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with Indivisible Agents

by

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# Location Choice in Two-Sided Markets with Indivisible Agents<sup>1</sup>

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## **Abstract**

Consider a model of location choice by two sorts of agents, called “buyers” and “sellers:” In the first period agents simultaneously choose between two identical possible locations; following this, the agents at each location play some sort of game with the other agents there. Buyers prefer locations with fewer other buyers and more sellers, and sellers have the reverse preferences. We study the set of possible equilibrium sizes for the two markets, and show that two markets of very different sizes can co-exist even if larger markets are more efficient. This extends the analysis of Ellison and Fudenberg [3] (EF), who ignored the constraint that the number of agents of each type in each market should be an integer, and instead analyzed the “quasi-equilibria” where agents are treated as infinitely divisible.

# 1 Introduction

In many economic models, agents benefit from interactions with agents whose characteristics are in some way different than their own. In exchange economies, agents gain from interactions with agents whose endowments or preferences are different; in economies with production, gains come about when producers interact with consumers, and in opposite-sex marriage “markets,” gains arise from men meeting women. If the agents must choose between several locations or markets in which to conduct these interactions, the question arises: Should we expect all activity to concentrate in a single market or location, or should we expect to see multiple active marketplaces?

To address this question, we consider a model of location choice by two sorts of agents, whom we will call “buyers” and “sellers.” In the first period of the model, agents simultaneously choose between two identical possible locations or markets. Following this, the agents at each location play some sort of game with the other agents there; the resulting payoffs are determined by the numbers of each type of agent who chose the same location. We suppose that buyers prefer markets with fewer other buyers and more sellers, and that sellers have the reverse preferences. Thus, when the numbers of buyers and sellers are even, there is an equilibrium where the two markets are exactly the same size.<sup>1</sup> Our question is whether this split-market equilibrium is indeed an isolated equilibrium, or whether there is a wide range (a “plateau”) of equilibrium size ratios for the two markets.

The traditional view (see e.g. [9]) was that the split-market equilibrium will be a knife-edge whenever there are increasing returns, that is when larger markets are more efficient. However, Ellison and Fudenberg [3] (EF) pointed out that in markets with a finite number of agents the effect of increasing returns can be offset by a “market impact effect”: When a seller contemplates a switch from market 1 to market 2, she takes into account that her joining market 2 will increase the seller-buyer ratio there, and thereby make that market less attractive for all sellers, including herself. It is true that this market impact effect typically vanishes as the number of agents in each market grows to infinity, but as EF point

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<sup>1</sup>More precisely, our results imply that when the numbers of buyers and sellers are even and sufficiently large, there is an equilibrium where the two markets are exactly the same size. We expect that typically this equilibrium will also be present even when the numbers of buyers and sellers is small, but have not tried to provide precise conditions under which this is true.

out, the effect of increasing returns typically vanishes as well. Moreover, EF argue that in many models of interest the effects vanish at the same rate. When this is true, EF show that the incentive constraints for equilibrium are consistent with two large markets being active even when their sizes are unequal, and they give a lower bound on the width of this plateau. However, they ignore the constraint that the number of agents of each type in each market should be an integer, and instead analyze the “quasi-equilibria” where agents are treated as infinitely divisible; their formal results say nothing about the existence of full equilibria. In the context of competing auction sites, Ellison, Fudenberg and Mobius (EFM) [4] show that when the number of buyers is large and buyer values are uniformly distributed, for any quasi-equilibrium size ratio  $\alpha$  of the game with  $B$  buyers and  $S$  sellers, there is a nearby  $S'$  for which the game with  $B$  buyers and  $S'$  sellers has an exact equilibrium with size ratio approximately  $\alpha$ . This shows that exact equilibria exist in some large economies, but says nothing about the number of equilibria for a typical pair  $(S, B)$ . Indeed, it leaves open the possibility that most pairs  $(S, B)$  have no exact equilibria other than complete tipping.

The point of this paper is to complete the analysis of tipping in large markets by providing a thorough analysis of equilibrium, paying attention to the constraint that each agent is indivisible. Put briefly, we have three main findings about economies (payoff functions and aggregate seller/buyer ratios) that satisfy the EF assumptions.

1. Despite the EF result about large economies always having a large quasi-equilibrium plateau, under some natural specifications of payoffs there are sequences  $\{S_n, B_n\}$  with  $B_n \rightarrow \infty$  that completely tip for all  $n$ . That is, at each point in the sequence, the only pure strategy equilibria are the ones in which all agents choose the same location.
2. For any  $\varepsilon > 0$ , there is a bound  $M$  such that if the aggregate buyer-seller and seller-buyer ratios are at least  $\varepsilon$  away from every integer, and the number of agents is at least  $M$ , then there is a range of exact equilibria. For fixed payoff functions the width of this range (expressed in terms of the proportion of agents in each market) is independent of the number of buyers and sellers when these are large.
3. For generic sequences  $\{S_n, B_n\}$  with  $B_n \rightarrow \infty$  and  $S_n/B_n$  convergent, there is a

plateau of exact equilibria for all large  $n$ . This plateau includes all of the size ratios predicted by the EF result, and typically includes others as well. Moreover, the asymptotic density of equilibria (i.e. the proportion of integers in a given range such that there is an equilibrium with that many buyers in one market, and the rest in the other) converges to a specific piecewise-linear function we identify. When there are fewer buyers than sellers, for almost all integer numbers of buyers in the center of the plateau, there is an equilibrium with that many buyers in market 1.

The set of exact equilibria described in point 2 can be much smaller than the set of quasi-equilibria delivered by the EF result. The genericity result in point 3 is proved by showing that the results obtain whenever  $S_n/B_n$  has an irrational limit. We believe that the theorem mentioned in point 2 can be extended to obtain larger and larger lower bounds on the equilibrium plateau by bounding the buyer-seller ratios away from  $1/2$ ,  $1/3$ ,  $2/3$ , etc. through the rational numbers, but this seems too cumbersome to be worth pursuing given the availability of the result in point 3.

The conclusions about the equilibrium plateau in point 3 apply whenever  $S_n/B_n$  converges to an irrational limit. Since the set of rational numbers has measure 0, this is analogous to the formulation of a generic sequence of purely competitive economies in, for example, Grodal [5] and H. Dierker [2]. The reason that sequences with an irrational limit are better behaved is closely related to the fact that the map that sends  $x$  to  $(x + y)_{\text{mod } 1}$  is an ergodic transformation if and only if  $y$  is irrational.

The reason that the result in point 3 delivers more size ratios than EF is that their result provided a condition for the incentive constraints to be satisfied with exactly the same seller/buyer ratio in each market. The proof of the new result, like the EFM characterization of the quasi-equilibrium set in the case of uniformly distributed buyer values, allows the seller/buyer ratios in the two markets to be slightly unequal, which in some cases leads to a substantially larger range of size ratios.

The result in point 3 is proved using nonstandard analysis, which is a way of formalizing and manipulating infinitesimal, and infinitely large, quantities. The statement of the result, however, is entirely standard and can be understood without any knowledge of nonstandard

analysis.<sup>2</sup>

## 2 The Geometry of Equilibrium

Consider a simple two-stage model of location choice. In the first stage  $S$  sellers and  $B$  buyers simultaneously choose between market 1 or market 2. If  $S_i$  sellers and  $B_i$  buyers chose market  $i$ , then the market game gives the sellers in market  $i$  an expected payoff of  $u_s(S_i, B_i)$  and the buyers an expected payoff of  $u_b(S_i, B_i)$ .<sup>3</sup> In a pure-strategy Nash equilibrium, there are four incentive constraints that must be satisfied: First, sellers in market 1 must be willing to stay in market 1, which is equivalent to the inequality

$$(S1) \quad u_s(S_1, B_1) \geq u_s(S_2 + 1, B_2)$$

We call this the  $S1$  constraint. Then there are the analogous constraints that sellers in market 2 must be willing to stay in market 2, buyers in market 1 must be willing to stay in market 1, and buyers in market 1 must be willing to stay in market 2. We call these constraints  $S2$ ,  $B1$ , and  $B2$  respectively.

Given the total numbers  $S$  of sellers and  $B$  of buyers, an allocation  $(S_1, S_2, B_1, B_2)$  is a pure-strategy Nash equilibrium if and only if

- (i) (*Incentive Compatibility*) the four incentive constraints  $S1, S2, B1, B2$  are satisfied.
- (ii) (*Feasibility*)  $S_1 + S_2 = S, B_1 + B_2 = B$ , and  $S_1, S_2, B_1$ , and  $B_2$  are all non-negative.
- (iii) (*Indivisibility*)  $S_1, S_2, B_1$ , and  $B_2$  are all integers.

EF call an allocation a *quasiequilibrium* if it satisfies (i) and (ii).

**Assumption 1** *There is a nonempty interval  $\Gamma = [\underline{\gamma}, \bar{\gamma}] \subset (0, \infty)$ , continuously differentiable functions  $F_s$  and  $F_b$ , and continuous functions  $G_s$  and  $G_b$  on  $\Gamma$  with  $F'_s < 0$  and*

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<sup>2</sup>For more information about nonstandard analysis, see Hurd and Loeb [6] and Anderson [1]. A metatheorem in nonstandard analysis guarantees the existence of a standard proof for the result; however, the standard proof is likely to be quite complex.

<sup>3</sup>Although the formal model we present is one of Nash equilibria in a one-stage game, the intended interpretation is that we are analyzing first-stage choices in the subgame-perfect equilibria of a two-stage game, where the payoff functions that this paper treats as primitives are computed as the Nash equilibria of the game at each market. These payoff functions will typically only be defined for integer numbers of agents; our assumptions are then on any extension of these functions to the reals.

