, \tilde{x}^{(j)}) = \int_0^{v_j(B)} (H_k(x))^{n-1} dx = k \int_0^{v_j(B)/k} (H_k(kx))^{n-1} dx. \tag{7.4}$$

On the other hand, the expected surplus to consumer j when good $i \in B$ is offered for sale separately is $ES_j(\{i\}, \tilde{x}^{(j)})$ $= \int_0^{\tilde{x}_{ij}} (H_1(x))^{n-1} dx$. By Theorem 6.4 and (6.2), $H_k(kx) < H_1(x)$ for all $x > 0$. This, together with (7.4), implies

$$ES_j(B, \tilde{x}^{(j)}) < k \int_0^{v_j(B)/k} (H_1(x))^{n-1} dx \tag{7.5}$$

if $v_j(B) > 0$. Since the function $(H_1(y))^{n-1}$ is increasing in $y \in \mathbf{R}_+$, from Theorem 3.C.1 in Marshall and Olkin (1979) we get that the function $F(y_1, ..., y_k) = \sum_{i=1}^k \int_0^{y_i} (H_1(x))^{n-1} dx$ is Schur-convex in $(y_1, ..., y_k) \in \mathbf{R}^k_+$. Therefore, from majorization comparisons (6.1) it follows that $F(y_1, ..., y_k) \geq F(\sum_{i=1}^k y_i/k, ..., \sum_{i=1}^k y_i/k)$ for all $(y_1, ..., y_k) \in \mathbf{R}^k_+$ (see also the proof of Theorem 5 in Palfrey, 1983). In particular,

$$k \int_0^{v_j(B)/k} (H_1(x))^{n-1} dx \leq \sum_{i \in B} \int_0^{\tilde{x}_{ij}} (H_1(x))^{n-1} dx = \sum_{i \in B} ES_j(\{i\}, \tilde{x}^{(j)}). \tag{7.6}$$

From (7.5) and (7.6) we get

$$ES_j(B, \tilde{x}^{(j)}) < \sum_{i \in B} ES_j(\{i\}, \tilde{x}^{(j)}) \tag{7.7}$$

if $v_j(B) > 0$ (clearly, (7.7) holds as equality if $v_j(B) = 0$). By (7.7), we have that if the seller follows a bundling decision $\mathcal{B} = \{B_1, ..., B_l\}$ such that $card(B_s) = k_s$, $s = 1, ..., l$, and $k_t \geq 2$ for at least one $t \in \{1, ..., l\}$, then the expected surplus $ES_j(\mathcal{B}, \tilde{x}^{(j)})$ to buyer j satisfies $ES_j(\mathcal{B}, \tilde{x}^{(j)}) = \sum_{s=1}^l ES_j(B_s, \tilde{x}^{(j)}) < \sum_{i=1}^m ES_j(\{i\}, \tilde{x}^{(j)}) =$ $ES_j(\underline{\mathcal{B}}, \tilde{x}^{(j)})$. The proof is complete.

Proofs of Theorems 5.1-5.4. Let $r \in (0, 2]$ and let c_i, $i \in M$, be arbitrary marginal costs of goods in M. Let the reservation prices $v(B)$ for bundles $B \in 2^M$ be given by $v(B) = v(g_r; B) = g_r\big(\sum_{i \in B} X_i\big)$ or by $v(B) = v(h_r; B) = h_r\big(\sum_{i \in B} X_i\big)$. Further, let $\mu \in \mathbf{R}$ and $p_{max} < \infty$. Suppose that the tastes X_i, $i \in M$, are i.i.d. r.v.'s such that $X_i \sim S_\alpha(\sigma, \beta, \mu)$, $i \in M$, for some $\sigma > 0$, $\beta \in [-1, 1]$ and $\alpha \in (0, r)$, $\beta = 0$ for $\alpha = 1$, or $X_i - \mu \sim \underline{\mathcal{CS}}(r)$, $i \in M$. We will show that the seller's profit maximizing bundling decision is $\underline{\mathcal{B}}$ if the prices per good $p_B < \mu$ for all bundles $B \in 2^M$, and is $\overline{\mathcal{B}}$ if $p_B > \mu$ for all $B \in 2^M$. For a bundle $B \in 2^M$, the profit maximizing price per good in the bundle is $p_B = \arg\max_{p \in [0, p_{max}]} \big(p - (1/k) \sum_{i \in B} c_i\big) P(v(B) \geq kp)$ and the seller's profit per good resulting from selling the bundle B (at the price per good p_B) is $E(\pi_B) = Jk\big(p_B - \sum_{i \in B} c_i\big) P(v(B) \geq kp_B)$, where $k = card(B)$ is the number of goods in B. For $i \in M$, let p_i be the price of good i in the case where the goods are sold separately (that is, in the case of the bundling decision $\underline{\mathcal{B}}$) and let, as in the proof of Theorem 4.1, π_i be the monopolist's profit from selling the good, namely, $p_i = p_{B_i}$ and $\pi_i = \pi_{B_i}$ with $B_i = \{i\}$. As in the setup of the optimal bundling problem in Section 5, in the case with $c_i = c$ for all $i \in M$, we write $\overline{p} = p_M$ for the price per good in the case where all the m goods are sold as a single bundle $B = M$ (that is, in the case of the bundling decision $\overline{\mathcal{B}}$) and $\underline{p}$ for the price of each good under unbundled sales (that is, $\underline{p} = p_B$ with $B = \{i\}$, $i \in M$).

