Incomplete Markets, Growth, and the Business Cycle

By

George-Marios Angeletos and Laurent E. Calvet

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George-Marios Angeletos and Laurent E. Calvet*
Harvard University and MIT, and Harvard University

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We introduce a Ramsey growth model with incomplete markets, decentralized production, and idiosyncratic technological risk. The combination of uninsurable shocks with the precautionary motive can slow down capital accumulation or give rise to persistent fluctuations even when agents are very patient and technology is strictly convex. The model generates closed-form expressions for the equilibrium dynamics under a finite or infinite horizon. Multiple steady states and poverty traps can arise from the endogeneity of the interest rate instead of the usual wealth effect. Depending on the economy's parameters, the local dynamics around a steady state are locally unique, totally unstable or locally undetermined, and the equilibrium path can be attracted to a limit cycle. In calibrated examples, financial incompleteness substantially slows down convergence to the steady state and thus increases the persistence of aggregate shocks.

Keywords: Idiosyncratic Risk, Precautionary Motive, Endogenous Fluctuations.

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1. Introduction

How does incomplete risk-sharing affect the level and volatility of macroeconomic activity? We investigate this question by introducing a Ramsey growth model with incomplete markets, decentralized production, and idiosyncratic technological risk. The combination of uninsurable shocks with the precautionary motive can slow down capital accumulation or give rise to persistent fluctuations, even when agents are very patient, technology is strictly convex, and the wealth distribution has no effect on endogenous aggregates.

We consider a GEI[1] growth economy with heterogeneous agents, who are both consumers and producers. Agents can invest in a private neoclassical technology with diminishing returns to scale. In contrast with the traditional Ramsey model, agents are exposed to *idiosyncratic* shocks in their productive investment, and possibly in some exogenous income. Agents can also trade in financial markets. They can borrow or lend a risk-free bond, and *partially* hedge their idiosyncratic risks by exchanging a finite number of risky assets. All securities are real and there are no constraints on short sales. A representative agent generally does not exist in our economy, but explicit aggregation is possible in the CARA-normal case. When markets are complete, the model reduces to the Ramsey growth model with identical agents, as in Cass (1965), Koopmans (1965), and Brock and Mirman (1972). On the other hand, with missing markets, inefficiencies arise because individual producers must bear idiosyncratic risks that cancel out at the aggregate level. This can lead to suboptimal investment, poverty traps, and endogenous fluctuations.

Under a finite horizon, the shocks received at the terminal date cannot be smoothed through time by borrowing and lending. Agents thus have a strong precautionary motive and accumulate large aggregate wealth in later periods. The anticipation of these movements generate fluctuations along the entire equilibrium path in some economies.

Rich dynamics also exist under an infinite horizon. For instance, multiple steady states can arise from the endogeneity of the interest rate. In a rich equilibrium, a low interest rate supports a high investment in the risky technology and a low level of marginal productivity. Conversely in a poor steady state or poverty trap, a high interest rate motivates agents to choose small amounts of the risky investment. Because the interest is endogenous, agents may thus prefer

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[1] GEI is the standard acronym for General Equilibrium with Incomplete markets.

to remain poor in equilibrium rather than bear the large investment risks that are required to increase their standards of living. Poverty traps thus originate from the endogeneity of the interest rate in the capital markets, a source of multiplicity that, to the best of our knowledge, is new to the literature. In contrast, earlier work has obtained poverty traps from wealth effects, building on the idea that poorer agents take less risks and can thus be trapped at low wealth levels (e.g., Banerjee and Newman, 1991). Our model assumes CARA preferences and therefore eliminates the influence of wealth on risky investment. The paper thus highlights that, in addition to the wealth effect, there is another channel through which financial incompleteness may affect the development of an economy: the impact of the interest-rate on risk taking.

The introduction of a new asset has an ambiguous effect on aggregate wealth in a steady state. On one hand, the new asset reduces technological risk, thus encouraging investment. On the other hand, financial innovation decreases the precautionary demand for savings, thereby increasing the interest rate and discouraging productive investment. In calibrated examples, the level of aggregate wealth is found to be the highest in economies with intermediate or high levels of financial sophistication.[2] The effect of financial innovation on the interest rate is similarly nonmonotonic in some economies.

The local dynamics around the steady state have a rich structure under incomplete markets. Depending on the economy's parameters, a steady state is locally unique, totally unstable or locally undetermined, and the equilibrium path can be attracted to a limit cycle. The complicated dynamics arise even when agents are very patient, suggesting that financial incompleteness is a useful substitute to technological nonconvexities in models of endogenous business cycles (Boldrin and Montrucchio, 1986; Benhabib and Farmer, 1994; Mitra, 1996).

Calibrated examples show that financial incompleteness has a substantial impact on the speed of convergence to the steady state. In the complete market framework, the transitional dynamics are determined by the agents' willingness

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[2] The literature has often suggested that financial innovation reduces the precautionary motive and may therefore be detrimental to growth (Mauro, 1995; Devereux and Smith, 1994). This type of argument ignores the adverse effect of production risk on the accumulation of physical or human capital. The present paper analyzes the trade-off between the two effects in a Ramsey framework with diminishing returns to capital accumulation, while a companion article (Angeletos and Calvet, 2000) and other pertinent work (e.g., Greenwood and Jovanovic, 1990; Obstfeld, 1994; Devereux and Smith, 1994) address these issues in the context of endogenous growth.

to smooth consumption across time. Financial incompleteness can slow down convergence through another effect: the sensitivity of interest rates to the precautionary motive. In an initially poor economy, agents make small risky investments and thus have a weaker precautionary motive than in the long run. The low demand for savings puts an upward pressure on the interest rate, which accentuates the low level of capital investment and slows down convergence. Conversely in a rich economy, high investment in the risky technology implies a strong precautionary motive and a low interest rate, thus intensifying the high level of capital. In both cases, prudence increases the persistence of aggregate shocks and has an amplification effect on productive investment.

This research emphasizes that technological and endowment risks have very distinct effects on aggregate economic activity. In general equilibrium, individual income risks are not entirely exogenous to the economy, but reflect the uncertainty associated with endogenous production and investment activities. This point was not considered by Krusell and Smith (1998) in their influential work on the role of incomplete markets in the neoclassical growth model. In contrast to their results, we show the presence of undiversifiable technological risks can have profound effects on aggregate investment, wealth, and volatility.

Our model has interesting policy implications. Completing financial markets may not only help improve welfare and reduce social inequalities through better risk-sharing; it may also stimulate investment and dampen the large financial volatility reported by Shiller (1981, 1989) and others. From an empirical perspective, financial incompleteness also has an *amplification* effect on the *propagation* of aggregate shocks. Small and serially independent shocks on the fundamentals of the economy may generate large and persistent fluctuations in the aggregates.[3]

Section 2 introduces the GEI Ramsey economy. We solve the individual decision problem and calculate the equilibrium path under a finite horizon in Section 3. In Section 4 we introduce the infinite-horizon economy and examine the comparative statics and local dynamics of the steady state. In Section 5 we calibrate the model around the steady state and analyze the persistence of aggregate shocks. We conclude in Section 6. Unless stated otherwise, all proofs are given in the

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[3] The calibration of standard real business cycle (RBC) models requires large and persistent exogenous aggregate shocks to match the observed data (e.g., Kydland and Prescott, 1982; Prescott, 1986). In light of our findings (especially Section 5), introducing financial incompleteness in the standard RBC framework may help reduce the magnitude and persistence of the exogenous shocks required to match the same data.

Appendix.

Literature Review

The paper brings together several branches of the literature. First, it adds financial incompleteness to the Ramsey growth model developed by Cass (1965), Koopmans (1965) and Brock and Mirman (1972). When the technology has decreasing returns to scale, the one-sector model is known to exhibit a unique saddle path in which future wealth is an increasing function of current wealth. This precludes nonmonotonicities or endogenous fluctuations along the equilibrium path. For this reason, researchers have considered the optimal growth problem with nonconvexities in production, and shown that complicated dynamics or endogenous cycles can arise (Boldrin and Montrucchio, 1986; Deneckere and Pelikan, 1986; Sorger, 1992). These fluctuations, however, only appear when agents are very impatient.[4] For instance, Mitra (1996) proves that a period-three cycle only exists if the psychological discount factor β is less than the constant $[(\sqrt{5}-1)/2]^2 \approx 0.38$. By contrast, our GEI growth economy generates deterministic fluctuations with a Cobb-Douglas technology for large values of the discount factor, such as $\beta = 0.99$.

The macro fluctuations exhibited in this paper are also reminiscent of the perfect foresight equilibria observed in deterministic economies with overlapping generations (Benhabib and Day, 1982; Grandmont, 1985) or credit market imperfections (Galor and Zeira, 1993; Kiyotaki and Moore, 1997; Aghion, Bacchetta and Banerjee, 1998; Aghion, Banerjee and Piketty, 1999). We contribute to this literature by showing that market incompleteness is an alternative source of fluctuations in aggregate output. This paper thus extends the results obtained in the incomplete-markets exchange economy of Calvet (1997). While in that model aggregate wealth is fixed by construction, the present paper shows that the combination of precautionary savings and financial incompleteness can lead to endogenous cycles in the real sector of the economy.

There is an extensive literature on decision theory under incomplete markets and an infinite horizon, which originates in the contributions of Bewley (1977), Yaari (1976), Schechtman (1976), and Schechtman and Escudero (1977). Under very general conditions, Chamberlain and Wilson (2000) show that when the exogenous interest rate is at least as large as the rate of time preference and the individual faces idiosyncratic income risks, her assets will eventually diverge to

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[4] Note that it is possible to observe fluctuations in a multi-sector growth model for more patient agents (Benhabib and Nishimura, 1979).

infinity due to the precautionary motive. In the limit, the individual accumulates an infinite buffer stock and can perfectly insulate her consumption from income shocks. These results, however, cannot be true in general equilibrium. In our model, like in Aiyagari (1994) and Krusell and Smith (1998), the interest rate is endogenous and converges to a level sufficiently low to support a finite accumulation of assets in the steady state. As a result, agents cannot fully self-insure along the equilibrium path.

The recent numerical literature has explored the robustness of the Ramsey model with respect to agent heterogeneity and idiosyncratic uncertainty.[5] Aiyagari (1994) considers the effect of exogenous endowment shocks on the steady state in the absence of aggregate and idiosyncratic technological risk.[6] Krusell and Smith (1998) extend Aiyagari's framework to the case of aggregate productivity shocks, but retain the assumption of no undiversifiable *idiosyncratic* technological risks. In their model, the behavior of the macroeconomic aggregates is almost perfectly described by the first moment of the wealth distribution, and the transitional dynamics are strikingly similar to those of the single-agent Ramsey economy.

Our model differs from these approaches in several respects. First, in earlier work, the aggregate capital stock and possibly a risk-free bond are the only available assets and cannot be sold short. By contrast, we assume that agents can trade a riskless asset and an arbitrary number of risky securities. There are no short-sales constraints, which allows us to disentangle the impact of borrowing constraints from missing assets or insurance markets. Second, and more importantly, we extend earlier papers by considering the effect of idiosyncratic *technological* risk. The introduction of idiosyncratic rather than aggregate investment risk generates critical market inefficiencies. The distinction between production and endowment risk drives the comparative statics of the steady state, helps us characterize the types of financial innovation that encourage or dampen economic growth, and generates novel implications for the propagation of aggregate shocks. Third, our model generates closed-form expressions for the aggregate dynamics.

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[5] See Ljungqvist and Sargent (2000) for an excellent discussion of these developments.