$F'_b > 0$  such that the approximations

$$\begin{aligned} u_s(\gamma B, B) &= F_s(\gamma) - \frac{G_s(\gamma)}{B} + o\left(\frac{1}{B}\right) \\ u_b(\gamma B, B) &= F_b(\gamma) - \frac{G_b(\gamma)}{B} + o\left(\frac{1}{B}\right) \end{aligned} \tag{1}$$

hold uniformly over  $\gamma \in \Gamma$  in the limit as  $B \rightarrow \infty$ .

We will maintain this assumption throughout the paper. EF argue that it is satisfied in many models of economic interest, including the Ellison, Fudenberg and Möbius [4] model of competing auctions, Krugman's [8] model of Marshallian labor market competition, and a two-population version of Pagano's [10] model of competing securities markets. They show that it allows the approximation of the incentive constraints at allocations with the same seller/buyer ratios in each market, and use this to prove the following result.

**Theorem 0** (*Ellison and Fudenberg [3]*) *Assume A1, and set*

$$r^*(\gamma) = \max\left(\left|\frac{2G_s(\gamma)}{-F'_s(\gamma)} + 1\right|, \left|\frac{2G_b(\gamma)}{\gamma F'_b(\gamma)} + 1\right|\right)$$

and

$$\alpha^*(\gamma) = \max\{0, 1/2 - 1/2r^*(\gamma)\}.$$

*Then for any  $\varepsilon > 0$ , there exists a  $\underline{B}$  such that for any integer  $B > \underline{B}$  and any integer  $S$  with  $S/B \in \Gamma$ , the model with  $B$  buyers and  $S$  sellers has a quasi-equilibrium with  $B_1$  buyers in market 1, for every  $B_1$  with  $B_1/B \in [\alpha^*(S/B) + \varepsilon, 1 - \alpha^*(S/B) - \varepsilon]$ .*

This result on its own says nothing about the set of full equilibria of the game. To fix ideas, we reproduce (as Figure 1) two illustrations from EFM that show the four constraints in an auction model with uniformly distributed buyer values.<sup>4</sup> Here the lines S1, S2, B1, B2 correspond to the points where the corresponding incentive constraints hold with equality. That is, along S1, each seller that is currently choosing market 1 is just indifferent between

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<sup>4</sup>In EFM, each seller has a single unit to sell, and reservation value 0 for keeping the item; each buyer wants to purchase a single unit, and buyer values are determined after location choice. Equilibrium thus payoffs in each market are determined by a uniform-price auction.

staying in market 1 or switching to market 2. Since sellers prefer there to be fewer other sellers, the points above this line are not consistent with equilibrium. Reasoning in this way about each of the incentive constraints, we see that the quasi-equilibrium set is the diamond-shaped region in the center of the figure. The exact equilibria, marked with stars, are the integer grid points that lie in the quasi-equilibrium set.

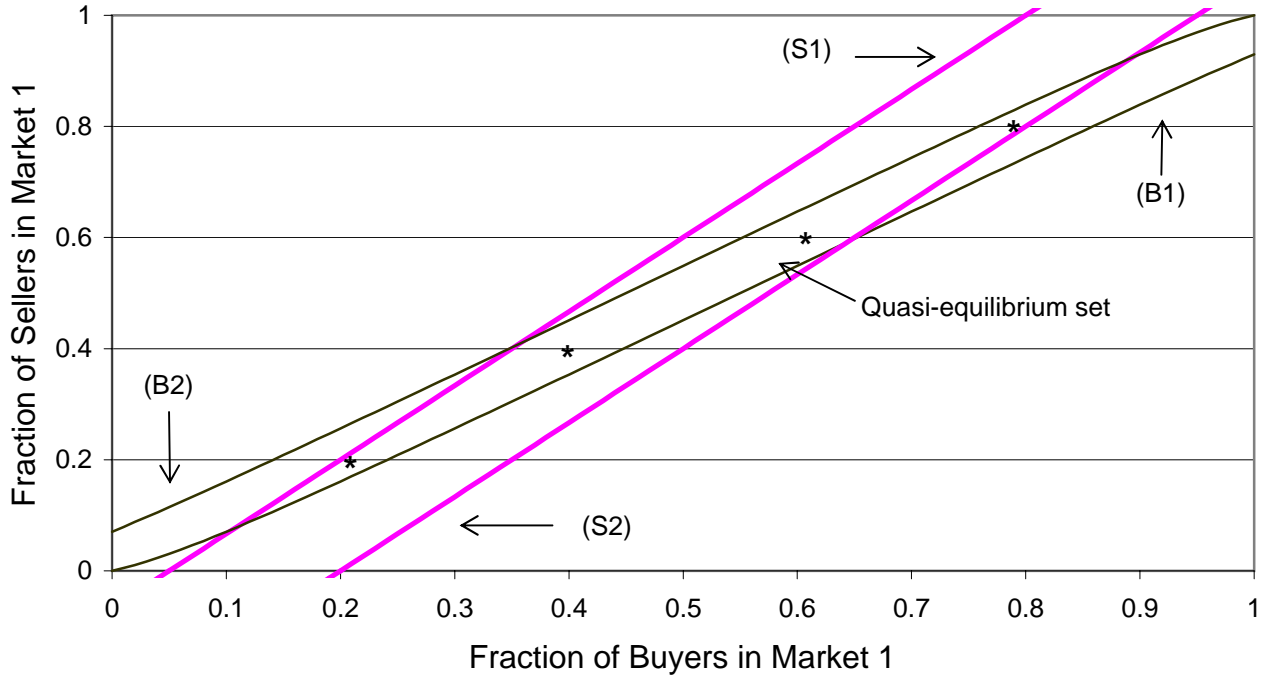
The top panel corresponds to a model with ten buyers and five sellers. In equilibrium the smaller market can have two buyers and one seller, or four buyers and two sellers; there is no equilibrium with three or five buyers in the smaller market. With three buyers in the smaller market, for example, there are a range of values of near one-and-a-half that satisfy the quasi-equilibrium conditions, but none of them satisfy the integer constraints. The bottom panel gives an idea of what happens as markets grows to thirty buyers and fifteen sellers. The quasi-equilibrium region looks much “flatter” because the impact of one agent moving from one market to another is smaller, so the utilities in the two markets need to be closer together in order to discourage agents whose equilibrium utility is lower from switching markets. (Lemma 4 makes a related observation: If  $(S_1, S_2, B_1, B_2)$  is a quasi-equilibrium, then the seller/buyer ratio in market 1 must equal the overall ratio  $S/B$  plus a term that goes to 0 at rate  $O(1/B_1)$ .) The exact equilibria are again marked with stars.

This leads us to the current question: when will there be exact equilibria? As the economy grows, the quasi-equilibrium set gets “narrower”, suggesting it might be harder to find an equilibrium at a given point, but at the same time there are more candidate integers in a given range of relative sizes, so the answer may not be obvious.

### 3 An Example with Complete Tipping

The main result of EF is that models satisfying Assumption 1 always have a large plateau of quasi-equilibria with two active markets when the number of agents is large. In this section, we show that the result does not always extend to a true equilibrium analysis. In particular, we provide a class of sequences of economies in which the increasing returns satisfy Assumption 1, but which have *no* pure strategy equilibria other than the completely “tipped” equilibria with all activity in one market.

## Quasi-equilibrium Set 10 Buyers and 5 Sellers



## 30 Buyers and 15 Sellers

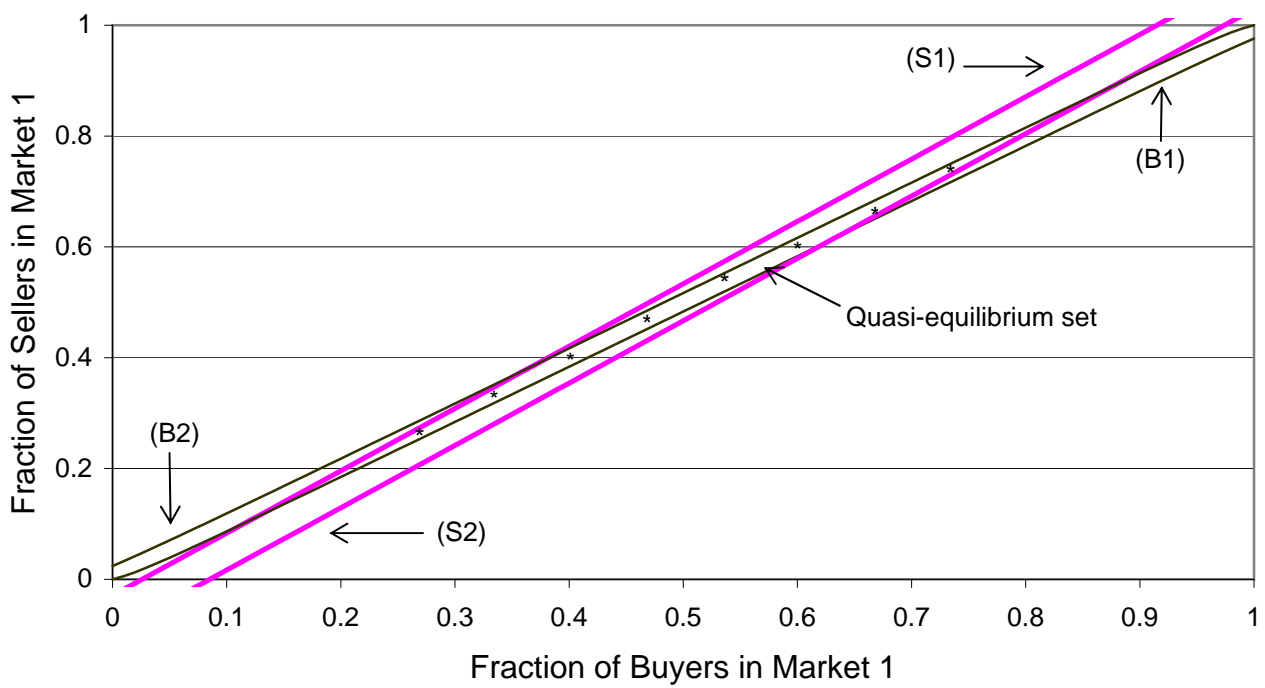


Figure 1: The quasiequilibrium and equilibrium sets in the EFM competing auction model.