Suppose that $p_B < \mu$ for all $B \in 2^M$. Then from Theorem 6.4 and relations (6.2) it follows that, for any bundle $B \in 2^M$ with the number of goods $card(B) = k \geq 2$, $E(\pi_B) = J\big(kp_B - \sum_{i \in B} c_i\big)P(v(B) \geq kp_B) = J\big(kp_B - \sum_{i \in B} c_i\big)P(\sum_{i \in B} X_i \geq (kp_B)^{1/r}) < J\sum_{i \in B}\big(p_B - c_i\big)P\big(X_i \geq (p_B)^{1/r}\big) \leq \sum_{i \in B} E(\pi_i)$. This implies that for any bundling decision $\mathcal{B} = \{B_1, ..., B_l\}$ such that $card(B_s) = k_s$, $s = 1, ..., l$, and $k_t \geq 2$ for at least one $t \in \{1, ..., l\}$, comparisons (7.3) hold.

Suppose now that $p_B > \mu$ for all $B \in 2^M$. Then using again Theorem 6.4 and relations (6.2) we get that, for any bundle $B \in 2^M$ with $card(B) = k \leq m - 1$, $E(\pi_B) = J\big(kp_B - \sum_{i \in B} c_i\big)P(\sum_{i \in B} X_i \geq (kp_B)^{1/r}) < J\big(kp_B - \sum_{i \in B} c_i\big)P(\sum_{i=1}^m X_i \geq (mp_B)^{1/r})$. Therefore, for any bundling decision $\mathcal{B} = \{B_1, ..., B_l\}$ such that $card(B_s) = k_s$, $s = 1, ..., l$, and $k_t \leq m - 1$ for at least one $t \in \{1, ..., l\}$,

$$E(\Pi_{\mathcal{B}}) = \sum_{s=1}^{l} E(\pi_{B_s}) < J\sum_{s=1}^{l}\big(k_s p_{B_s} - \sum_{i \in B_s} c_i\big)P(\sum_{i=1}^{m} X_i \geq (mp_B)^{1/r}) =$$

$$J\sum_{s=1}^{l} k_s\big(p_{B_s} - (1/m)\sum_{i=1}^{m} c_i\big)P(\sum_{i=1}^{m} X_i \geq (mp_B)^{1/r}) \leq \sum_{s=1}^{l}(k_s/m)E(\Pi_{\overline{\mathcal{B}}}) = E(\Pi_{\overline{\mathcal{B}}}). \tag{7.8}$$

From (7.3) and (7.8) we get that the profit maximizing bundling decision is $\underline{\mathcal{B}}$ if $p_B > \mu$ for all $B \in 2^M$ and is $\overline{\mathcal{B}}$ if $p_B < \mu$ for all $B \in 2^M$.

Clearly, the condition that $p_B > \mu$ for all $B \in 2^M$ is satisfied if $c_i \geq \mu$ for all $i \in M$. Furthermore, in the case of identical marginal costs $c_i = c$, $i \in M$, the condition that $p_B > \mu$ for all $B \in 2^M$ holds if $\underline{p} > \mu$. Indeed, suppose that this not the case and there exists a bundle $B \in 2^M$ with $card(B) = k > 1$ and $p_B \leq \mu$. Then, as above, we get $kE(\pi_1) = Jk(\underline{p} - c)P(X_1 \geq (\underline{p})^{1/r}) < Jk(\underline{p} - c)P(\sum_{i=1}^k X_i \geq (k\underline{p})^{1/r}) \leq E(\pi_B)$. On the other hand, $E(\pi_B) = Jk(p_B - c)P(\sum_{i=1}^k X_i \geq (k\underline{p})^{1/r}) < Jk(p_B - c)P(X_1 \geq (p_B)^{1/r}) \leq kE(\pi_1)$, which is a contradiction. Similarly, we get that if $c_i = c$, $i \in M$, then $\overline{p} < \mu$ implies that $p_B < \mu$ for all $B \in 2^M$. This completes the proof of Theorem 5.2. Theorem 5.4 follows from Theorem 5.2 with $r = 1$. Theorems 5.1 and 5.3 could be proven in a similar way, with the use of Theorems 6.1 and 6.3 instead of Theorem 6.2. The proof is complete.

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