[6] In related work, Scheinkman and Weiss (1986) consider an incomplete market economy with endogenous labor supply but no capital accumulation. Because agents have a disutility for labor, the poor are more willing to work than the rich, and aggregate production is higher when the more productive agents have little wealth. Fluctuations in our model have a profoundly different origin, and arise *even though macro variables are always independent of the distribution of wealth.*

We can therefore analyze equilibrium multiplicity and the local dynamics around a steady state. Unlike earlier models, we can also calculate the growth paths when the economy has a finite horizon. Finally, our model generates rich dynamics in financial prices, capital investment and aggregate output.

2. A Ramsey Model with Incomplete Markets

This section introduces the one-sector growth model with incomplete markets. We successively specify technologies, preferences, asset markets and the equilibrium concept.

2.1. Agents and Idiosyncratic Risks

We consider a dynamic economy in discrete time with a finite or an infinite horizon T. The economy is stochastic and all random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Individuals $h = 1, ..., H$ are born at date $t = 0$, and live and consume a single consumption good in dates $t = 0, .., T$.

Each individual is an entrepreneur who owns his own stock of capital and operates his own production scheme.[7] The technology is standard neoclassical, convex, and requires neither adjustment costs nor indivisibilities in investment. These assumptions would lead to the standard neoclassical growth model, if it were not for the following: production is subject to (partially undiversifiable) *idiosyncratic* uncertainty. An investment of k_t^h units of capital at date t yields $B_{t+1}^h F(k_t^h) + (1-\delta)k_t^h$ units of the consumption good at date $t+1$. The constant $\delta \in [0,1]$ denotes the depreciation rate of capital. The production function F is increasing, strictly concave, and satisfies the Inada conditions.[8] The total factor productivity B_{t+1}^h is a *multiplicative* random shock specific to individual h, called *technological* or *investment risk*. It is the key ingredient of the model.

We also find it useful to introduce two other sources of income. First, producers have access to a storage technology with gross rate of return $\rho \in [0,1]$. An investment of s_t^h units of the good in the storage technology yields ρs_t^h with

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[7] The difficulties associated to multiple ownership under incomplete markets are well known (Grossman and Hart, 1979).

[8] We neglect the labor-leisure choice, and implicitly assume a concave technology $\widehat{F}(K, L)$ with constant returns to scale. We normalize labor to $L = 1$, denote by $k = K/L$ the capital intensity, and consider the reduced production function $F(k) \equiv \widehat{F}(k, 1)$. The function F satisfies the conditions $F' > 0$, $F'' < 0$, $F(0) = 0$, $F'(0) = +\infty$, $F(+\infty) = +\infty$, and $F'(+\infty) = 0$.

certainty at date $t+1$. Second, entrepreneurs receive a stochastic endowment stream $\{e_t^h\}_{t=0}^T$. The *additive* shock e_t^h models risks that are outside the control of individuals and do not affect production capabilities. The additive endowment risk e_{t+1}^h and the multiplicative shock B_{t+1}^h are both *idiosyncratic*, but only the latter affects the return on investment.

The idiosyncratic shocks B_{t+1}^h allow us to capture the impact on growth of a wide range of technological risks. The uncertainty of an entrepreneurial project obviously influences specific investment in capital or R&D. Similarly, the riskiness of a worker's human capital, or of the matching between his skills and the firm, affects important decisions such as the supply of labor,[9] education, learning by doing, job search, career choices, etc. In this sense, the model helps analyze how a large class of idiosyncratic risks influences the accumulation of physical, human or intangible capital. Our approach thus contrasts with the usual simplifying hypothesis that income shocks have a purely exogenous origin (Aiyagari, 1994; Krusell and Smith, 1998). While the exogeneity assumption may be reasonable in a life-cycle framework, general equilibrium modelling should recognize that labor income is substantially determined by idiosyncratic production shocks. Our analysis will show that the distinction between endowment risk and investment risk is critical, and should prompt the reassessment of some arguments appearing in the literature.

2.2. CARA-Normal Specification

The model is tractable in the CARA-normal case. The idiosyncratic shocks B_{t+1}^h and e_{t+1}^h are $\mathcal{F}_{t+1}$–measurable, unknown at t and revealed at $t+1$. They are assumed to be jointly normal. We complement this distributional assumption by the specification of preferences. Consider a fixed $A>0$, and $u(c) \equiv -\exp(-Ac)/A$. From a consumption stream $\{c_t^h\}_{t=0}^T$, agent h derives the utility

$$U_0^h = \sum_{t=0}^{T} \beta^t u(c_t^h) = -\frac{1}{A} \sum_{t=0}^{T} \beta^t \exp(-Ac_t^h).$$

Since individual consumption is stochastic, at each period t agents maximize their expected utility conditional on available information.

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[9] See Marcet, Obiols-Homs and Weil (2000) for a recent discussion of the labor supply in the neoclassical growth context.

The CARA-normal specification implies that aggregate demand and equilibrium macro variables are independent from the distribution of wealth. This allows us to achieve explicit aggregation and derive new channels through which financial incompleteness affects economic activity. Our approach thus complements the vast literature on CRRA preferences, where the distribution of wealth, an infinite-dimensional object, enters the state space of the economy. Wealth effects then typically prevent the existence of analytical solutions and approximate numerical simulations are the only way to proceed. The CARA-normal specification permits us to overcome this analytical obstacle, but does not seem to invalidate our result that incomplete markets can generate complicated dynamics. With CRRA preferences or borrowing constraints, the wealth distribution would be an additional channel through which idiosyncratic uncertainty affects aggregate behavior, as has been emphasized in the literature (Galor and Zeira, 1993; Aiyagari, 1994). Perhaps surprisingly, we show that rich dynamics can arise *even* in the absence of such effects.

2.3. The Asset Structure

Individual risks can be partially hedged by trading a limited set of short-lived securities $i = 0, .., N$. Each asset i is worth $\pi_{i,t}$ units of the good at date t, and yields a random amount of consumption $d_{i,t+1}$ at date $t+1$. Security $i = 0$ is riskless, in the sense that $d_{0,t+1} \equiv 1$ with certainty in date $t+1$. The quantity $R_t \equiv 1/\pi_{0,t}$ then denotes the gross interest rate between t and $t+1$, and $r_t = R_t - 1$ is the corresponding net rate. Assets are in zero net supply,[10] there are no short sales constraints, and default is not allowed.[11] It is convenient to stack asset prices and payoffs in the vectors $\pi_t = (\pi_{i,t})_{i=0}^N$ and $d_{t+1} = (d_{i,t+1})_{i=0}^N$. Without loss of generality, we assume that $(d_{i,t+1})_{i=0}^N$ is an orthonormal family of $L^2(\Omega)$, implying that risky assets have zero expected payoffs and are mutually uncorrelated. At the outset of every period t, investors are informed of the realization of the asset payoffs d_t and idiosyncratic shocks $\{(B_t^h, e_t^h)\}_{h=1}^H$.[12] Information is thus symmetric

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[10] A positive asset supply could be introduced by considering a government issuing public debt, or by opening the economy to international asset markets. We do not expect these alternatives to alter the significance of our results.

[11] Default or short-sales constraints would introduce kinks and nonconvexities that would presumably strengthen the failure of the complete-markets framework.

[12] The results of this paper would not be modifed under the weaker assumptions that income shocks are privately observed and that the structure of the economy is common knowledge. We

across agents and generates a filtration $\{\mathcal{F}_t\}_{t=0}^T$. Conditional on the information available to her, each investor h selects a portfolio $\theta_t^h = (\theta_{i,t}^h)_{i=0}^N$ at date t.

We assume by construction that all the assets traded in one period are *short-lived*, in the sense that they only deliver payoffs in the next period. In the next sections, we will show that there is no risk premium in equilibrium, and that the interest rate sequence $\{R_t\}_{0 \leq t < T}$ is deterministic and known in advance to all agents. For this reason, equilibrium allocations and prices do not change if we introduce a *long-lived* security delivering one unit of the good every period. Such a security, called a *perpetuity,* is worth $\pi_L(t) = \sum_{s=t}^{T-1} 1/(R_t ... R_s)$ at date t after delivery of the period's coupon.[13]

Asset returns and the idiosyncratic risks of all agents are assumed to be jointly normal. For simplicity, we also consider that the vectors $\{d_{t+1}, (B_{t+1}^h, e_{t+1}^h)_{1 \leq h \leq H}\}$ are i.i.d. across time, and that the idiosyncratic shocks (B_{t+1}^h, e_{t+1}^h) are identically distributed across agents.[14] For all h, t, we can project the idiosyncratic risks B_{t+1}^h and e_{t+1}^h on the asset span available at date t:

$$B_{t+1}^h = B + \sum_{i=1}^N \kappa_i^h d_{i,t+1} + \eta_{t+1}^h, \tag{2.1}$$

$$e_{t+1}^h = e + \sum_{i=1}^N \xi_i^h d_{i,t+1} + \varepsilon_{t+1}^h, \tag{2.2}$$

where $B \equiv \mathsf{E} B_{t+1}^h$, $e \equiv \mathsf{E} e_{t+1}^h$, $\kappa_i^h \equiv Cov(B_{i,t}^h, d_{i,t})$, and $\xi_i^h \equiv Cov(e_{i,t}^h, d_{i,t})$. The residuals η_{t+1}^h and ε_{t+1}^h represent the undiversifiable component of the investment and endowment shocks to individual h. They are assumed to be identically dis-

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use a stronger assumption in the text to emphasize that the endogenous fluctuations observed in equilibrium do not originate from information assymmetries.

[13] More generally, traders can dynamically replicate a large class of long-lived risky assets.

[14] This assumption simplifies notation and permits us to focus on the leading case of a fully stationary economy. As in the deterministic Ramsey model (see for instance Barro and Sala-i-Martin, 1995), decision-theoretic and equilibrium calculations easily extend to economies with an exogenous growth rate $g \geq 0$. Given a stationary random process (B_t^h, e_t^h, d_t), consider production returns, endowments, and asset returns of the form $\widehat{B}_t^h = (1+g)^t B_t^h$, $\widehat{e}_t^h = (1+g)^t e_t^h$, and $\widehat{d}_t = (1+g)^t d_t$. We also assume that the CARA coefficient decays at rate g; the utility in period t is thus $u_t(c) = -\exp(-\widehat{A}_t c)/\widehat{A}_t$, where $\widehat{A}_t = A(1+g)^{-t}$. Under these assumptions, the equilibrium equations derived in Sections 3 and 4 directly apply to the detrended values of consumption and investment.

tributed across investors, and the corresponding variances

$$\sigma_P^2 \equiv Var(\eta_{t+1}^h | \mathcal{F}_t) = Var(\eta_{t+1}^h),$$
$$\sigma_E^2 \equiv Var(\varepsilon_{t+1}^h | \mathcal{F}_t) = Var(\varepsilon_{t+1}^h),$$

are useful measures of financial incompleteness. The residual shocks are thus identically distributed $\binom{\eta_t^h}{\varepsilon_t^h} \sim \mathcal{N}\left[0, \begin{pmatrix} \sigma_P^2 & 0 \\ 0 & \sigma_E^2 \end{pmatrix}\right]$, independent across time, and satisfy the cross-sectional restrictions $\sum_h \eta_t^h = \sum_h \varepsilon_t^h = 0$ for all t. Because there is no risk premium in equilibrium, risky assets will play only one role in the model − the definition of the uninsurable risks η_t^h and ε_t^h. Economies without risky assets are therefore important special cases of our model, which naturally arise when states, and thus individual shocks, are not publicly observed.[15]

2.4. GEI Equilibrium

For any quotation $\{\pi_t\}_{t=0}^T$ of asset prices, each agent chooses a contingent plan $\{c_t^h, k_t^h, s_t^h, \theta_t^h, w_t^h\}_{t=0}^T$ adapted to the filtration $\{\mathcal{F}_t\}_{t=0}^T$. Under an infinite horizon $(T = \infty)$, the possibility of Ponzi games must be addressed. In our model, it is sufficient to impose

Assumption 1. *When the time horizon is infinite, an admissible consumption plan satisfies* $\beta^t \mathsf{E}_0 \exp(-A w_t^h) \to 0$ *as* $t \to \infty$.