**Example 1** Let  $u_s$  and  $u_b$  be utility functions satisfying Assumption 1. Suppose that  $u_s(\gamma B, B)$  is increasing in  $B$  and decreasing in  $\gamma$ , and that  $u_b(\gamma B, B)$  is increasing in  $B$  and  $\gamma$ . Then, regardless of the size of  $B$ , a model with  $B$  buyers and  $S = B + 1$  sellers has no pure strategy Nash equilibrium with two active markets.

Proof

Suppose to the contrary that there is an equilibrium with  $0 < B_1 < B$  and  $0 < S_1 < S$  sellers in market 1, and  $B_2 = B - B_1$  buyers and  $S_2 = S - S_1$  sellers in market 2, and assume without loss of generality that  $B_1 \leq B_2$ .

We proceed to rule out all possible relationships between  $S_1$  and  $B_1$

- If  $S_1 < B_1$ , then a buyer in market 1 will defect. After the defection market 2 will be larger (have more buyers) and have a more favorable seller/buyer ratio for buyers.
- If  $S_1 = B_1$ , then a buyer in market 1 will defect. Market 2 is larger after the defection. Also,  $S_1 = B_1$  implies  $S_2 = B_2 + 1$ , so after the defection the the ratio of sellers to buyers in market 2 becomes one, the same as in market 1. Hence, the buyer is better off after the defection
- If  $S_1 = B_1 + 1$  and  $B_1 = B_2$ , then a buyer in market 2 will defect.  $S_2 = B_2$  so the situation is exactly as in the case above, but with the labels reversed.
- If  $S_1 = B_1 + 1$  and  $B_1 < B_2$ , then a seller in market 1 will defect. Market 2 is larger, and even after the defection its seller/buyer ratio is lower:  $(S_2 + 1)/B_2 = (B_2 + 1)/B_2 < (B_1 + 1)/B_1 = S_1/B_1$ .
- If  $S_1 > B_1 + 1$ , then a seller in market 1 will defect. Market 2 is at least as large, and even after the deviation it will have a lower seller/buyer ratio.

■

In some sense the assumption that  $S = B + 1$  in the example is a worst case for the existence of equilibria. Equilibrium requires that the seller-buyer ratios be very close in the two markets. When market 1 and market 2 are about the same size, the fact that we have to put the one extra seller somewhere means we have to have about one-half too

few sellers in one market and about one-half too many sellers in the other. The “market impact” of moving to the large market is one seller or one buyer, so it doesn’t outweigh this. This suggests that finding equilibria may be easier when the seller-buyer is bounded away from 1, as then the impact of the “rounding error” is less severe. More generally, one way of looking at the example is that the size of the set of equilibria will depend on how closely the overall seller/buyer ratio can be approximated in the two markets when we are restricted to integer allocations.

## 4 A Sufficient Condition for a Plateau of Equilibria in Large Finite Economies

In this section we provide sufficient conditions for the existence of a plateau of true equilibria. The sufficient conditions illustrate that the nonexistence example was special. In the example, the seller-buyer ratio is very close to one, but not equal to one. The main proposition of this section shows that if the ratio of the number of agents on the two sides of the market is not almost exactly an integer (and the number of agents is large), then there is a nondegenerate plateau of untipped equilibria. It also shows that within the equilibrium plateau the equilibria are not too far apart.

Let  $\mathbf{N}$  be the set of non-negative integers. The main result of this section is

**Theorem 1** *Fix functions  $u_b$  and  $u_s$  satisfying Assumption 1. There are constants  $\underline{\alpha}, k_1, k_2, k_3, k_4$  with  $\underline{\alpha} < 1/2$  such that for any  $\varepsilon > 0$ , any  $\alpha \in [\underline{\alpha}, 1 - \underline{\alpha}]$ , and any positive integers  $B$  and  $S$  satisfying*

- (i)  $S/B \in \Gamma$ ,
- (ii)  $|B/S - n| > \varepsilon$  and  $|S/B - n| > \varepsilon$  for all  $n \in \mathbf{N}$ , and
- (iii)  $B + S > k_1 + k_2/\varepsilon$ ,

*the model with  $B$  buyers and  $S$  sellers has an equilibrium with  $B_1$  buyers and  $S_1$  sellers in market 1 for some  $B_1, S_1$  with  $|B_1/B - \alpha| < (k_3 + k_4/\varepsilon)/B$  and  $|S_1/S - \alpha| < (k_3 + k_4/\varepsilon)/S$ .*

The proof includes more information on the constants  $k_1, k_2, k_3$ , and  $k_4$ , including explicit formulas for some of them. The following corollary is a simpler statement about

the existence of an equilibrium plateau.

**Corollary 1** *Fix functions  $u_b$  and  $u_s$  satisfying Assumption 1. Then there is  $\underline{\alpha} < 1/2$  such that for any  $\varepsilon > 0$  and any  $\delta > 0$  there is an  $M$  such that if*

(i)  $S/B \in \Gamma$ ,

(ii)  $|B/S - n| > \varepsilon$  and  $|S/B - n| > \varepsilon$  for all  $n \in \mathbf{N}$ , and

(iii)  $B + S > M$ ,

then for every  $\alpha \in [\underline{\alpha}, 1 - \underline{\alpha}]$ , the model with  $B$  buyers and  $S$  sellers has an equilibrium with  $B_1$  buyers and  $S_1$  sellers in market 1 for some  $B_1, S_1$  with  $|B_1/B - \alpha| < \delta$  and  $|S_1/S - \alpha| < \delta$ .

#### Proof of Theorem 1

We proceed in two main steps. The first step is to show that the quasiequilibrium set contains a parallelogram in  $B$ - $S$  space that includes all points on the line segment defined by  $B_1/B = S_1/S$  and  $B_1/B \in [\underline{\alpha}, 1 - \underline{\alpha}]$  and that is not “too thin.” The second step is to show that every point in the parallelogram is within a specified distance of an integer point that is also within the parallelogram. The integer points are true equilibria.

To simplify the exposition we will prove the result only for pairs with  $S \leq B$ . “Sellers” and “buyers” are completely symmetric in the model, so the argument for  $B \leq S$  would be identical.

**Lemma 1** *There exist  $\underline{\alpha}$  and  $\underline{N}$  with  $\underline{\alpha} < 1/2$  such that for all  $B$  and  $S$  with  $B + S > \underline{N}$  and  $\gamma \equiv S/B \in \Gamma$ , it is a quasi-equilibrium of the model with  $B$  buyers and  $S$  sellers to have  $B_1$  buyers and  $S_1$  sellers in market 1 for any  $B_1$  and  $S_1$  with and any  $B_1, S_1$  with  $B_1/B \in [\underline{\alpha}, 1 - \underline{\alpha}]$  and  $|S_1/S - B_1/B| \leq 1/4B$ .*

Remark: When we graph the space of allocations as in Figures 1 and 2, with  $B_1/B$  on the x-axis and  $S_1/S$ , on the y-axis, the lemma says that the quasiequilibrium set contains the parallelogram bounded by  $(\underline{\alpha}, \underline{\alpha} \pm 1/4B)$  and  $(1 - \underline{\alpha}, 1 - \underline{\alpha} \pm 1/4B)$ .

#### Proof of Lemma 1

We will show that buyers in market 1 do not wish to switch to market 2, i.e. that the  $B1$  constraint is satisfied. The argument for the  $B2$  constraint is identical. The arguments for the  $S1$  and  $S2$  constraints are very similar.