An *admissible* plan also satisfies in any date-event the budget constraints

$$\begin{aligned} & c_t^h + k_t^h + s_t^h + \pi_t \cdot \theta_t^h = w_t^h, \\ & w_{t+1}^h = e_{t+1}^h + B_{t+1}^h F(k_t^h) + (1-\delta) k_t^h + \rho s_t^h + d_{t+1} \cdot \theta_t^h, \end{aligned} \tag{2.3}$$

with the convention that $s_T^h = 0$ and $\theta_T^h = 0$ when T is finite. The variable w_t^h, called *wealth*, thus represents the trader's net credit or debt position at date t and is independent of the income expected in later periods.

Definition 1. *A* GEI equilibrium *consists of a price sequence* $\{\pi_t\}_{t=0}^T$ *and a collection of admissible plans* $(\{c_t^h, k_t^h, s_t^h, \theta_t^h, w_t^h\}_{t=0}^T)_{1 \le h \le H}$ *such that:*

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[15] By construction, a trader does not extract relevant information from observing the shocks of other agents. When only the riskless asset is traded, the same equilibrium path thus arises whether endowments are privately or publicly observed.

(i) Each agent's plan is optimal.
(ii) Asset markets clear in every date-event: $\sum_{h=1}^{H} \theta_t^h = 0$.

In the next sections, we show that the equilibrium calculation is straightforward within this framework.

3. Equilibrium under a Finite Horizon

We derive in this section the equilibrium equations of our Ramsey economy when the time horizon is finite.

3.1. Decision Theory

Consider a fixed entrepreneur h facing an exogenous, deterministic path of asset prices $\{\pi_t\}_{t=0}^{T}$. We denote $J(w,t)$ as the indirect utility of wealth along the exogenous price path, and $J_w(w,t)$ as the marginal utility. At date t, the investor maximizes $u(c_t^h) + \beta \mathsf{E}_t J(w_{t+1}^h, t+1)$ subject to the budget constraints (2.3). This problem has a closed form solution under

Recursive Condition. *The marginal utility of wealth satisfies*

$$J_w(w, t+1) = \exp[-A(a_{t+1} w + b_{t+1})]$$

in period $t+1$*, where* $a_{t+1} > 0$ *and* b_{t+1} *are two constant numbers.*

There exists an arbitrage opportunity if $R_t < \rho$: the agent wants to borrow an infinite amount at rate R_t and invest it in the storage technology. The existence of an optimal decision thus requires that $R_t \geq \rho$. The investor does not use the storage technology if $R_t > \rho$, and is indifferent between the bond and storage when $R_t = \rho$. In the latter case, only the sum $\theta_{0,t}^h + \rho s_t^h$ is determined by the decision problem.

Let $\Phi(k) \equiv BF(k) + (1-\delta)k + e$ denote the expected revenue next period when investing k units of capital. Because investors are facing idiosyncratic uncertainty, we also consider the "risk-adjusted" production function

$$G(k,a) \equiv BF(k) + (1-\delta)k + e - \frac{Aa}{2}[F(k)^2\sigma_P^2 + \sigma_E^2],$$

whose derivative with respect to capital satisfies

$$G_k(k,a) \equiv F'(k)[B - AaF(k)\sigma_P^2] + 1 - \delta.$$

We can then prove

Theorem 1. *When* $B+R_t\sum_{i=1}^N \pi_{i,t}\kappa_i^h > 0$, *the optimal capital stock* k_t^h *is strictly positive and satisfies*

$$G_k(k_t^h, a_{t+1}) + R_t F'(k_t^h)\sum_{i=1}^N \pi_{i,t}\kappa_i^h = R_t. \tag{3.1}$$

The agent does not use the storage technology if $R_t > \rho$, *and chooses a portfolio of the bond and the storage technology if* $R_t = \rho$. *The individual consumption-portfolio decision is determined by*

$$\begin{aligned} c_t^h &= a_t w_t^h + b_t^h, \\ \theta_{i,t}^h &= -\xi_i^h - \kappa_i^h F(k_t^h) - R_t\pi_{i,t}/(Aa_{t+1}),\ \forall i = 1, ..., N, \\ \theta_{0,t}^h + \rho s_t^h &= (a_t/a_{t+1})w_t^h - R_t b_t - R_t k_t^h - R_t \textstyle\sum_{i=1}^N \pi_{i,t}\theta_{i,t}^h, \end{aligned} \tag{3.2}$$

where the quantities a_t *and* b_t^h *are defined by*

$$\begin{aligned} a_t &= \frac{1}{1+(a_{t+1}R_t)^{-1}}, \\ b_t^h &= (a_t/a_{t+1}R_t)[b_{t+1}^h + a_{t+1}G(k_t^h, a_{t+1}) - \ln(\beta R_t)/A - a_{t+1}R_t k_t^h] + \\ &\quad + a_t \textstyle\sum_{i=1}^N \pi_{i,t}\left[\xi_i^h + \kappa_i^h F(k_t^h) + R_t\pi_{i,t}/(2Aa_{t+1})\right], \end{aligned} \tag{3.3}$$

The utility of wealth at t satisfies $J_w(w,t) = \exp[-A(a_{t+1}w + b_{t+1})]$ *and is therefore of the CARA-type.*

Because the agent has a precautionary motive, the function G, the intercept b_t^h and the consumption level c_t^h decrease with the variances σ_E^2 and σ_P^2 of the uninsurable shocks.

In the absence of a risk premium, the price of risky assets is zero: $\pi_{i,t} = 0$ $(i = 1, .., N)$, as will be the case in equilibrium.[16] The optimal capital stock then satisfies the reduced condition

$$G_k(k_t^h, a_{t+1}) = R_t. \tag{3.4}$$

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[16] The equilibrium risk premium would not be zero if there were aggregate uncertainty, in which case equilibrium consumption would be correlated with asset returns.

Although the production function $F(k)$ is strictly concave, the risk-adjusted function $G(k,a) = \Phi(k) - Aa[F(k)^2\sigma_P^2 + \sigma_E^2]/2$ need not be concave when $\sigma_P > 0$. The function $G_k(k,a)$ can thus be non-monotonic in k, and the optimality condition (3.4) can have multiple solutions. The optimal capital stock is easily selected among the solution set under

Assumption 2. *The technology satisfies at least one of the following conditions:*
(i) $\rho \geq 1 - \delta$
(ii) $F(k) = k^\alpha$, $0 < \alpha < 1$
(iii) The function $[F(k)]^2$ *is convex.*

We can then prove

Proposition 1. *Under Assumption 2, the optimal capital stock is the smallest solution to equation (3.4):* $k_t^h = \min\{k : G_k(k, a_{t+1}) = R_t\}$.[17]

We now turn to the equilibrium calculation.

3.2. Equilibrium Analysis

Let C_t, W_t, K_t, and S_t denote the population average of consumption, wealth, capital and storage investment in a given date t. The initial wealth $\sum_{h=1}^H w_0^h/H$ is an exogenous parameter of the economy, which is conveniently denoted by $\overline{W}_0$. For simplicity, we focus on *deterministic* equilibria, in which the terminal wealth W_T is known with certainty at date $t = 0$. We can then prove

Theorem 2. *The macroeconomic variables* C_t, W_t, K_t, S_t *and* R_t *are deterministic along an equilibrium path. All agents have identical marginal propensities to consume* a_t *and choose identical levels of risky investment:* $k_t^h = K_t$ *for all* h. *The risk premium is equal to zero. When* $B > 0$, *an equilibrium path satisfies in*

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[17] In general, the optimal capital stock need not be the first solution to (3.4), and switches between solutions are a potential source of rich dynamics. We find it more striking to exhibit fluctuations in the absence of such effects.

every $t \leq T-1$ *the relations:*

$$R_t = G_k(K_t, a_{t+1}) \equiv F'(K_t)\left[B - Aa_{t+1}F(K_t)\sigma_P^2\right] + 1 - \delta, \tag{3.5}$$

$$W_{t+1} = \Phi(K_t) + \rho S_t \equiv BF(K_t) + (1-\delta)K_t + e + \rho S_t, \tag{3.6}$$

$$a_t = 1/[1 + (a_{t+1}R_t)^{-1}], \tag{3.7}$$

$$C_{t+1} - C_t = \tfrac{1}{A}\ln(\beta R_t) + \tfrac{A}{2}a_{t+1}^2\left[F(K_t)^2\sigma_P^2 + \sigma_E^2\right] \tag{3.8}$$

$$W_t = C_t + K_t + S_t, \tag{3.9}$$

and the boundary conditions $W_0 = \overline{W}_0,\ a_T = 1,\ K_T = 0.$[18]

The equilibrium equations contain rich economic effects. When the technological risk cannot be fully hedged ($\sigma_P > 0$), the interest rate is equal to the *risk-adjusted* marginal return $F'(K_t)\left[B - Aa_{t+1}F(K_t)\sigma_P^2\right] + 1 - \delta$. Under incomplete markets, risk aversion thus creates an interaction between the *current* investment K_t and the *future* marginal propensity a_{t+1}. We also note that the consumption growth $C_{t+1} - C_t$ increases with the variance of future consumption $Var_t(c_{t+1}^h) = a_{t+1}^2\left[F(K_t)^2\sigma_P^2 + \sigma_E^2\right]$. This is a well-known consequence of the precautionary motive (Leland, 1968; Sandmo, 1970; Caballero, 1990).

Equilibrium paths can be calculated by a backward recursion. Consider the macroeconomic vector $x_t = (a_t, C_t, W_t, K_t, S_t, R_t)$, and the reduced *state vector* $z_t = (a_t, C_t, W_t)$. We easily show

Proposition 2. *For any state vector* $z_{t+1} = (a_{t+1}, C_{t+1}, W_{t+1}) \in (0,1] \times \mathsf{R} \times [e, +\infty)$, *there exists a unique macroeconomic vector* $x_t = (a_t, C_t, W_t, K_t, S_t, R_t) \in (0,1) \times \mathsf{R}^2 \times \mathsf{R}_+^3$ *satisfying recursive relations* (3.5) – (3.9).

Proof. Let K_t^* denote the smallest solution to the equation $G_K(K, a_{t+1}) = \rho$, and let $W_{t+1}^* = \Phi(K_t^*)$. Any solution x_t can be calculated by following procedure. *Step* 1. If $W_{t+1} < W_{t+1}^*$, equations (3.5) and (3.6) imply that $K_t = \Phi^{-1}(W_{t+1})$, $S_t = 0$, and $R_t = G_k(K_t, a_{t+1}) > \rho$. On the other hand if $W_{t+1} \geq W_{t+1}^*$, we infer from (3.5) and (3.6) that $R_t = \rho$, $K_t = K_t^*$, and $S_t = \rho^{-1}[W_{t+1} - \Phi(K_t)]$.

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[18] When $B = 0$, $\delta = 1$ and $\rho = 0$, there is no production and no technology to transfer real resources intertemporally: $C_t = W_t = e$ and $K_t = S_t = 0$ for all t. The model then reduces to *SPEC*, the exchange economy of Calvet (1997).

Step 2. Equations (3.7) – (3.9) then assign unique values to a_t, C_t and W_t. We observe that $(a_t, K_t, S_t, R_t) \in (0,1) \times \mathsf{R}^*_+ \times \mathsf{R}_+ \times [\rho, +\infty)$, while no simple restrictions seem to be imposed on current consumption and wealth. ¥

The equilibrium dynamics is thus fully characterized by the three-dimensional vector z_t.[19] The proposition defines the recursive mappings $x_t = \widetilde{H}(z_{t+1})$ and

$$z_t = H(z_{t+1}). \tag{3.10}$$

We also note that the condition $W_{t+1} \in [e, +\infty)$ does not guarantee that $W_t \in [e, +\infty)$.

The equilibrium calculation is very similar to the method used for the traditional Ramsey model. At the terminal date, individuals consume all their wealth, and the state variables satisfy the relations $a_T = 1$ and $C_T = W_T$. For any choice of $W_T \in [e, +\infty)$, we set $z_T = (1, W_T, W_T)$ and recursively calculate the implied path $z_{t-1} = H(z_t)$. Some terminal wealth levels W_T imply that $W_t < e$ and the algorithm stops at an instant $t > 0$. Other values W_T generate an entire path $\{z_t\}_{t=0}^T$ and an initial wealth level W_0. The recursion thus defines a function $W_0(W_T)$, whose domain is a subset of $[e, +\infty)$. Since the initial wealth $\overline{W}_0 \in \mathsf{R}$ is exogenously fixed, an equilibrium corresponds to a terminal wealth level W_T such that $W_0(W_T) = \overline{W}_0$. We can then easily prove

Theorem 3. *There exists a GEI equilibrium in any economy.*

The equilibrium calculation is thus very straightforward. We guess a final wealth W_T, calculate a path, and check that the implied $W_0(W_T)$ is equal to the exogenous wealth $\overline{W}_0$ of the economy.

The difference between the complete and incomplete market case stems from the nonmonotonicities embedded in the equilibrium recursion. For instance, when future wealth W_{t+1} is low ($W_{t+1} < W^*_{t+1}$), an increase in W_{t+1} implies a higher capital stock $K_t = \Phi^{-1}(W_{t+1})$ and a lower interest rate R_t, which leads to an increase in the component $-\ln(\beta R_t)/A$ of current consumption. Under complete markets, these are the only effects and current consumption C_t and wealth W_t are increasing functions of future wealth W_{t+1}. When markets are incomplete,

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[19] We note that the dimensionality of our system is influenced by the financial structure and the aggregate productivity of capital. More specifically, our three-dimensional model reduces to: a *two*-dimensional system in $(C_\mathrm{t}, W_\mathrm{t})$ when $B > 0$ and markets are complete ($\sigma_\mathsf{P} = \sigma_\mathsf{E} = 0$); a *one*-dimensional system in a_t when agents do not produce ($B = 0$, $\delta = 1$, $\rho = 0$).

however, the higher capital investment K_t implies that individual producers are bearing more production risk and have a stronger precautionary motive. As a result, current consumption C_t and wealth W_t can be *decreasing* functions of future wealth W_{t+1}.

Numerical simulations confirm that such non-monotonicities do occur along equilibrium paths. For instance, Figure 2 illustrates the function $W_0(W_T)$ for appropriate values of the economy's parameters. We observe that $W_0(W_T)$ is non-monotonic, and infer

Proposition 3. *There robustly exist multiple equilibria in some economies.*

By contrast, it is straightforward to check that the function $W_0(W_T)$ is increasing and that there exists a unique equilibrium path when markets are complete ($\sigma_P = \sigma_E = 0$).

For simplicity, we now focus on economies with a single equilibrium, and calculate the unique growth path using the equilibrium recursion. Figure 1 displays an economy with large endogenous fluctuations, even though there is no aggregate uncertainty. This graph, as well as Figure 2, illustrate the kind of complex dynamics that the introduction of incomplete markets may generate in an otherwise standard neoclassical economy. Such behaviors do not necessarily require a high degree of impatience. In Figure 1, we obtain endogenous fluctuations along the unique equilibrium path for a high psychological discount factor ($\beta = 0.99$) and a low rate of capital depreciation ($\delta = 0.1$). Note that endogenous cycles and non-monotonicities require substantial undiversifiable idiosyncratic risks in these two examples.

These results suggest that market incompleteness can help obtain endogenous fluctuations for reasonable parameter values in earlier models considered in the literature. For instance, large investor impatience is required to obtain cycles in one-sector growth models with non-convex technologies (Boldrin and Montrucchio, 1986; Deneckere and Pelikan, 1986; Sorger, 1992; Mitra, 1996). The results of Figure 1 and Section 4 indicate that financial incompleteness is a useful substitute for investor impatience in the calibration of these stationary economies. Similarly, incomplete markets could help generate fluctuations in endogenous growth models (Benhabib and Farmer, 1994) for reasonable levels of investor impatience, uninsurable risks and increasing returns.

Figures 3 and 4 consider the case of small market incompleteness. We choose an economy with $T = 1,000$ periods and parameters $A = 10$, $B = 1$, $\beta = 0.95$,

$\alpha = 0.40$, $\delta = 0.05$, $\rho = 1-\delta$, $e = 0$, $\sigma_E = 0$, and $\sigma_P = 0.50$.[20] Because uninsurable risks are now relatively small, we may expect that the GEI equilibrium remains relatively close to the complete-market growth path. In fact, the GEI growth path combines two features: a turnpike property and a strong precautionary effect around the terminal date. Starting from a low level of initial wealth, the economy accumulates wealth and remains many periods in the neighborhood of the GEI steady state.[21] This is evident in Figure 3, which plots $\log(W_t/W_\infty)$, the natural logarithm of the ratio between equilibrium wealth W_t and the steady-state level W_∞. In the traditional Ramsey model, aggregate wealth progressively decreases around the terminal date. Under incomplete markets, however, the last shocks cannot be easily smoothed by borrowing and lending, and individuals have a very strong precautionary motive before the terminal date. As a result, aggregate wealth overshoots the steady state before being consumed in the very last periods.

4. Equilibrium under Infinite Horizon

We now extend the model to an infinite horizon. We successively discuss decision theory, perfect foresight equilibria, and the properties of the steady state.

4.1. Decision Theory

Given a deterministic sequence of asset prices $\{\pi_t\}_{t=0}^{+\infty}$, we calculate the optimal decision of an individual investor by taking the pointwise limit of the finite horizon problem. In every period, let $\pi_L(t) = \sum_{s=t+1}^{+\infty} 1/(R_t..R_{s-1})$ denote the price of a perpetuity, $a_t = [1 + \pi_L(t)]^{-1}$ the marginal propensity to consume, and K_t the smallest solution to equation (3.5). We also consider $M_t^h = G(K_t, a_{t+1})/R_t - K_t - (Aa_{t+1}R_t)^{-1} \ln(\beta R_t) + \sum_{i=1}^{N} \pi_{i,t}[\xi_i^h + \kappa_i^h F(K_t) + R_t \pi_{i,t}/(2Aa_{t+1})]$ and

$$b_t^h = a_t \left(M_t^h + \sum_{s=t+1}^{+\infty} \frac{M_s^h}{R_t...R_{s-1}} \right).$$

We guarantee the convergence of the series by:

Assumption 3. *The sequences $\{R_t\}_{t=0}^{+\infty}$, $\{\pi_t\}_{t=0}^{+\infty}$ and $\{\pi_L(t)\}_{t=0}^{+\infty}$ are bounded.*

This assumption is sufficient (but not necessary) for the existence of a_t and b_t^h.

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[20] The full calibration of the model is discussed in Section 5.

[21] We discuss the steady state of the infinite-horizon economy in the next sections.

Theorem 5. *Under Assumptions* $[1]-[3]$*, the indirect utility of wealth is CARA every period:*

$$J(W,t) = -(Aa_t)^{-1}\exp[-A(a_t W + b_t^h)],$$

and consumption-portfolio decision rules satisfy (3.2).

Since the optimal decision is the limit of the finite horizon problem, Ponzi schemes are ruled out in the strongest conceivable way, along any possible path.

4.2. Perfect Foresight Equilibria

We now introduce

Definition 2. *A* perfect foresight equilibrium (PFE) *is a GEI equilibrium in which prices are deterministic and satisfy Assumption* [3].

It is straightforward to show

Proposition 4. *In a PFE, the risk premium is zero and the sequence* $\{z_t\}_{t=0}^{\infty}$ *satisfies the recursion* $z_t = H(z_{t+1})$ *in every period.*

By Assumption [3], the sequence $a_t = 1/[1+\pi_L(t)]$ is bounded away from 0 in a PFE. Conversely, consider a sequence $\{z_t = (a_t, C_t, W_t)\}_{t=0}^{\infty}$ satisfying equilibrium recursion (3.10) and the condition $\inf_t a_t > 0$. The perpetuity price $\pi_L(t) = 1/a_t - 1$, the corresponding interest rate R_t and the price sequence $\pi_t = (1/R_t, 0, ..0)$ are bounded across t. We infer from Theorem 5 that each agent has a unique optimal plan and that markets are clearing in every date-event. The set of PFEs thus identifies with the set of sequences satisfying the equilibrium recursion (3.10) and the condition $\inf_t a_t > 0$.

4.3. Steady State

A *steady state* is a PFE in which the state vector z_t is invariant through time: $z_t = z_\infty$ for all t. The vector z_∞ is then a fixed point of the recursion mapping H. Assumption [3] implies that $R_\infty > 1$, $a_\infty = 1 - R_\infty^{-1}$, and the storage technology is not used in the steady state ($S_\infty = 0$). The steady interest rate and capital stock are then determined by the system:

$$\begin{aligned} \ln(R_\infty \beta) &= -A^2(1-R_\infty^{-1})^2\left[F(K_\infty)^2\sigma_P^2+\sigma_E^2\right]/2, \\ R_\infty &= F'(K_\infty)\left[B-A(1-R_\infty^{-1})F(K_\infty)\sigma_P^2\right]+1-\delta. \end{aligned} \tag{4.1}$$

The first equation corresponds to the stationarity of consumption, and the second to the optimality of the capital investment K_∞. This system allows us to analyze existence, multiplicity and comparative statics. We first show

Theorem 6. *There exists a steady state in every economy.*

We note that $R_\infty \leq 1/\beta$, and that the upper bound $1/\beta$ is reached when markets are complete. This result is a possible solution to the low risk-free rate puzzle, which has been extensively discussed in the literature (e.g. Weil, 1992; Aiyagari, 1994; Constantinides and Duffie, 1996; Heaton and Lucas, 1996).

Multiple steady states can also arise under incomplete markets. As shown in the Appendix, system (4.1) implicitly defines two decreasing functions $K_1(R)$ and $K_2(R)$, which cross several times in some cases.[22]

Theorem 7. *There robustly exists a multiplicity of steady states in some economies.*

A steady state with a high interest rate also has low capital and aggregate wealth, and can be interpreted as a poverty trap. In earlier work (e.g. Banerjee and Newman, 1991), poverty traps originate in a wealth effect: poor agents are unwilling to take risk and make large investments in the productive technology. In our model, investment is independent of individual wealth and the poverty trap arises from the capital markets. When the endogenous interest rate is high, investors are unwilling to invest large amounts in the high-yield risky technology and the economy is stuck at a low wealth.

The comparative statics can easily be derived from system (4.1). In the following theorem, we consider for simplicity an economy with a unique steady state.[23]

Theorem 8. *In the neighborhood of a unique steady state, the capital stock K_∞ locally increases with the endowment risk σ_E, the discount factor β, and the productivity parameters B and $1-\delta$. On the other hand, the production risk*

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[22] It is easy to show that the number of steady states is generically odd.

[23] See the Appendix for a discussion of multiple steady states.

σ_P and the coefficient of absolute risk aversion A have ambiguous effects on the steady level of capital.