We begin with the “hardest” case for the B1 constraint: when  $S_1 = (\alpha - 1/4B)S$  and  $B_1 = \alpha B$ . In this case we need to show that

$$u_b(\alpha - 1/4B)S, \alpha B \geq u_b(1 - \alpha + 1/4)S, (1 - \alpha)B + 1).$$

Given Assumption 1, this will be satisfied whenever  $B + S$  is greater than some  $\underline{N}$  if

$$\begin{aligned} & F_b\left(\gamma\left(1 - \frac{1}{4\alpha B}\right)\right) - G_b\left(\gamma\left(1 - \frac{1}{4\alpha B}\right)\right) \frac{1}{\alpha B} > \\ & F_b\left(\gamma\left(1 - \frac{3}{4(1-\alpha)B}\right)\right) - G_b\left(\gamma\left(1 - \frac{3}{4(1-\alpha)B}\right)\right) \frac{1}{(1-\alpha)B} \end{aligned}$$

whenever  $B$  is greater than some  $\underline{B}$ . (The restriction to  $S/B \in [\underline{\gamma}, \bar{\gamma}] \subset (0, \infty)$  implies that  $B$  is always large when  $B + S$  is large.) There exists  $\underline{B}$  so that this is true for all  $B > \underline{B}$  for all  $\gamma \in \Gamma$  and all  $\alpha \in [\underline{\alpha}, \bar{\alpha}]$  we have

$$\left(-\frac{1}{4\alpha} + \frac{3}{4(1-\alpha)}\right) \gamma F'_b(\gamma) > \left(\frac{1}{\alpha} - \frac{1}{1-\alpha}\right) G_b(\gamma),$$

which is equivalent to

$$\frac{\alpha - 1/4}{1 - 2\alpha} > \frac{G_b(\gamma)}{\gamma F'_b(\gamma)}.$$

We can choose  $\underline{\alpha} \in (1/4, 1/2)$  so that this is true uniformly over  $\gamma$ . Note that for the same  $\underline{\alpha}$  the equation we would have derived had we started with any larger  $S_1$  will also be satisfied due to the monotonicity of  $F_b$ . Hence, having chosen an  $\underline{\alpha}$  satisfying this equation we can then choose  $\underline{B}$  so that B1 is satisfied for all  $B_1$  and  $S_1$  in the parallelogram provided that  $B > \underline{B}$ , which completes the proof of Lemma 1. ■

Given the result of lemma 1 it suffices to prove the following lemma about parallelograms containing grid points. Lemma 2 concludes that there exist integers  $B_1$  and  $S_1$  satisfying several properties. The first two of these are that  $(B_1/B, S_1/S)$  belongs to the parallelogram described in the Lemma 1, which implies that it is a true equilibrium to have  $B_1$  buyers and  $S_1$  sellers in market 1. The last two indicate we can find such equilibria close to every point of the plateau identified in the theorem.

**Lemma 2** *Assume  $\underline{\alpha} < 1/2$ . Then there are constants  $k_1, k_2, k_3, k_4$  such that for any  $\varepsilon > 0$ , any  $\alpha \in [\underline{\alpha}, 1 - \underline{\alpha}]$  and any  $S, B$  satisfying (i)  $S \leq B$ , (ii)  $S/B \in \Gamma$ , (iii)  $|B/S - n| > \varepsilon$*

for all  $n \in N$ , and (iv)  $S > k_1 + k_2/\varepsilon$ , there exist integers  $S_1$  and  $B_1$  such that  $B_1/B \in [\underline{\alpha}, 1 - \underline{\alpha}]$ ,  $|S_1/S - B_1/B| < 1/4B$ ,  $|B_1/B - \alpha| < (k_3 + k_4/\varepsilon)(1/B)$ , and  $|S_1/S - \alpha| < (k_3 + k_4/\varepsilon)(1/S)$ .

Proof of Lemma 2

Fix some  $\alpha \in [\underline{\alpha}, 1 - \underline{\alpha}]$ . We assume here that  $\alpha > 1/2$ . (A symmetric argument applies for  $\alpha < \frac{1}{2}$ .)

Let  $B_1^0$  be the integer closest to  $\alpha B$  in the set of integers  $B'$  such that  $B'/B \in (\underline{\alpha}, 1 - \underline{\alpha})$  (This set will be non-empty for  $k_1$  sufficiently large given the assumption that  $S/B \in \Gamma$ .)

We now define a sequence of integer pairs  $(S_1^k, B_1^k)$  by repeatedly reducing  $S_1$  by one and  $B_1$  by an integer close to  $B/S$ . We will choose  $(S_1, B_1)$  to be the first point in the sequence that satisfies  $|S_1^k/S - B_1^k/B| < 1/4B$ . The key observation is that this always occurs for some  $k$  with  $k \leq 4 + 1/\varepsilon$ .

Let  $S_1^1 = \lfloor (B_1^0/B)S \rfloor$ . Note that  $S_1^1/S \leq B_1^0/B$ . Let  $B_1^1$  be the largest integer with  $B_1^1/B \leq S_1^1/S + 1/4B$ .

If  $S_1^1/S < B_1^1/B + 1/4B$ , then choose we will choose  $S_1 = S_1^1$  and  $B_1 = B_1^1$ . Note in this case that  $S_1$  and  $B_1$  satisfy  $|S_1/S - B_1/B| < 1/4B$ .

Otherwise, if for some  $k \geq 1$  we have  $S_1^k/S \geq B_1^k/B + 1/4B$ , then define  $S_1^{k+1}$  and  $B_1^{k+1}$  by  $S_1^{k+1} = S_1^k - 1$ , and

$$B_1^{k+1} = \begin{cases} B_1^k - \lfloor B/S \rfloor & \text{if } B/S - \lfloor B/S \rfloor \leq S_1^k/S - B_1^k/B + 1/4B \\ B_1^k - \lfloor B/S \rfloor - 1 & \text{if } B/S - \lfloor B/S \rfloor > S_1^k/S - B_1^k/B + 1/4B \end{cases}$$

We then choose  $(S_1, B_1) = (S_1^k, B_1^k)$  where  $k$  is the smallest index for which  $|S_1^k/S - B_1^k/B| \leq 1/4B$ .

Claim: This choice of  $(S_1, B_1)$  is well-defined and the index  $k$  satisfies  $k \leq 3 + 1/\varepsilon$ .

To show this, we write  $x$  for  $B/S - \lfloor B/S \rfloor$  and define  $a^k \equiv B(S_1^k/S - B_1^k/B - 1/4B)$ . Note that  $(S_1^k, B_1^k)$  are the desired  $(S_1, B_1)$  if  $a^k \in (-1/2, 0]$ .

The definition of  $B_1^1$  as the largest integer satisfying  $B_1^1/B \leq S_1^1/S + 1/4B$  implies that  $(B_1^1 + 1)/B > S_1^1/S + 1/4B$ ,  $a^1 = B(S_1^1/S - B_1^1/B - 1/4B) \in [-1/2, 1/2)$ .

If  $(S_1^1, B_1^1)$  is not the desired pair of integers, then  $a^1 \in (0, 1/2)$ . For  $a^k \in (0, 1/2)$  the expressions for  $S_1^{k+1}, B_1^{k+1}$  can be rewritten as

$$S_1^{k+1} = S_1^k - 1$$

$$B_1^{k+1} = \begin{cases} B_1^k - \lfloor B/S \rfloor & \text{if } x \leq a^k + 1/2 \\ B_1^k - \lfloor B/S \rfloor - 1 & \text{if } x > a^k + 1/2 \end{cases}$$

Hence,

$$a^{k+1} = \begin{cases} B \left( \frac{S_1^{k+1}}{S} - \frac{B_1^{k+1}}{B} - \frac{1}{4B} \right) = B \left( \frac{S_1^k}{S} - \frac{1}{S} - \frac{B_1^k}{B} - \lfloor B/S \rfloor - \frac{1}{4B} \right) = a^k - x & \text{if } x \leq a^k + 1/2 \\ a^k + (1 - x) & \text{if } x > a^k + 1/2 \end{cases}$$

We complete the proof of our claim by considering two cases:

Case 1:  $x \leq 1/2$ .

In this case the dynamics are  $a^{k+1} = a^k - x$  whenever  $a^k > 0$ . Hence we have  $a^k \in [-1/2, 0)$  for  $k = 1 + \left\lceil \frac{a^1 - x}{x} \right\rceil < 2 + 1/x \leq 1 + 2/\varepsilon$ .

Case 2:  $x > 1/2$ .

If  $a^1 \in [-1/2, 0]$ , then  $(S_1^1, B_1^1)$  has the desired property. If  $a^1 \in [x - 1/2, 1/2)$ , then  $a^2 = a^1 - x \in [-1/2, 0]$ , so  $(S_1^2, B_1^2)$  has the desired property. Finally, if  $a^1 \in (0, x - 1/2)$ , then  $a^2 = a^1 + 1 - x$ , and  $a^{j+1} = a^j + 1 - x$  for all  $j$  with  $a^j \in (0, x - 1/2)$ . Hence,  $a^{1 + \lceil (x - 1/2 - a^1)/(1 - x) \rceil} \in (x - 1/2, 1/2)$ . Hence  $(S^k, B^k)$  has the desired property for  $k = 2 + \lceil (x - 1/2 - a^1)/(1 - x) \rceil \leq 2 + \lceil 1/2(1 - x) \rceil \leq 3 + 1/2\varepsilon$ .

The statement of the Lemma requires that  $(S_1, B_1)$  have four properties (which hold uniformly given some choice of  $k_1, k_2, k_3$ , and  $k_4$ ).

The first of these was that  $B_1/B \in [\underline{\alpha}, 1 - \underline{\alpha}]$ . It is immediate from our construction that  $B_1 \in [B_1^0 - k\lfloor B/S \rfloor, B_1^0]$  for  $k \leq 3 + 1/\varepsilon$ . The fact that  $B_1^0/B \leq 1 - \underline{\alpha}$  implies  $B_1/B \leq 1 - \underline{\alpha}$ . Given that  $\alpha \geq 1/2$  we have  $B_1/B \geq 1/2 - 1/B - (3 + 1/\varepsilon)(1/S + 1/B)$ . This is greater than  $\underline{\alpha}$  when  $B > k_1 + k_2/\varepsilon$  given an appropriate choice of  $k_1$  and  $k_2$ .

The second of these was that  $|S_1/S - B_1/B| < 1/4B$ . We chose  $(S_1, B_1)$  so that this holds.