The recent literature on the neoclassical model with incomplete markets (Aiyagari, 1994; Krusell and Smith, 1998) has considered idiosyncratic risk in exogenous income ($\sigma_E > 0$), but not in productive investment ($\sigma_P = 0$). The theorem shows that the distinction between the two types of risks has important consequences for the equilibrium steady state. We first note that both forms of idiosyncratic risk increase the precautionary demand for savings, which tends to reduce the interest rate and *increase* capital investment. This explains why K_∞ increases with σ_E, as in Aiyagari (1994). We note, however, that uninsurable technological shocks also reduce the risk-adjusted return to investment and can thus *discourage* capital accumulation. These conflicting effects lead to a non-monotonic relation between σ_P and K_∞ in some economies. These results help us understand the possible long run effects of financial innovation on the macroeconomy.[24] While better insurance against endowment risk always reduces output through the precautionary effect, aggregate activity may actually be increased by new hedging instruments for technological risks.

We could now present numerical simulations illustrating how the capital stock and the interest rate vary with σ_P. However, the interpretation of these results would be problematic because the elasticity of intertemporal substitution varies inversely with consumption under CARA preferences.[25] We thus postpone the numerical calculations to Section 5, where we modify the model to match a constant elasticity of substitution at the steady state.

4.4. Local Dynamics

The local dynamics around the steady state can be examined by linearizing the recursion mapping H defined by $(3.5) - (3.9)$. We observe that wealth W_t is the only predetermined variable in the model, while consumption and marginal propensity are free to adjust. When markets are complete, the stable manifold

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[24] In Angeletos and Calvet (2000), we find a similar non-monotonicity between long-run growth and market incompleteness in an endogenous-growth context.

[25] Since each agent h has relative risk aversion $-u''(c_t^h)c_t^h/u'(c_t^h) = Ac_t^h$, the coefficient of relative risk-aversion has cross-sectional mean $\gamma_t \equiv \frac{1}{H}\sum_h Ac_t^h = AC_t = -u''(C_t)C_t/u'(C_t)$. The coefficient $\gamma_\infty = AC_\infty$ at the steady state therefore varies with σ_P, and it is difficult to disentangle the effect of σ_P from changes in the intertemporal elasticity of substitution. Section 5 presents a small modification of the model that solves this difficulty.

has dimension 1 and the system is locally determined. These results need not hold under incomplete markets. Denoting by $\#ST$ the number of steady states, we easily show

Theorem 9. *When $\#ST = 1$, the steady state is either locally determined or locally undetermined depending on the parameters of the economy.*

This establishes that our economy generates robust indeterminacy even when agents are very patient (e.g. $\beta = 0.95$). We also note that it is not easy to rule out this form of indeterminacy, because there is no obvious focal equilibrium on which agents can coordinate. Consistent with the results of Section 3, this suggests that market incompleteness could help generate indeterminacy and endogenous fluctuations for reasonable parameter values in calibrated models with increasing returns to scale (Benhabib and Farmer, 1994).

We now examine Ramsey economies with multiple steady states. For the case $\#ST = 3$, the Appendix provides an economy in which the equilibria with the lowest and highest interest rate exhibit local uniqueness, while the middle steady state is totally unstable.

Theorem 10. *When $\#ST > 1$, some steady states can be totally unstable.*

We can also find examples of flip bifurcations. This implies the existence of cycles of period 2 on an open set of economies.

Theorem 11. *Some economies robustly contain attracting cycles.*

Note that these cycles can arise in economies with reasonably patient investors ($\beta = 0.95$). Financial incompleteness thus allows us to eliminate the high discounting required to obtain cycles in stationary economies with nonconvexities in production (Boldrin and Montrucchio, 1986; Deneckere and Pelikan, 1986; Sorger, 1992). All these results illustrate the kind of complicated equilibrium dynamics that financial incompleteness generates in the neoclassical Ramsey framework under infinite horizon. In the next section we focus on reasonably calibrated economies with a unique steady state and examine the effect of market incompleteness on the transitional dynamics.

5. Calibrated Steady State and Persistence

We now present calibrated results for the equilibrium growth path. A slightly modified utility function is first introduced to facilitate the comparison between our model and the CRRA specification. We then analyze the comparative statics of the steady state and present simulations on the speed of convergence.

5.1. Extension: Calibrating CARA to CRRA

One difficulty with the standard CARA utility is that the intertemporal elasticity of substitution becomes very large at very low consumption levels.[26] To remedy this problem, we denote by $C_\infty > 0$ the mean consumption in the steady state and assume that agents have utilities of the form:

$$u(c_t^h) = -\frac{1}{(\gamma / C_\infty)} \exp\left[-\frac{\gamma c_t^h}{C_\infty}\right].$$

The coefficient of relative risk aversion is now equal to γ at the steady state C_∞, which allows us to calibrate the model.

This utility has two possible interpretations. We can either view it as a behavioral assumption that risk aversion is determined by the relative level of current consumption to the steady state. Preferences are then endogenous to the economy. Alternatively, we interpret the hypothesis as a practical restriction on the parameter space of the model introduced in the previous sections. In both cases, the new specification allows us to produce calibrated results that are directly comparable with standard RBC models.

The equilibrium dynamics is given by Theorem 2 with the coefficient $A = \gamma / C_\infty$. The main difference with the previous approach lies in the calculation of the steady state. We know that $R_\infty > 1$, $a_\infty = 1 - R_\infty^{-1}$, $S_\infty = 0$ and $W_\infty = \Phi(K_\infty) = C_\infty + K_\infty$, which implies that $A = \gamma / [\Phi(K_\infty) - K_\infty]$. The steady interest rate and capital stock are determined by the system:

$$\begin{aligned} \ln(\beta R_\infty) &= -\tfrac{\gamma^2 (1 - R_\infty^{-1})^2}{2[\Phi(K_\infty) - K_\infty]^2} \left[F(K_\infty)^2 \sigma_P^2 + \sigma_E^2\right], \\ R_\infty &= F'(K_\infty)\left[B - \tfrac{\gamma(1 - R_\infty^{-1}) F(K_\infty) \sigma_P^2}{\Phi(K_\infty) - K_\infty}\right] + 1 - \delta. \end{aligned} \tag{5.1}$$

Assuming for simplicity that $e = 0$, we show in the Appendix:

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[26] Alternatively, the coefficient of relative risk aversion increases with the level of consumption.

Theorem 12. *For any $\sigma_P \geq 0$ and $\sigma_E > 0$, there exists a steady state with $K_\infty > 0$ and $1 < R_\infty < 1/\beta$.*

Similar to Theorem 8, system (5.1) implies that the capital stock K_∞ tends to increase with the endowment risk σ_E, the discount factor β, and the productivity parameters B and $1-\delta$. On the other hand, the production risk σ_P and the coefficient of relative risk aversion γ have ambiguous effects.

5.2. Comparative Statics with a Cobb-Douglas Technology

In order to calibrate the economy, we now assume that the technology is Cobb-Douglas: $F(K) = K^\alpha$. We set $e = 0$, and normalize the endowment risk to be proportional to the steady-state output: $[Var(\varepsilon_t^h)]^{1/2} = \sigma_E F(K_\infty)$. Under these hypotheses, the TFP is only a scale factor of the capital stock that can be normalized to $B = 1$. We now interpret σ_P and σ_E as percentage rates. For instance when $\sigma_P = 0.25$, the standard deviation of gross output, $\sigma_P F(K_\infty)$, represents 25% of the mean production $F(K_\infty)$.

Let $Q \equiv F(K)/K = K^{\alpha-1}$ denote the ratio of output to capital. Since $F'(K) = \alpha Q$ and $\Phi(K) - K = (Q-\delta)K$, the steady-state system (5.1) can be rewritten:

$$\begin{aligned} \ln(\beta R_\infty) &= -\tfrac{1}{2}\Gamma_\infty^2(\sigma_P^2 + \sigma_E^2), \\ R_\infty &= \alpha Q_\infty(1 - \Gamma_\infty \sigma_P^2) + 1 - \delta, \end{aligned} \tag{5.2}$$

where

$$\Gamma_\infty \equiv \gamma \frac{R_\infty - 1}{R_\infty} \frac{Q_\infty}{Q_\infty - \delta} > 0.$$

Under complete markets ($\sigma_P = \sigma_E = 0$), the system implies $R_\infty = 1/\beta$ and $Q_\infty = (1 - \beta + \delta\beta)/(\alpha\beta)$ like in the standard Ramsey model. Consistent with Section 4, the interest rate is smaller ($R_\infty < 1/\beta$) when idiosyncratic risks are partially undiversifiable.

We loosely interpret Γ_∞ as an 'effective' degree of prudence and risk aversion, which measures the effect of σ_P or σ_E on savings and investment. When σ_P^2 and σ_E^2 are close to 0, the first-order variations in R_∞ and Q_∞ are obtained by keeping Γ_∞ constant in (5.2). It is then straightforward to work out the comparative statics of the steady state.[27]

More generally, we can analyze the monotonicities around any financial structure by slightly adapting the arguments of Section 4.[28] An increase in the endowment risk σ_E affects the precautionary savings motive but not the risk-adjusted return to capital investment. As a result, the general-equilibrium effect of an increase in σ_E is to reduce both R_∞ and Q_∞ and thereby increase K_∞. This is precisely what Aiyagari (1994) showed.

The effect of the productivity risk σ_P is fundamentally different for two reasons. First, an increase in σ_P reduces the risk-adjusted return to capital and has therefore a direct adverse effect on investment demand. Second, even when σ_P increases, agents can potentially bear *less* risk by simply reducing their investment in the risky technology. In equilibrium, the precautionary effect of an increase in σ_P may thus be moderated by a decrease in capital investment.

When σ_P increases, the investment and precautionary motives have opposite effects on the capital stock K_∞. The dominant force depends on both preferences (prudence and risk aversion, γ) and technology (strength of diminishing returns, α, and magnitude of risk, σ_P and σ_E). Numerical simulations suggest the following pattern as we vary σ_P. For a low to moderate risk aversion γ, the investment effect always dominates; in this case, completing the markets unambiguously increases capital accumulation. On the other hand, the capital stock is a single-peaked function of σ_P when the risk aversion γ is sufficiently high.

Figures 5 and 6 illustrate the monotonicities of the capital stock K_∞ and the net interest rate $r_\infty \equiv R_\infty - 1$ for typical parameters of RBC models in annual frequency: $\alpha = 0.4$, $\delta = 0.05$, and $\beta = 0.95$. In each graph, the solid line corresponds to $\sigma_E = 0$ and the dashed one to $\sigma_E = 25\%$. When $\gamma = 2$, capital monotonically decreases as σ_P varies from 0 to 100% (Figure 5). On the other hand for a higher risk aversion ($\gamma = 10$), the relation between σ_P and capital is non-monotonic (Figure 6). We also observe that investment risk has a small effect on capital for the chosen parameters. At $\sigma_P = 100\%$, the capital stock K_∞ is about 10% lower than its complete-market value when $\gamma = 2$. Market incompleteness has a small effect because idiosyncratic shocks are very short and transitory. Investors can efficiently self-insure by borrowing and lending, and the

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[27] When σ_{P}^2 is close to zero, the first-order variations of the steady state satisfy $d(\ln R_\infty) = -\Gamma_\infty^2 d(\sigma_{\mathsf{P}}^2)/2$ and $d(\ln Q_\infty) = \Gamma_\infty(1-\gamma/\gamma_{\mathsf{crit}})d(\sigma_{\mathsf{P}}^2)$, where $\gamma_{\mathsf{crit}} = 2\alpha(Q_\infty - \delta)/(R_\infty - 1)$. The capital stock thus decreases as we move away from complete markets if and only if $\gamma < \gamma_{\mathsf{crit}}$.

[28] See the proof of Theorem 12 in the Appendix.

lack of insurance has no substantial impact on the macroeconomic aggregates.

Financial incompleteness does matter, however, if idiosyncratic shocks are more persistent or investment is subject to adjustment costs. Persistence in idiosyncratic shocks and high setup costs seem to be empirically valid assumptions, especially if we think of human capital development, large R&D projects, or other investments that involve specialization, indivisibilities, and long horizons. To capture these effects, we simply increase the length of a time period, and view the interval between t and $t+1$ as the horizon of an investment project. Consider for instance an investment horizon of 5 years, which approximately corresponds to the length of a large investment project, or the length of a college or graduate degree.[29] As previously, we assume that the capital share is $\alpha = 0.4$, and that the discount rate and the depreciation rate are both approximately 5% per year ($\beta = 0.75$ and $\delta = 0.25$ over a 5-year period). With an intermediate risk aversion ($\gamma = 6$), Figure 7 illustrates that uninsurable risks have a substantial effect on the steady state.[30] At $\sigma_P = 100\%$, the capital stock K_∞ is now about half what it would be under full insurance. Figure 8 reports similar results using a higher capital share ($\alpha = 0.7$) intended to capture investment in both physical and human capital (and other intangibles). The effect of uninsured risk is again very strong. These examples suggest that insurance problems can have substantial effects on capital accumulation when markets are incomplete, as is the case in most of the developing world.

The simulations also provide valuable insights on the interaction between additive and multiplicative risks. Recall that in Figures $5-8$, the dashed lines correspond to $\sigma_E = 25\%$ and the solid ones to $\sigma_E = 0$. We observe that the steady state is less sensitive to σ_E as the productivity risk σ_P increases (see especially Figures 7 and 8). This is because when σ_P is large, individuals are already holding a buffer stock that can be used to insure against both investment and endowment risks. The precautionary effect of σ_P similarly diminishes with σ_E, implying that the investment effect dominates more easily when there is already a lot of precautionary saving in the economy.

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[29] When an investment project is sensitive to political risk, a period of four or five years also corresponds to the time interval between elections.

[30] We interpret the length of a time period to be one year in Figures 5 and 6, and five years in Figures 7 and 8. For simplicity, however, the graphs report the steady state quantities K_∞ and r_∞ corresponding to a time period of the model, and not their annualized equivalents.

5.3. Convergence and Persistence

We now show that convergence to the steady state can be slower under incomplete markets than with a complete financial structure. We demonstrate this effect by examining the rate of convergence to the steady state. As discussed in the Appendix, we calculate the stable eigenvalue λ of the linearized system and approximate the local dynamics by $\log(K_{t+1}/K_\infty) = \lambda \log(K_t/K_\infty)$. The quantity $1-\lambda$ is then called the *convergence rate*. Market incompleteness slows down convergence to the steady state if $1-\lambda$ decreases with σ_P. Numerical simulations show that such is the case, at least near $\sigma_P = 0$, for a wide range of plausible parameter values. We also observe that the speed of convergence is sometimes non-monotonic for large values of σ_P.

For the typical RBC parameters used in Figure 5 ($\alpha = 0.4$, $\delta = 0.05$, $\beta = 0.95$, $\sigma_E = 0$ and $\gamma = 2$), the convergence rate decreases monotonically as σ_P varies from 0 to 100%. The magnitude of the decline, however, is very modest. The convergence rate equals 5.4% under complete markets, and 5.2% when $\sigma_P = 100\%$. As in Figure 5, the insurance problem is negligible with this calibration. Shocks are short, transitory, and easy to spread through intertemporal smoothing.

The convergence rate falls substantially, however, if idiosyncratic risks are large and persistent. Consider for instance a 5-year investment project involving both physical and human capital. As in Figure 8, we calibrate the model with $\alpha = 0.7$, $\beta = 0.75$ (discount rate $\approx 5\%$ per year), $\delta = 0.25$ ($\approx 5\%$ per year), $\sigma_E = 0$, and $\gamma = 6$. Figure 9 illustrates how investment risk influences the local dynamics around the steady state. When σ_P increases from 0 to 25%, the convergence rate falls from 3.4% to 2.4%, and the corresponding half-life of a shock[31] increases from 20 to 30 periods. This is a very substantial increase, which suggests that undiversifiable productivity shocks can play a useful role for the calibration of RBC models (Kydland and Prescott, 1982; Prescott, 1986; Cooley, 1995). We expect that financial incompleteness should significantly reduce the persistence of the exogenous aggregate shocks used in these models to match aggregate time series.

The intuition underlying these results is quite straightforward. In the complete market Ramsey framework, a negative wealth shock has some persistence because agents seek to smooth consumption across time. Under incomplete markets, convergence is also slowed down by the effect of the precautionary motive on interest

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[31] The half-line T is defined by the equation $\lambda^{\mathsf{T}} = 1/2$, or equivalently $T = -\log_2 \lambda$.

rates. Consider for instance an economy starting at a low wealth level. Because agents make small risky investments, they have a weaker precautionary demand for savings than in the long run. This puts a downward pressure on the interest rate, which accentuates the low level of capital investment and slows down convergence. Conversely in a rich economy, high investment in the risky technology implies a strong precautionary motive and a low interest rate, which intensifies the high level of capital. In both cases, prudence increases the persistence of an aggregate shock, and has an amplification effect on investment and interest rates. We observe that these results crucially depends on the riskiness of production and thus could not be observed in the models of Aiyagari (1994) and Krusell and Smith (1998). An empirical investigation of a cross-section of countries would be a natural test of our results.

6. Conclusion

This paper introduces an incomplete market economy with decentralized production and idiosyncratic technological risk. We assume that there are no short sales constraints, and that the absence of certain assets is the only source of market imperfection in the economy. To simplify the analysis, we do not consider the possibility of aggregate uncertainty. Macroeconomic aggregates and financial prices are then deterministic but can possibly follow complicated dynamics and cycles. Because it influences risk-taking, market incompleteness plays a critical role on the level and volatility of aggregate activity, *even though* the wealth distribution plays no role in the equilibrium dynamics of our model. The interaction between the precautionary motive and investment risk is thus a powerful source of poverty traps and volatility in incomplete-market economies.

The paper also contains interesting comparative statics properties. First, there can exist multiple steady states because agents may prefer to remain poor rather than bear the large investment risks that are required to increase their standards of living. The poverty trap hinges on the endogeneity of the interest rate rather than the typical wealth effect. In addition, the introduction of a new asset (or a new insurance contract) has an ambiguous effect on the level of aggregate wealth in the steady state. On one hand, the new security may reduce technological risk and thus encourage investment. On the other hand, financial innovation dampens the precautionary demand for savings, increases the interest rate, and thus discourages productive investment. Overall, the level of aggregate wealth is found to be the highest in economies with intermediate or high levels of financial sophistication. We note that the positive effect of financial innovation would be stronger if agents had utility functions, such as CRRA, in which the reduction in the endowment risk could also encourage agents to invest more in the risky technology.[32]

This work suggests several directions for further research. Work in progress attempts to introduce aggregate uncertainty in our model. This will allow us to embed our results in a stochastic RBC framework and characterize the impulse response of the economy to aggregate shocks. Another paper, Angeletos and Calvet (2000), considers market incompleteness in an endogenous growth context. There is then an interaction between the financial structure, the investment specialization patterns, and the long-run growth rate of the economy. These various projects may help us further understand the interaction between financial markets and the real sector of the economy.

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[32] Kimball (1993) shows that a background risk negatively affects investement in an independent risky project when agents have decreasing absolute risk aversion and decreasing absolute prudence.

7. Appendix

Proof of Theorem 1

An entrepreneur chooses consumption c_t^h, capital expenditure k_t^h, storage s_t^h and asset holdings θ_t^h that maximize

$$u(c_t^h) + \beta\,\mathsf{E}_t J\left[e_{t+1}^h + B_{t+1}^h F(k_t^h) + (1-\delta)k_t^h + \rho s_t^h + d_{t+1}\cdot\theta_t^h,\ t+1\right]$$

subject to the budget equality $c_t^h + k_t^h + s_t^h + \pi_t \cdot \theta_t^h = w_t^h$, and the non-negativity constraints: $k_t^h \geq 0$, $s_t^h \geq 0$. It is convenient to consider $\widehat{\theta}_{0,t}^h = \rho s_t^h + \theta_{0,t}^h$.

The First Order Conditions to this problem are

$$u'(c_t^h) = \beta R_t\,\mathsf{E}_t J_w, \tag{7.1}$$

$$\pi_{i,t}u'(c_t^h) = \beta\,\mathsf{E}_t(d_{i,t+1}J_w) \quad (1 \leq i \leq N), \tag{7.2}$$

$$u'(c_t^h) = \beta\,\mathsf{E}_t\left\{\left[B_{t+1}^h F'(k_t^h) + 1 - \delta\right] J_w\right\} + \lambda_t^h, \tag{7.3}$$

where λ_t^h is the Lagrange multiplier associated to the non-negativity of capital. Since $d_{i,t+1}$ and w_{t+1}^h are jointly normal, Stein's lemma implies

$$\begin{aligned}\mathsf{E}_t(d_{i,t+1}J_w) &= -Aa_{t+1}Cov_t(d_{i,t+1}; w_{t+1}^h)\,\mathsf{E}_t J_w \\ &= -Aa_{t+1}\left[\theta_{i,t}^h + \kappa_i^h F(k_t^h) + \xi_i^h\right]\,\mathsf{E}_t J_w,\end{aligned}$$

and equation (7.2) can be rewritten:

$$\pi_{i,t}u'\left(c_t^h\right) = -\beta Aa_{t+1}\left[\theta_{i,t}^h + \kappa_i^h F(k_t^h) + \xi_i^h\right]\,\mathsf{E}_t J_w. \tag{7.4}$$

By equation (7.1), this relation simplifies to $R_t\pi_{i,t} = -Aa_{t+1}\left[\theta_{i,t}^h + \kappa_i^h F(k_t^h) + \xi_i^h\right]$ and therefore

$$\theta_{i,t}^h = -\xi_i^h - \kappa_i^h F(k_t^h) - R_t\pi_{i,t}/(Aa_{t+1}).$$

Individual wealth in period $t+1$ then reduces to

$$\begin{aligned}w_{t+1}^h &= F(k_t^h)(B + \eta_{t+1}^h) + (1-\delta)k_t^h + e + \varepsilon_{t+1}^h + \widehat{\theta}_{0,t}^h - \frac{R_t}{Aa_{t+1}}\sum_{i=1}^N \pi_{i,t}d_{i,t+1} \\ &= \Phi(k_t^h) + \widehat{\theta}_{0,t}^h + F(k_t^h)\eta_{t+1}^h + \varepsilon_{t+1}^h - \frac{R_t}{Aa_{t+1}}\sum_{i=1}^N \pi_{i,t}d_{i,t+1}.\end{aligned} \tag{7.5}$$

By Stein's lemma, the FOC (7.3) simplifies to

$$u'(c_t^h) = \beta \left[BF'(k_t^h) + 1 - \delta\right] \mathsf{E}_t J_w - \beta F'(k_t^h) Cov_t(B_{t+1}^h, w_{t+1}^h) A a_{t+1} \mathsf{E}_t J_w + \lambda_t^h.$$

Since $Cov(B_{t+1}^h, w_{t+1}^h) = F(k_t^h)\sigma_P^2 - R_t \sum_{i=1}^N \pi_{i,t}\kappa_i^h/(Aa_{t+1})$ (from equation (7.5)), we obtain

$$u'(c_t^h) = \beta \left[G_k(k_t^h, a_{t+1}) + R_t F'(k_t^h) \sum_{i=1}^N \pi_{i,t}\kappa_i^h\right] \mathsf{E}_t J_w + \lambda_t^h.$$

If $B + R_t \sum_{i=1}^N \pi_{i,t}\kappa_i^h > 0$, we infer that $k_t^h > 0$ and $\lambda_t^h = 0$. We then divide this equation by (7.1) and obtain optimality condition (3.1).