The third is that  $|B_1/B - \alpha B| < (k_3 + k_4/\varepsilon)(1/B)$ . That we can choose  $k_3$  and  $k_4$  so that this is true again follows easily from  $B_1 \in [B_1^0 - k\lfloor B/S \rfloor, B_1^0]$  for  $k \leq 3 + 1/\varepsilon$ . The upper bound gives  $B_1/B \leq B_1^0/B \leq \lceil \alpha B/B \rceil < \alpha + (1/B)$ . The lower bound gives  $B_1/B < \alpha - 1/B - (3 + 1/\varepsilon)(\lfloor B/S \rfloor)$ . Using  $\lfloor B/S \rfloor < 2(B/S)$ , this is greater than  $\underline{\alpha} - (k_3 + k_4/\varepsilon)(1/B)$  for  $k_3 = 7$  and  $k_4 = 2$ .

The argument for the fourth condition is similar, but simpler because  $S_1^k = S_1^1 - (k - 1)$ .

This finishes the proof of lemma 2, and thus proves the theorem. ■

Intuitively, what is going on in the lemma is that if we graph the parallelogram with  $B_1/B$  on the x-axis and  $S_1/S$  on the y-axis, the parallelogram covers the section of the line  $y = x$  between  $\underline{\alpha}$  and  $1 - \underline{\alpha}$  and has height (and thus width)  $1/2B$ . The fact that its width is only  $1/2B$  means that when we look for an equilibrium with  $S_1^1$  sellers in market 1 we find that it doesn't exist because  $B_1^1 < S_1^1 - 1/4B$ , but if we added one more buyer we would jump across the parallelogram to a point with  $B_1^1 + 1 > S_1^1 + 1/4B$ . Now, however, think about what happens if we just reduce  $S_1^1$  by one. Because  $B/S$  is not an integer, at this  $y$  value the parallelogram doesn't pass between the grid points at the same place. Instead it is as if it is translated by  $B/S - \lceil B/S \rceil$ . When this quantity is some  $\varepsilon$  close to zero we may have to translate the parallelogram by this amount  $1/\varepsilon$  times before we hit a grid point, but we will certainly hit one by this time.

Our theorem shows that we get a plateau of true equilibria whenever the seller-buyer ratio (and the buyer-seller ratio) is not too close to an integer. The example of the previous section was one where  $S_n/B_n$  was becoming closer and closer to one as  $n$  increased. It is closeness to an integer, not being exactly equal to an integer, that can lead to nonexistence. If the buyer-seller ratio is exactly equal to an integer, then an analogue to Theorem 1 would hold, and in fact we would obtain an even stronger result: there would be an exact equilibrium for every  $S_1$  in the specified range.<sup>5</sup> We should also note that having the buyer-seller ratio close to an integer does not always lead to nonexistence of split-market equilibria. For example, there will be a plateau of equilibria even when  $S_n/B_n \rightarrow 1$  provided that  $S_n + B_n$  is even for all  $n$ , since that permits us to find an integer pair in the parallelogram of quasiequilibria.

## 5 The Density of the Set of Equilibria

The following theorem provides a characterization of the equilibrium set for sequences of finite economies, and shows that there is a wide plateau of exact equilibria whenever the limit ratio  $\gamma$  is irrational. The theorem states that the equilibrium set fills out a plateau of width bounded away from zero, and specifies the density of the set of equilibria. If there are fewer sellers than buyers, then there exists  $\alpha^* < \frac{1}{2}$  depending on  $\gamma$  such that

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<sup>5</sup>This is true if  $S \leq B$ . When  $B \leq S$  there would be an exact equilibrium for every  $B_1$  in the range.

for essentially all integers  $S_1$  in  $(S\alpha^*, S(1 - \alpha^*))$ , there is an inequilibrium with  $S_1$  sellers in market 1. If there are fewer buyers than sellers, then for essentially all integers  $B_1$  in  $(B\alpha^*(\gamma), B(1 - \alpha^*(\gamma)))$ , there is an inequilibrium with  $B_1$  sellers in market 1.

Although the proof uses nonstandard analysis, the statement is completely standard and can be understood without any knowledge of nonstandard analysis.

In the following definition,  $\mathcal{N}(B, S)$  is essentially the set of  $B_1$  such that in the model with  $B$  buyers and  $S$  sellers, there is an equilibrium with  $B_1$  buyers in market 1.<sup>6</sup>

**Definition 1** Let  $\mathcal{N}(B, S)$  denote the set of all  $B_1 \in \{0, 1, \dots, B\}$  such that there exists  $S_1 \in \{0, 1, \dots, S\}$  such that

1. the market with  $B$  buyers and  $S$  sellers has a Nash equilibrium with  $B_1$  buyers and  $S_1$  sellers in market 1; and

2. if we define  $S_2 = S - S_1$  and  $B_2 = B - B_1$ , then

$$\left\{ \frac{S_1}{B_1}, \frac{S_2}{B_2}, \frac{S_1 + 1}{B_1}, \frac{S_2 + 1}{B_2}, \frac{S_1}{B_1 + 1}, \frac{S_2}{B_2 + 1} \right\} \subset \Gamma$$

The following definition specifies, for each  $\gamma$ , a piecewise linear function of  $\alpha$  which is strictly positive on an open interval containing  $\alpha = \frac{1}{2}$ . We shall see that the density of the set  $\mathcal{N}(B, S)$  converges to this piecewise linear function.

**Definition 2** Let

$$\begin{aligned} T(\gamma) &= \frac{G_s(\gamma)}{F'_s(\gamma)} - \frac{G_b(\gamma)}{F'_b(\gamma)} \\ \alpha^*(\gamma) &= \max \left\{ 0, \frac{1}{2} - \frac{\gamma+1}{2|1+\gamma-2T(\gamma)|} \right\} \\ \alpha^{**}(\gamma) &= \max \left\{ 0, \frac{1}{2} - \frac{|\gamma-1|}{2|1+\gamma-2T(\gamma)|} \right\} \\ H(\alpha, \gamma) &= \begin{cases} 0 & \text{if } \alpha \in [0, \alpha^*(\gamma)] \\ \frac{(\alpha - \alpha^*(\gamma)) \min\{1, \gamma\}}{\alpha^{**}(\gamma) - \alpha^*(\gamma)} & \text{if } \alpha \in (\alpha^*(\gamma), \alpha^{**}(\gamma)) \\ \min\{1, \gamma\} & \text{if } \alpha \in [\alpha^{**}(\gamma), 1 - \alpha^{**}(\gamma)] \\ \frac{(1 - \alpha - \alpha^*(\gamma)) \min\{1, \gamma\}}{\alpha^{**}(\gamma) - \alpha^*(\gamma)} & \text{if } \alpha \in (1 - \alpha^{**}(\gamma), 1 - \alpha^*(\gamma)) \\ 0 & \text{if } \alpha \in [1 - \alpha^{**}(\gamma), 1] \end{cases} \end{aligned} \quad (2)$$

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<sup>6</sup> $\mathcal{N}(B, S)$  is actually a slightly smaller set than the one just described, because of condition 2. If  $\frac{S_1}{B_1} \notin \Gamma$ , or  $\frac{S_2}{B_2} \notin \Gamma$ , or a defection by one player could take the ratio of the sellers to buyers in that player's new market outside of  $\Gamma$ , then Assumption 1 gives us no control over payoffs, and hence makes it impossible to prove that a given pair  $(B_1, S_1)$  is not an equilibrium. Because the set we identify is no larger than the set of equilibria, the density of the set of all equilibria, including equilibria with ratios outside  $\Gamma$ , will be at least as great as indicated in Theorem 2.

**Theorem 2** Suppose Assumption 1 holds,  $S_n \rightarrow \infty$ ,  $B_n \rightarrow \infty$ , and  $\frac{S_n}{B_n} \rightarrow \gamma \in (\underline{\gamma}, \bar{\gamma}) \setminus \mathbf{Q}$ . If  $\varepsilon_n \rightarrow 0$ , and  $\varepsilon_n B_n \rightarrow +\infty$ , then

$$\frac{|\{B_{n1} \in \mathbf{N} \cap [B_n(\alpha - \varepsilon_n), \alpha + \varepsilon_n] : B_{n1} \in \mathcal{N}(B_n, S_n)\}|}{2\varepsilon_n B_n} \rightarrow H(\alpha, \gamma)$$

uniformly over  $\alpha \in [0, 1]$ .

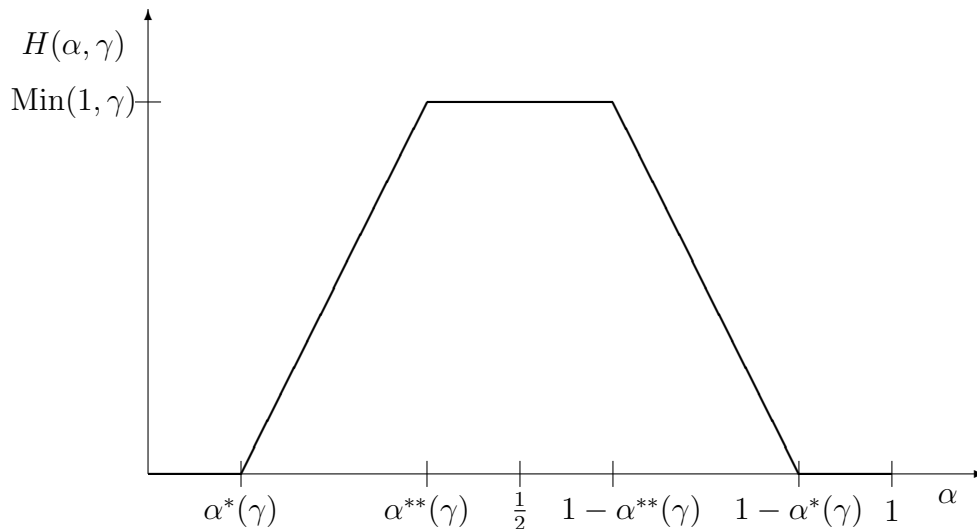


Figure 2: The density-of-equilibria function  $H(\alpha, \gamma)$ .