We finally turn to the FOC associated to the riskless asset. Since

$$\mathsf{E}_t J_w = \exp\left[(Aa_{t+1})^2 Var_t(w_{t+1}^h)/2 - Aa_{t+1}\mathsf{E}_t(w_{t+1}^h) - Ab_{t+1}\right],$$

equation (7.1) simplifies to

$$\begin{aligned} 0 &= Ac_t^h + \ln(\beta R_t) + \frac{1}{2}(Aa_{t+1})^2\left[F(k_t^h)^2\sigma_P^2 + \sigma_E^2\right] + \frac{R_t^2}{2}\sum_{i=1}^N (\pi_{i,t})^2 \\ &\quad - Aa_{t+1}\left[\Phi(k_t^h) + \widehat{\theta}_{0,t}^h\right] - Ab_{t+1}, \end{aligned} \tag{7.6}$$

or equivalently

$$\widehat{\theta}_{0,t}^h = \frac{c_t^h}{a_{t+1}} + \frac{\ln(\beta R_t)}{Aa_{t+1}} - G(k_t^h, a_{t+1}) - \frac{b_{t+1}}{a_{t+1}} + \frac{R_t^2}{2Aa_{t+1}}\sum_{i=1}^N (\pi_{i,t})^2.$$

The individual budget constraint

$$\begin{aligned} w_t^h &= k_t^h + \left(1 + \frac{1}{a_{t+1}R_t}\right)c_t^h + \frac{\ln(\beta R_t)}{Aa_{t+1}R_t} - \frac{G(k_t^h, a_{t+1})}{R_t} - \frac{b_{t+1}}{a_{t+1}R_t} \\ &\quad + \frac{R_t}{2Aa_{t+1}}\sum_{i=1}^N (\pi_{i,t})^2 - \sum_{i=1}^N \pi_{i,t}\left[\xi_i^h + \kappa_i^h F(k_t^h) + R_t\pi_{i,t}/(Aa_{t+1})\right] \end{aligned}$$

then implies

$$c_t^h = a_t \left\{ \begin{array}{c} w_t^h - k_t^h - \ln(\beta R_t)/(Aa_{t+1}R_t) + G(k_t^h, a_{t+1})/R_t + b_{t+1}/(a_{t+1}R_t) + \\ + \sum_{i=1}^N \pi_{i,t}\left[\xi_i^h + \kappa_i^h F(k_t^h) + R_t\pi_{i,t}/(2Aa_{t+1})\right] \end{array} \right\}.$$

By the envelope theorem $J_w(w,t) = \exp(-Ac_t^h)$, the Recursive Condition is satisfied at date t.

Proof of Proposition 1

Under condition (iii), the function $G(k,a)$ is concave in k and equation (3.4) has a unique solution. More generally, consider an arbitrary production function and let $k_0 = F^{-1}[B/(Aa_{t+1}\sigma_P^2)]$. The function $G_k(k,a_{t+1}) = F'(k)[B - Aa_{t+1}F(k)\sigma_P^2] + 1 - \delta$ is decreasing and larger than $1-\delta$ when $k \in (0,k_0]$, and is strictly smaller than $1-\delta$ on $[k_0,+\infty)$. The equation $G_k(k,a_{t+1}) = R_t$ has thus a unique solution when $R_t \geq 1-\delta$, as is always the case under condition (i).

We now consider a Cobb-Douglas technology $F(k) = k^\alpha$ and an arbitrary interest rate R_t. We note that

$$\begin{aligned} G_{kk}(k,a_{t+1}) &= F''(k)[B - Aa_{t+1}\sigma_P^2 F(k)] - Aa_{t+1}\sigma_P^2[F'(k)]^2 \\ &= \alpha k^{\alpha-2}\left[(1-2\alpha)Aa_{t+1}\sigma_P^2 F(k) - (1-\alpha)B\right]. \end{aligned}$$

When $\alpha > 1/2$, the function G_k is strictly decreasing in the capital stock, and satisfies $\lim_{k\to 0} G_k(k,a_{t+1}) = +\infty$, $\lim_{k\to+\infty} G_k(k,a_{t+1}) = -\infty$; the equation $G_k(k,a_{t+1}) = R_t$ has thus a unique solution for all $R_t > 0$. On the other hand when $\alpha < 1/2$, the function $G_k(k,a_{t+1})$ is decreasing for low values of k, reaches a minimum $R_{\min}(a_{t+1})$, and converges to $1-\delta$ as $k \to +\infty$. Equation (3.4) has therefore two solutions when $R_{\min}(a_{t+1}) < R_t < 1-\delta$, and no solution if $R_t < R_{\min}(a_{t+1})$.

When equation (3.4) has two solutions k_1 and k_2 ($k_2 > k_1$), we observe that

$$\begin{aligned} G(k_2,a_{t+1}) &= G(k_1,a_{t+1}) + \int_{k_1}^{k_2} G_k(k,a_{t+1})dk \\ &< G(k_1,a_{t+1}) + (k_2-k_1)R_t. \end{aligned}$$

Any admissible plan of the form (c,k_2,s,θ) is therefore suboptimal.

Proof of Theorem 2

We show the theorem by a backward recursion. At date $t = T$, agents have identical propensities $a_T = 1$ and make no risky investment: $k_T^h = 0$. Assume now that the properties of the theorem hold at instants $t+1,..,T$.

Step 1. We first want to show that there is no risk premium in equilibrium: $\pi_{i,t} = 0$ for all i. It is convenient to define the truncated price vector $\widehat{\pi}_t = (\pi_{i,t})_{i=1}^N$ and the individual vectors $\kappa^h = (\kappa_i^h)_{i=1}^N$. Since there is no aggregate multiplicative shock, we know that $\sum_{h=1}^{H} \kappa^h = 0$ and therefore $\sum_{h=1}^{H} \widehat{\pi}_t \cdot \kappa^h = 0$. Without loss of generality, we index households so that

$$\widehat{\pi}_t \cdot \kappa^1 \geq ... \geq \widehat{\pi}_t \cdot \kappa^L \geq 0 \geq \widehat{\pi}_t \cdot \kappa^{L+1} \geq ... \geq \widehat{\pi}_t \cdot \kappa^H.$$

The corresponding capital stocks provided by equation (3.1) then satisfy $k_t^1 \geq ... \geq k_t^H$. For any $i \in \{1, .., N\}$, the market clearing of security i implies

$$\frac{1}{H} \sum_{h=1}^{H} \theta_{i,t}^h = -\frac{1}{H} \sum_{h=1}^{H} F(k_t^h) \kappa_i^h - \frac{R_t \pi_{i,t}}{A a_{t+1}} = 0,$$

or equivalently $\pi_{i,t} = -A a_{t+1} \sum_{h=1}^{H} F(k_t^h) \kappa_i^h / (H R_t)$. This yields the vector equality $\widehat{\pi}_t = -A a_{t+1} \sum_{h=1}^{H} F(k_t^h) \kappa^h / (H R_t)$. Since

$$\|\widehat{\pi}_t\|^2 = -\frac{A a_{t+1}}{H R_t} \left[\sum_{h=1}^{L} F(k_t^h)\, \widehat{\pi}_t \cdot \kappa^h + \sum_{h=L+1}^{H} F(k_t^h)\, \widehat{\pi}_t \cdot \kappa^h \right],$$

we infer $\|\widehat{\pi}_t\|^2 \leq -A a_{t+1} \left[\sum_{h=1}^{L} F(k_t^L)\, \widehat{\pi}_t \cdot \kappa^h + \sum_{h=L+1}^{H} F(k_t^{L+1})\, \widehat{\pi}_t \cdot \kappa^h \right] / (H R_t)$, or

$$\|\widehat{\pi}_t\|^2 \leq -\frac{A a_{t+1}}{H R_t} [F(k_t^L) - F(k_t^{L+1})] \sum_{h=1}^{L} \widehat{\pi}_t \cdot \kappa^h \leq 0.$$

We conclude that $\widehat{\pi}_t = 0$ in any equilibrium.

Step 2. In equilibrium agents make the same investment $k_t^h = K_t > 0$ in the risky technology. Each agent purchases $\theta_{i,t}^h = -\xi_i^h - \kappa_i^h F(K_t)$ units of each risky asset; she thus sells at no cost the tradable components of her production and endowment risks. Equations (3.5) and (3.7) are implied by individual decision, and equations (3.6) and (3.9) are obtained by aggregating the budget constraints. We note that equations (3.5) and (3.6) uniquely determine R_t, K_t and S_t. The first-order condition $u'(c_t^h) = \beta R_t \mathsf{E}_t u'(c_{t+1}^h)$ implies

$$0 = A c_t^h + \ln(\beta R_t) + \frac{1}{2} A^2 a_{t+1}^2 \left[F(K_t)^2 \sigma_P^2 + \sigma_E^2 \right] - A c_{t+1}^h.$$

Aggregation of this relation then yields equation (3.8). We finally note that the macro aggregates are the same in all states of date t.

Proof of Theorem 3

Consider $a \in \mathsf{R}$ and the mapping

$$\varphi(W_T) = \begin{cases} W_0(W_T) & \text{if } W_0(W_T) \text{ is defined and larger than } a, \\ a & \text{otherwise.} \end{cases}$$

Since φ is continuous on its domain $[e, +\infty)$, its image is an interval of the real line. When $W_T \to e$, productive investment K_{T-1} and s_{T-1} decline to zero, implying that $R_{T-1} \to +\infty$, $a_{T-1} \to 1$, $c_{T-1} \to -\infty$, $W_{T-1} \to -\infty$, and $\varphi(e) = a$. On the other hand when $W_T \to +\infty$, we observe that $R_t = \rho$ for all t and $\varphi(W_T) \to +\infty$. This establishes that $\varphi[e, +\infty) = [a, +\infty)$. Since the argument applies to any choice of a, we conclude that $\varphi(W_T) = W_0$ has a solution for any $W_0 \in \mathsf{R}$.

Proof of Theorem 5

In order to check that b_t^h is well-defined, we consider the decomposition

$$M_t^h = X_t^h + Y_t^h,$$

where $X_t^h = G(K_t, a_{t+1})/R_t - K_t - \ln(\beta R_t)/(A a_{t+1} R_t)$, and $Y_t^h = \sum_{i=1}^N \pi_{i,t}[\xi_i^h + \kappa_i^h F(K_t) + R_t \pi_{i,t}/(2 A a_{t+1})]$. By Assumption [3], the sequences $\{a_t\}_{t=0}^\infty$ and $\{R_t\}_{t=0}^\infty$ are contained in compact intervals of the form $[\underline{a}, 1]$ and $[\rho, \overline{R}]$, where $\underline{a} > 0$. This implies that the infinite sequences $\{K_t\}$ and $\{X_t^h\}_{t=0}^\infty$ are bounded. We also infer from Assumption [3] the boundedness of $\{Y_t^h\}_{t=0}^\infty$ and $\{M_t^h\}_{t=0}^\infty$. The deterministic sequence $\{b_t^h\}_{t=0}^\infty$ is thus well-defined and bounded.