Figure 2 contains a graph of the function  $H(\alpha, \gamma)$  for the application considered in Figure 1: the EFM competing auction model with  $S/B = \frac{1}{2}$ . In this case, we find  $\alpha^*(\gamma) = 1/8$ ,  $\alpha^{**}(\gamma) = 3/8$ , and  $\text{Min}(1, \gamma) = 1/2$ . Hence, when the number of buyers and sellers is large the equilibrium set includes splits with  $B_1/B$  ranging from about  $1/8$  to  $7/8$ . The equilibrium set reaches a maximum density of  $1/2$  for values of  $\alpha$  between  $3/8$  and  $5/8$ . This means that there are true equilibria for about half of the  $B_1$  in this range. There are only half as many sellers as buyers, so this means that there is an equilibrium for almost all  $S_1$  between  $\frac{3}{8}S$  and  $\frac{5}{8}S$ .

Before proving the theorem we first need some notation and three lemmas. We will suppose throughout that  $B_1, B_2, S_1, S_2 \in \mathbf{N}$ ,  $B_1 + B_2 = B$ ,  $S_1 + S_2 = S$ .

**Lemma 3** Define  $\gamma_1 = \frac{S_1}{B_1}$ ,  $\gamma_2 = \frac{S_2}{B_2}$ ,  $\gamma = \frac{S}{B}$ , and  $\alpha_1 = \frac{B_1}{B}$ . Then,

$$\begin{aligned}\gamma_2 - \gamma_1 &= \frac{\gamma - \gamma_1}{1 - \alpha_1} \\ \gamma_2 - \gamma &= \frac{B_1}{B_2}(\gamma - \gamma_1) \\ \frac{1}{B_2} &= \frac{\alpha_1}{(1 - \alpha_1)B_1} \\ \frac{1}{B_1} - \frac{1}{B_2} &= \frac{1 - 2\alpha_1}{(1 - \alpha_1)B_1}\end{aligned}$$

Proof of Lemma 3

A simple formula for  $\gamma$  is

$$\frac{B_1}{B}\gamma_1 + \frac{B_2}{B}\gamma_2 = \frac{B_1}{B}\left(\frac{S_1}{B_1}\right) + \frac{B_2}{B}\left(\frac{S_2}{B_2}\right) = \frac{S_1 + S_2}{B} = \frac{S}{B} = \gamma.$$

This implies  $B_1\gamma_1 + B_2\gamma_2 = B\gamma = B_1\gamma + B_2\gamma$ , which gives  $B_1(\gamma_1 - \gamma) = -B_2(\gamma_2 - \gamma)$ , and

$$\gamma_2 - \gamma = \frac{B_1}{B_2}(\gamma - \gamma_1)$$

The second identity follows from

$$\begin{aligned}\gamma_2 - \gamma_1 &= \gamma_2 - \gamma + \gamma - \gamma_1 \\ &= \left(1 + \frac{B_1}{B_2}\right)(\gamma - \gamma_1) \\ &= \left(1 + \frac{\alpha_1 B}{(1 - \alpha_1)B}\right)(\gamma - \gamma_1) \\ &= \frac{\gamma - \gamma_1}{1 - \alpha_1}\end{aligned}$$

The third identity comes from

$$\frac{1}{B_2} = \frac{1}{(1 - \alpha_1)B} = \frac{\alpha_1}{(1 - \alpha_1)B_1}.$$

The fourth comes from

$$\frac{1}{B_1} - \frac{1}{B_2} = \frac{1}{B_1} - \frac{\alpha_1}{(1 - \alpha_1)B_1} = \frac{1 - 2\alpha_1}{(1 - \alpha_1)B_1}.$$

■

At this point in the proof, we begin to use nonstandard analysis. Let  ${}^*\mathbf{N}$  be the set of nonstandard integers, so that every  $n \in {}^*\mathbf{N} \setminus \mathbf{N}$  is an infinite nonstandard natural number. Given any finite nonstandard number  $\alpha$ , let  ${}^\circ\alpha$  denote the standard part of  $\alpha$ , the unique standard real number infinitely close to  $\alpha$ .

**Lemma 4** *Suppose  $S, B \in {}^*\mathbf{N} \setminus \mathbf{N}$ ,  ${}^\circ\gamma \in (\underline{\gamma}, \bar{\gamma})$ , and  ${}^\circ\alpha_1 \in (0, 1)$ , Then if  $\gamma_1 \neq \gamma + O\left(\frac{1}{B_1}\right)$ , there is no equilibrium with  $B_1$  buyers and  $S_1$  sellers in market 1.*

Proof of Lemma 4

We may assume without loss of generality that  $\gamma_1 > \gamma_2$ . Consider a seller contemplating switching from market 1 to market 2. The change in the seller's payoff resulting from the switch is

$$\begin{aligned} & F_s\left(\gamma_2 + \frac{1}{B_2}\right) - \frac{G_s\left(\gamma_2 + \frac{1}{B_2}\right)}{B_2} - F_s(\gamma_1) + \frac{G_s(\gamma_1)}{B_1} + o\left(\frac{1}{B_1}\right) \\ & \geq \max\{F'_s(\gamma') : \gamma' \in \Gamma\} \left(\gamma_2 - \gamma_1 + \frac{1}{B_2}\right) + O\left(\frac{1}{B_1}\right) \\ & > 0 \end{aligned}$$

provided  ${}^\circ\gamma_1 \in (\underline{\gamma}, \bar{\gamma})$  and  ${}^\circ\gamma_2 \in (\underline{\gamma}, \bar{\gamma})$ , which shows there is no equilibrium with  $B_1$  buyers and  $S_1$  sellers in market 1. ■

The previous lemma formalizes the observation that the “diamond” of quasi-equilibria becomes flatter as the economy grows. The next step is to characterize which splits of agents between the two markets are compatible with the incentive constraints. EF gave a sufficient condition for incentive compatibility that assumed  $\gamma_1 = \gamma_2 = \gamma$ . The next lemma improves that condition by considering all  $\gamma$ 's that are consistent with the previous lemma.

**Lemma 5** *Suppose  $S, B \in {}^*\mathbf{N} \setminus \mathbf{N}$ ,  ${}^\circ\gamma \in (\underline{\gamma}, \bar{\gamma})$ , and  ${}^\circ\alpha_1 \in (0, 1)$ , Then*

1. *If  $\gamma_1 = \gamma + O\left(\frac{1}{B_1}\right)$ ,*

$${}^\circ(S_1 - \gamma B_1) - \frac{G_s({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F'_s({}^\circ\gamma)} \in ({}^\circ\alpha_1 - 1, {}^\circ\alpha_1) \quad (3)$$

*and*

$${}^\circ(S_1 - \gamma B_1) - \frac{G_b({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F'_b({}^\circ\gamma)} \in (-{}^\circ\alpha_1 {}^\circ\gamma, (1 - {}^\circ\alpha_1) {}^\circ\gamma) \quad (4)$$

*then there is an equilibrium with  $B_1$  buyers and  $S_1$  sellers in market 1.*

2. Suppose

$$\circ \left( \frac{S_1}{B_1} \right) \in (\underline{\gamma}, \bar{\gamma}) \text{ and } \circ \left( \frac{S_2}{B_2} \right) \in (\underline{\gamma}, \bar{\gamma})$$

If

$$\circ(S_1 - \gamma B_1) - \frac{G_s(\circ\gamma)(1 - 2\circ\alpha_1)}{F'_s(\circ\gamma)} \notin [\circ\alpha_1 - 1, \circ\alpha_1]$$

or

$$\circ(S_1 - \gamma B_1) - \frac{G_b(\circ\gamma)(1 - 2\circ\alpha_1)}{F'_b(\circ\gamma)} \notin [-\circ\alpha_1 \circ\gamma, (1 - \circ\alpha_1) \circ\gamma]$$

then there is no equilibrium with  $B_1$  buyers and  $S_1$  sellers in market 1.

### Proof of Lemma 5

We need to consider the four incentive compatibility constraints corresponding to the curves S1, S2, B1, and B2. First, consider a seller contemplating switching from market 1 to market 2. The change in the seller's payoff resulting from the switch is

$$\begin{aligned} & F_s \left( \gamma_2 + \frac{1}{B_2} \right) - \frac{G_s \left( \gamma_2 + \frac{1}{B_2} \right)}{B_2} - F_s(\gamma_1) + \frac{G_s(\gamma_1)}{B_1} + o \left( \frac{1}{B_1} \right) \\ &= F'_s(\gamma_1) \left( \gamma_2 - \gamma_1 + \frac{1}{B_2} \right) + o \left( \gamma_2 - \gamma_1 + \frac{1}{B_2} \right) \\ &+ G_s(\gamma_1) \left( \frac{1}{B_1} - \frac{1}{B_2} \right) + o \left( \frac{1}{B_2} \right) + o \left( \frac{1}{B_1} \right) \\ &= F'_s(\gamma_1) \left( \gamma_2 - \gamma_1 + \frac{1}{B_2} \right) + G_s(\gamma_1) \left( \frac{1}{B_1} - \frac{1}{B_2} \right) + o \left( \frac{1}{B_1} \right) \\ &= F'_s(\gamma_1) \left( \frac{\gamma - \gamma_1}{1 - \alpha_1} + \frac{\alpha_1}{(1 - \alpha_1)B_1} \right) + G_s(\gamma_1) \left( \frac{1 - 2\alpha_1}{(1 - \alpha_1)B_1} \right) + o \left( \frac{1}{B_1} \right) \\ &= F'_s(\gamma) \left( \frac{\gamma - \gamma_1}{1 - \alpha_1} + \frac{\alpha_1}{(1 - \alpha_1)B_1} \right) + G_s(\gamma) \left( \frac{1 - 2\alpha_1}{(1 - \alpha_1)B_1} \right) + o \left( \frac{1}{B_1} \right) \end{aligned}$$

The incentive compatibility constraint requires that this be less than or equal to zero, i.e.

$$\begin{aligned} F'_s(\gamma) \left( \frac{\gamma - \gamma_1}{1 - \alpha_1} + \frac{\alpha_1}{(1 - \alpha_1)B_1} \right) &< -G_s(\gamma) \left( \frac{1 - 2\alpha_1}{(1 - \alpha_1)B_1} \right) + o \left( \frac{1}{B_1} \right) \\ \frac{\gamma - \gamma_1}{1 - \alpha_1} + \frac{\alpha_1}{(1 - \alpha_1)B_1} &> -\frac{G_s(\gamma)(1 - 2\alpha_1)}{F'_s(\gamma)(1 - \alpha_1)B_1} + o \left( \frac{1}{B_1} \right) \\ \gamma - \gamma_1 + \frac{\alpha_1}{B_1} &> -\frac{G_s(\gamma)(1 - 2\alpha_1)}{F'_s(\gamma)B_1} + o \left( \frac{1}{B_1} \right) \\ \gamma_1 - \gamma &< \frac{G_s(\gamma)(1 - 2\alpha_1)}{F'_s(\gamma)B_1} + \frac{\alpha_1}{B_1} + o \left( \frac{1}{B_1} \right) \end{aligned}$$

Now, consider a seller contemplating switching from market 2 to market 1. By symmetry, the incentive compatibility constraint is

$$\begin{aligned}\gamma_2 - \gamma &< \frac{G_s(\gamma)(1 - 2\alpha_2)}{F'_s(\gamma)B_2} + \frac{\alpha_2}{B_2} + o\left(\frac{1}{B_2}\right) \\ \frac{\alpha_1}{1 - \alpha_1}(\gamma - \gamma_1) &< \alpha_1 \frac{G_s(\gamma)(2\alpha_1 - 1)}{(1 - \alpha_1)F'_s(\gamma)B_1} + \frac{\alpha_1}{B_1} + o\left(\frac{1}{B_1}\right) \\ \gamma - \gamma_1 &< \frac{G_s(\gamma)(2\alpha_1 - 1)}{F'_s(\gamma)B_1} + \frac{1 - \alpha_1}{B_1} + o\left(\frac{1}{B_1}\right) \\ \gamma_1 - \gamma &> \frac{G_s(\gamma)(1 - 2\alpha_1)}{F'_s(\gamma)B_1} + \frac{\alpha_1 - 1}{B_1} + o\left(\frac{1}{B_1}\right)\end{aligned}$$

Thus, the incentive compatibility constraint for sellers is

$$\gamma_1 \in \gamma + \frac{G_s(\gamma)(1 - 2\alpha_1)}{F'_s(\gamma)B_1} + \left( \frac{\alpha_1 - 1}{B_1} + o\left(\frac{1}{B_1}\right), \frac{\alpha_1}{B_1} + o\left(\frac{1}{B_1}\right) \right)$$

or equivalently

$$\begin{aligned}\frac{S_1}{B_1} - \gamma - \frac{G_s(\gamma)(1 - 2\alpha_1)}{F'_s(\gamma)B_1} &\in \left( \frac{\alpha_1 - 1}{B_1} + o\left(\frac{1}{B_1}\right), \frac{\alpha_1}{B_1} + o\left(\frac{1}{B_1}\right) \right) \\ S_1 - \gamma B_1 - \frac{G_s(\gamma)(1 - 2\alpha_1)}{F'_s(\gamma)} &\in (\alpha_1 - 1 + o(1), \alpha_1 + o(1))\end{aligned}$$

This is clearly satisfied provided

$${}^\circ(S_1 - \gamma B_1) - \frac{G_s({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F'_s({}^\circ\gamma)} \in ({}^\circ\alpha_1 - 1, {}^\circ\alpha_1)$$

and clearly not satisfied provided

$${}^\circ(S_1 - \gamma B_1) - \frac{G_s({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F'_s({}^\circ\gamma)} \notin [{}^\circ\alpha_1 - 1, {}^\circ\alpha_1]$$

Now, consider a buyer contemplating switching from market 1 to market 2. The change

in the buyer's payoff resulting from the switch is

$$\begin{aligned}
& F_b \left( \frac{B_2}{B_2+1} \gamma_2 \right) - \frac{G_b \left( \frac{B_2}{B_2+1} \gamma_2 \right)}{B_2+1} - F_b(\gamma_1) + \frac{G_b(\gamma_1)}{B_1} + o \left( \frac{1}{B_1} \right) \\
&= F'_b(\gamma_1) \left( \gamma_2 - \gamma_1 - \frac{\gamma_2}{B_2} \right) + o \left( \gamma_2 - \gamma_1 - \frac{\gamma_2}{B_2} \right) \\
&+ G_b(\gamma_1) \left( \frac{1}{B_1} - \frac{1}{B_2} \right) + o \left( \frac{1}{B_2} \right) + o \left( \frac{1}{B_1} \right) \\
&= F'_b(\gamma_1) \left( \gamma_2 - \gamma_1 - \frac{\gamma_2}{B_2} \right) + G_b(\gamma_1) \left( \frac{1}{B_1} - \frac{1}{B_2} \right) + o \left( \frac{1}{B_1} \right) \\
&= F'_b(\gamma_1) \left( \frac{\gamma - \gamma_1}{1 - \alpha_1} - \frac{\alpha_1 \gamma_2}{(1 - \alpha_1) B_1} \right) + G_b(\gamma_1) \left( \frac{1 - 2\alpha_1}{(1 - \alpha_1) B_1} \right) + o \left( \frac{1}{B_1} \right) \\
&= F'_b(\gamma) \left( \frac{\gamma - \gamma_1}{1 - \alpha_1} - \frac{\alpha_1 \gamma_2}{(1 - \alpha_1) B_1} \right) + G_b(\gamma) \left( \frac{1 - 2\alpha_1}{(1 - \alpha_1) B_1} \right) + o \left( \frac{1}{B_1} \right)
\end{aligned}$$

The incentive compatibility constraint requires that this be less than or equal to zero, i.e.

$$\begin{aligned}
F'_b(\gamma) \left( \frac{\gamma - \gamma_1}{1 - \alpha_1} - \frac{\alpha_1 \gamma_2}{(1 - \alpha_1) B_1} \right) &< -G_b(\gamma) \left( \frac{1 - 2\alpha_1}{(1 - \alpha_1) B_1} \right) + o \left( \frac{1}{B_1} \right) \\
\frac{\gamma - \gamma_1}{1 - \alpha_1} - \frac{\alpha_1 \gamma_2}{(1 - \alpha_1) B_1} &< -\frac{G_b(\gamma) (1 - 2\alpha_1)}{F'_b(\gamma) (1 - \alpha_1) B_1} + o \left( \frac{1}{B_1} \right) \\
\gamma - \gamma_1 - \frac{\alpha_1 \gamma_2}{B_1} &< -\frac{G_b(\gamma) (1 - 2\alpha_1)}{F'_b(\gamma) B_1} + o \left( \frac{1}{B_1} \right) \\
\gamma_1 - \gamma &> \frac{G_b(\gamma) (1 - 2\alpha_1)}{F'_b(\gamma) B_1} - \frac{\alpha_1 \gamma}{B_1} + o \left( \frac{1}{B_1} \right)
\end{aligned}$$

By symmetry, the incentive compatibility constraint for a buyer in market 2 is

$$\begin{aligned}
\gamma_2 - \gamma &> \frac{G_b(\gamma) (1 - 2\alpha_2)}{F'_b(\gamma) B_2} - \frac{\alpha_2 \gamma}{B_2} + o \left( \frac{1}{B_2} \right) \\
\left( \frac{\alpha_1}{1 - \alpha_1} \right) (\gamma - \gamma_1) &> \left( \frac{\alpha_1}{1 - \alpha_1} \right) \frac{G_b(\gamma) (2\alpha_1 - 1)}{F'_b(\gamma) B_1} - \frac{\alpha_1 \gamma}{B_1} + o \left( \frac{1}{B_1} \right) \\
\gamma - \gamma_1 &> \frac{G_b(\gamma) (2\alpha_1 - 1)}{F'_b(\gamma) B_1} - \frac{(1 - \alpha_1) \gamma}{B_1} + o \left( \frac{1}{B_1} \right) \\
\gamma_1 - \gamma &< -\frac{G_b(\gamma) (2\alpha_1 - 1)}{F'_b(\gamma) B_1} + \frac{(1 - \alpha_1) \gamma}{B_1} + o \left( \frac{1}{B_1} \right) \\
\gamma_1 - \gamma &< \frac{G_b(\gamma) (1 - 2\alpha_1)}{F'_b(\gamma) B_1} + \frac{(1 - \alpha_1) \gamma}{B_1} + o \left( \frac{1}{B_1} \right)
\end{aligned}$$

so the incentive compatibility constraint for buyers is

$$\gamma_1 \in \gamma + \frac{G_b(\gamma) (1 - 2\alpha_1)}{F'_b(\gamma) B_1} + \left( -\frac{\alpha_1 \gamma}{B_1} + o \left( \frac{1}{B_1} \right), \frac{(1 - \alpha_1) \gamma}{B_1} + o \left( \frac{1}{B_1} \right) \right)$$

As in the case of sellers, this is satisfied provided

$${}^\circ(S_1 - \gamma B_1) - \frac{G_b({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F'_b({}^\circ\gamma)} \in ({}^\circ\alpha_1, (1 - {}^\circ\alpha_1)^\circ\gamma)$$

and not satisfied provided

$${}^\circ(S_1 - \gamma B_1) - \frac{G_b({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F'_b({}^\circ\gamma)} \notin [{}^\circ\alpha_1, (1 - {}^\circ\alpha_1)^\circ\gamma]$$

■

**Remark:** The previous lemma characterizes the allocations of agents to each market that are consistent with the incentive constraints. Since the result uses the approximation of the utility functions provided by Assumption 1, it is interesting to note that it gives exactly the range of solutions that EFM obtained for the exact utility functions that arise with uniformly distributed buyer values in their auction model.

Proof of Theorem 2

Suppose  $\gamma \in (\underline{\gamma}, \bar{\gamma}) \setminus \mathbf{Q}$ . Consider three sequences  $S_n$ ,  $B_n$  and  $\varepsilon_n$  satisfying the hypotheses of the theorem. Fix any  $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ . Let  $S = S_n$ ,  $B = B_n$ ,  $\hat{\gamma} = \frac{S_n}{B_n}$ ,  $K = \lfloor \varepsilon_n B_n \rfloor$  and  $\mathbf{K} = \{-K, -K + 1, \dots, -1, 0, 1, \dots, K - 1, K\}$ . Notice that  ${}^\circ\hat{\gamma} = \gamma$  and  $N \in {}^*\mathbf{N} \setminus \mathbf{N}$ .

For  $\alpha \in {}^*[0, 1]$ , let  $L(\alpha)$  denote the Lebesgue measure of the intersection

$$[(\alpha - 1, \alpha) + T(\hat{\gamma})(1 - 2\alpha)] \cap (-\alpha\hat{\gamma}, (1 - \alpha)\hat{\gamma})$$

Notice that  $L$  is piecewise linear in  $\alpha$ , and that  $L(\frac{1}{2}) = \min\{1, \hat{\gamma}\}$ . The two intervals  $(\alpha - 1, \alpha) + T(\hat{\gamma})(1 - 2\alpha)$  and  $(-\alpha\hat{\gamma}, (1 - \alpha)\hat{\gamma})$  are of length 1 and  $\hat{\gamma}$ , and they are both centered at 0 when  $\alpha_1 = \frac{1}{2}$ . The first interval moves linearly with slope  $1 - 2T(\hat{\gamma})$  in response to changes in  $\alpha$ , while the second interval moves linearly with slope  $-\hat{\gamma}$ , so the movement of the first interval, relative to the second, is linear with slope  $1 + \hat{\gamma} - 2T(\hat{\gamma})$  in response to changes in  $\alpha$ . The two intervals cease to be nested when the magnitude of the relative movement equals  $\frac{|1 - \hat{\gamma}|}{2}$ , i.e. when

$$\alpha = \frac{1}{2} \pm \frac{|1 - \hat{\gamma}|}{2|1 + \hat{\gamma} - 2T(\hat{\gamma})|}$$

The two intervals cease to intersect when the magnitude of the relative movement equals  $\frac{1 + \hat{\gamma}}{2}$ , i.e. when

$$\alpha = \frac{1}{2} \pm \frac{1 + \hat{\gamma}}{2|1 + \hat{\gamma} - 2T(\hat{\gamma})|}$$

Therefore,

$$L(\alpha) = {}^*H(\alpha, \hat{\gamma})$$

for all  $\alpha \in {}^*[0, 1]$ .

Given  $k \in \mathbf{K}$ , let

$$B_k = \lfloor \alpha B \rfloor + k \text{ and } \alpha_k = \frac{B_k}{B}$$

Since the interval  $[\alpha_k - 1, \alpha_k)$  has length 1, there is exactly one  $S_k$  such that

$$S_k - \hat{\gamma}B_k - \frac{G_s(\hat{\gamma})(1 - 2\alpha_k)}{F'_s(\hat{\gamma})} \in [\alpha_k - 1, \alpha_k)$$

and for this  $S_k$ , we have

$$\begin{aligned} & S_k - \hat{\gamma}B_k - \frac{G_s(\hat{\gamma})(1 - 2\alpha_k)}{F'_s(\hat{\gamma})} - \alpha_k + 1 \\ &= -{}^*r \left( \hat{\gamma}B_k - \frac{G_s(\hat{\gamma})(1 - 2\alpha_k)}{F'_s(\hat{\gamma})} - \alpha_k + 1 \right) \\ &\simeq -{}^*r \left( \hat{\gamma}B_k - \frac{G_s(\gamma)(1 - 2\alpha)}{F'_s(\gamma)} - \alpha + 1 \right) \\ & S_k - \hat{\gamma}B_k - \frac{G_b(\hat{\gamma})(1 - 2\alpha_k)}{F'_b(\hat{\gamma})} + \alpha_k \hat{\gamma} \\ &= S_k - \hat{\gamma}B_k - \frac{G_s(\hat{\gamma})(1 - 2\alpha_k)}{F'_s(\hat{\gamma})} - \alpha_k + 1 + T(\hat{\gamma})(1 - 2\alpha_k) + \alpha_k(1 + \hat{\gamma}) - 1 \\ &= -{}^*r \left( \hat{\gamma}B_k - \frac{G_s(\hat{\gamma})(1 - 2\alpha_k)}{F'_s(\hat{\gamma})} - \alpha_k + 1 \right) + T(\hat{\gamma})(1 - 2\alpha_k) + \alpha_k(1 + \hat{\gamma}) - 1 \\ &\simeq -{}^*r \left( \hat{\gamma}B_k - \frac{G_s(\gamma)(1 - 2\alpha)}{F'_s(\gamma)} - \alpha + 1 \right) + T(\gamma)(1 - 2\alpha) + \alpha(1 + \gamma) - 1 \end{aligned}$$

For  $k_0 \in \mathbf{K}$  and  $k \in \mathbf{N}$ ,

$$\hat{\gamma}B_{k_0+k} = \hat{\gamma}B + \hat{\gamma}(k_0 + k) = \hat{\gamma}B_{k_0} + \hat{\gamma}k \simeq \hat{\gamma}B_{k_0} + \gamma k$$

so by the Spillover Principle, we may pick  $K_1 \in {}^*\mathbf{N} \setminus \mathbf{N}$ ,  $\frac{K_1}{K} \simeq 0$ , such that

$$\hat{\gamma}B_{k_0+k_1} \simeq \hat{\gamma}B_{k_0} + \gamma k_1$$

for  $k \in \mathbf{K}_1 = \{-K_1, -K_1 + 1, \dots, -1, 0, 1, \dots, K_1 - 1, K_1\}$  and  $k_0 \in \mathbf{K}$ .

Since  $\gamma$  is irrational, translation (modulo one) by  $\gamma$  is an ergodic transformation of  $[0, 1]$ , so for any interval  $(a, b) \subset [0, 1)$  and any  $k_0 \in \mathbf{K}$ ,

$$\frac{|\{k \in \mathbf{K}_1 : {}^\circ * r(\hat{\gamma} B_{k_0+k}) \in (a, b)\}|}{2K_1 + 1} \simeq b - a$$

hence

$$\begin{aligned} & \frac{|\{k \in k_0 + \mathbf{K}_1 : B_k \in \mathcal{N}(B, S)\}|}{2K_1 + 1} \\ & \simeq \frac{\left| \left\{ k \in k_0 + \mathbf{K}_1 : {}^\circ \left( S_k - \hat{\gamma} B_k - \frac{G_b(\gamma)(1-2\alpha)}{F'_b(\gamma)} + \alpha\gamma \right) \in (0, \gamma) \right\} \right|}{2K_1 + 1} \\ & \simeq L(\alpha) = {}^* H(\alpha, \hat{\gamma}) \simeq H(\alpha, \gamma) \end{aligned}$$

We may write  $\mathbf{K}$  as a union of sets of the form  $k_0 + \mathbf{K}_1$ , so

$$\frac{|\{k \in \mathbf{K} : B_k \in \mathcal{N}(B, S)\}|}{2K_1 + 1} \simeq H(\alpha, \gamma)$$

The result follows immediately. ■

## 6 Conclusion

In much of the economics literature it is customary to model a large population using a continuum of agents. Ellison and Fudenberg [3] argued that this is not appropriate in a class of models of two-sided location choice. They noted that a broad range of outcomes can be consistent with the no-deviation constraints in some situations that might be regarded as having only “fully tipped” outcomes when analyzed with a continuum-of-agents approximation. [3] studied only the incentive constraints, and ignored the constraint that the number of agents of each type in each market must be an integer. In this paper, we have extended their analysis to fully take discreteness into account. We find that in the generic case the equilibrium set fills out an interval strictly containing the “quasi-equilibrium” set described in EF, but that there are non-generic cases where the equilibrium set is much smaller. In addition to extending the earlier result, this paper bolsters the message that taking discreteness into account can be important even in very large economies. One suggestion for future research would be to examine the implications of the discreteness of agents in other classes of models. Another would be to examine

equilibrium selection in models of the type discussed here. We know that these models have a dense plateau of equilibria. It would be interesting to know more about whether good evolutionary arguments can be made to regard some of the equilibria as more likely than others.

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