We now check that $J(W, Z) \equiv -(A a_0)^{-1} \exp\left[-A(a_0 W + b_0^h)\right]$ coincides with the value function. For any real number W, any admissible plan $\{c_t', k_t', s_t', \theta_t', W_t'\}_{t=0}^\infty$ such that $W_0' = W$ satisfies

$$\begin{aligned} J(W, Z) &= \underset{\{c,\theta,W_1\}}{Max} \left[u(c) + \beta \mathsf{E} J(W_1, \mathcal{S}Z)\right] \\ &\geq u(c_0') + \beta \mathsf{E}_0 J(W_1', \mathcal{S}Z) \\ &\geq u(c_0') + \beta \mathsf{E}_0 u(c_1') + \beta^2 \mathsf{E}_0 J(W_2', \mathcal{S}^2 Z), \end{aligned}$$

and by repetition

$$J(W, Z) \geq \mathsf{E}_0 \left[\sum_{t=0}^{T-1} \beta^t u(c_t')\right] + \beta^T \mathsf{E}_0 J(W_T', \mathcal{S}^T Z). \tag{7.7}$$

Since $\{\exp(-Ab_T^h)/a_T\}_{T=0}^{\infty}$ is bounded and $\exp(-Aa_T W'_T) \leq 1+\exp(-AW'_T)$, Assumption [1] implies that $\beta^T \mathsf{E}_0 J(W'_T, \mathcal{S}^T Z) = -(Aa_T)^{-1}\beta^T \mathsf{E}_0 \exp\left[-A(a_T W'_T + b_T^h)\right]$ converges to zero as T goes to infinity. Letting T go to infinity in (7.7), we obtain

$$J(W,Z) \geq \mathsf{E}_0 \left[\sum_{t=0}^{+\infty} \beta^t u(c'_t)\right]. \tag{7.8}$$

This proves that $J(W,Z)$ is an upper bound to the value function. It is then easy to check that the consumption plan defined by (3.2) is admissible and reaches $J(W,Z)$.

Proof of Theorems 6 and 8

Letting $\varphi(R) \equiv A^{-2}\,(1-1/R)^{-2}\ln(\frac{1}{R\beta})$, we rewrite system (4.1) as:

$$\varphi(R) = \left[F(K)^2\sigma_P^2 + \sigma_E^2\right]/2 \tag{7.9}$$

$$F'(K)\left[B - A\sigma_P^2\left(1-\frac{1}{R}\right)F(K)\right] + 1 - \delta - R = 0. \tag{7.10}$$

The function $\varphi(R)$ decreases on $(1, 1/\beta]$ and satisfies $\varphi(1,1/\beta] = [0,+\infty)$. Equation (7.9) thus defines the decreasing function

$$K_1(R) \equiv \left[\frac{2\varphi(R)-\sigma_E^2}{\sigma_P^2}\right]^{1/(2\alpha)},$$

which maps $(1, \varphi^{-1}(\sigma_E^2/2)]$ onto $[0,+\infty)$. Similarly, equation (7.10) implicitly defines the decreasing function $K_2(R)$ that maps $[1,+\infty)$ onto $(0,K^*]$, where $K^* = (F')^{-1}(\delta/B)$.

Consider the function $\Delta(R) = K_1(R) - K_2(R)$. When $R \to 1$, we observe that $K_1(R) \to +\infty$, $K_2(R) \to K^*$, and therefore $\Delta(R) \to +\infty$. We also note that $\Delta[\varphi^{-1}(\sigma_E^2/2)] = -K_2[\varphi^{-1}(\sigma_E^2/2)] < 0$. The graphs of the functions K_1 and K_2 therefore intersect and there exists at least one steady state.

Finally, we analyze the monotonicity of the steady state with respect to the economy's parameters. We consider the case $|K'_2(R)| < |K'_1(R)|$. An increase in σ_E or β pushes down the function $K_1(R)$ and leaves the function $K_2(R)$ unchanged. The steady state is therefore characterized by a lower interest rate and a higher capital stock. Similarly, an increase in $1-\delta$ and B pushes up $K_2(R)$, also leading to a lower interest rate and a higher capital stock. We note that an increase in A or σ_P pushes down both $K_1(R)$ and $K_2(R)$ and can have ambiguous effects, as is verified in simulations.

Proof of Theorem 7

It is enough to provide an example of multiplicity. We numerically check that there exist three equilibria when $A = 1.2$, $B = 20$, $\alpha = 0.95$, $\beta = 0.05$, $\delta = 0.5$, $\rho = 1 - \delta$, $\sigma_P^2 = 85$, $\sigma_E = 0$, $e = 0$.

Proof of Theorem 9, 10 and 11: Local Stability of the Steady State

Around the steady state, we know that there is no storage, and the iterated function H is implicitly defined by:

$$\begin{aligned}
K_t &= \Phi^{-1}(W_{t+1}), \\
R_t &= F'(K_t)\left[B - A a_{t+1} F(K_t)\sigma_P^2\right] + 1 - \delta, \\
a_t &= 1/[1 + (a_{t+1}R_t)^{-1}], \\
C_t &= C_{t+1} - \ln(\beta R_t)/A - A a_{t+1}^2\left[F(K_t)^2\sigma_P^2 + \sigma_E^2\right]/2, \\
W_t &= C_t + K_t.
\end{aligned}$$

The function H has Jacobian

$$J = \begin{bmatrix} \frac{\partial a_t}{\partial a_{t+1}} & 0 & \frac{\partial a_t}{\partial W_{t+1}} \\ \frac{\partial C_t}{\partial a_{t+1}} & 1 & \frac{\partial C_t}{\partial W_{t+1}} \\ \frac{\partial C_t}{\partial a_{t+1}} & 1 & \frac{\partial (C_t + K_t)}{\partial W_{t+1}} \end{bmatrix}.$$

We observe that $\partial K_t/\partial W_{t+1} = 1/\Phi'(K_t) > 0$, and[33]

$$\begin{aligned}
\frac{\partial R_t}{\partial a_{t+1}} &= -AF(K_t)F'(K_t)\sigma_P^2 \leq 0, \\
\frac{\partial R_t}{\partial W_{t+1}} &= \frac{1}{\Phi'(K_t)}\left\{F''(K_t)\left[B - A a_{t+1}F(K_t)\sigma_P^2\right] - a_{t+1}[F'(K_t)]^2 A\sigma_P^2\right\} < 0.
\end{aligned}$$

Let $\varphi(v) = 1/(1 + v^{-1})$. We note that $\varphi(v) = v/(1+v)$ and therefore $\varphi'(v) = 1/(1+v)^2 = [\varphi(v)/v]^2$. Since $a_t = \varphi(a_{t+1}R_t)$, we infer that

$$\begin{aligned}
\frac{\partial a_t}{\partial a_{t+1}} &= \left(\frac{a_t}{a_{t+1}R_t}\right)^2\left(R_t - a_{t+1}\left|\frac{\partial R_t}{\partial a_{t+1}}\right|\right), \\
\frac{\partial a_t}{\partial W_{t+1}} &= -\left(\frac{a_t}{a_{t+1}R_t}\right)^2 a_{t+1}\left|\frac{\partial R_t}{\partial W_{t+1}}\right| < 0.
\end{aligned}$$

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[33] Since $R_\infty > 1$, we infer that $B - A a_{t+1} F(K_t)\sigma_P^2 > 0$ on a neighborhood of the steady state.

Future propensity a_{t+1} has an ambiguous effect on current propensity a_t. There is both a positive *direct* effect (due to the complementarity of future and current consumption) and a negative *indirect* effect (the precautionary motive causes a decline in the current interest rate R_t). Since $C_t = C_{t+1} - \ln(\beta R_t)/A - Aa_{t+1}^2 \left[F(K_t)^2\sigma_P^2 + \sigma_E^2\right]/2$, we infer that

$$\frac{\partial C_t}{\partial a_{t+1}} = \frac{1}{AR_t}\left|\frac{\partial R_t}{\partial a_{t+1}}\right| - Aa_{t+1}[F(K_t)^2\sigma_P^2 + \sigma_E^2]$$
$$\frac{\partial C_t}{\partial W_{t+1}} = \frac{1}{AR_t}\left|\frac{\partial R_t}{\partial W_{t+1}}\right| - Aa_{t+1}^2\sigma_P^2\frac{F(K_t)F'(K_t)}{\Phi'(K_t)}$$

An increase in a_{t+1} and W_{t+1} leads to a decline in the current interest rate, which has a positive effect on current consumption. On the other hand, the increase in a_{t+1} and W_{t+1} implies that the agent bears more risk between time t and time $t+1$. The precautionary motive can then lead to a decrease in current consumption and current wealth, which may generate endogenous fluctuations.

The characteristic polynomial $P(x) = \det(J - xI)$ can be rewritten as

$$P(x) = \left(\frac{\partial a_t}{\partial a_{t+1}} - x\right)\left\{x^2 - \left[1 + \frac{\partial(C_t + K_t)}{\partial W_{t+1}}\right]x + \frac{\partial K_t}{\partial W_{t+1}}\right\} + x\frac{\partial C_t}{\partial a_{t+1}}\frac{\partial a_t}{\partial W_{t+1}}.$$

The roots of P are the eigenvalues of the *backward* dynamical system.[34] Since $P(-\infty) = +\infty$ and $P(+\infty) = -\infty$, there always exists a real eigenvalue. Simple calculation shows that $P(1) > 0$ if and only if $|K_2'(R_\infty)| < |K_1'(R_\infty)|$. Thus when there is a unique steady state, the Jacobian matrix J has an eigenvalue in $(1, +\infty)$, and the dimension of the stable manifold is at least 1.

When markets are complete ($\sigma_P = \sigma_E = 0$), we know that

$$\frac{\partial a_t}{\partial a_{t+1}} = \beta,\ \frac{\partial C_t}{\partial a_{t+1}} = 0,\ \frac{\partial C_t}{\partial W_{t+1}} > 0,$$

and the characteristic polynomial is of the form $P(x) = (\beta - x)\,Q(x)$, where

$$Q(x) = x^2 - \left[1 + \frac{\partial(C_t + K_t)}{\partial W_{t+1}}\right]x + \frac{\partial K_t}{\partial W_{t+1}}.$$

This implies that $x = \beta$ is an eigenvalue, which is contained in the interval $(0, 1)$. We observe that $Q(0) > 0$ and $Q(1) < 0$, and infer the polynomial Q has one root

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[34] The eigenvalue λ considered in Section 5 thus satisfies $P(1/\lambda) = 0$.

in the interval $(0,1)$ and one root in $(1,+\infty)$. Overall, the Jacobian matrix J has two eigenvalues in the interval $(0,1)$, and one eigenvalue larger than 1. The stable manifold has thus dimension 1.

Consider the parameter values: $\beta = 0.95$, $A = 2$, $B = 10$, $\alpha = 0.35$, $\delta = \rho = 0.1$, $\sigma_E = 0$. We check numerically that the characteristic polynomial has one eigenvalue $x_1 > 1$ and two eigenvalues in the unit circle when $\sigma_P \leq 30.5$. On the other hand when $\sigma_P = 31.5$, the steady state is unique and there is one eigenvalue in each of the interval $(-\infty,-1)$, $(-1,1)$ and $(1,+\infty)$. This establishes Theorem 9. We also note that the system undergoes a flip bifurcation as σ_P varies between 30.5 and 31.5, which proves Theorem 11.

Finally, consider the economy $A = 1.2$, $B = 20$, $\alpha = 0.95$, $\beta = 0.05$, $\delta = 0.5$, $\rho = 1-\delta$, $\sigma_P^2 = 85$, $\sigma_E = 0$, $e = 0$ examined in the proof of Theorem 7. This economy has three steady states, and it can be checked numerically that the steady state with the intermediate interest rate is totally unstable, thus establishing Theorem 10.

Proof of Theorem 12: Analysis of the Calibrated Model

The steady state is defined by the system (5.1). Since $R_\infty > 1$, the second equation implies that $BF'(K_\infty) + 1 - \delta > 1$, or equivalently $K_\infty < K^* = (F')^{-1}(\delta/B)$. We find it useful to consider the decreasing function $\psi(R) = 2\gamma^{-2}(1-R^{-1})^{-2}\ln[1/(R\beta)]$, which maps $(1,\beta^{-1}]$ onto $[0,+\infty)$. Let

$$\chi(K) \equiv \frac{F(K)^2\sigma_P^2 + \sigma_E^2}{[\Phi(K)-K]^2}.$$

For any $K \in [0,\infty)$, the first equation of (5.1) has unique solution

$$R_1(K) = \psi^{-1}[\chi(K)] \in (1,\beta^{-1}].$$

It is easy to check that

$$\chi'(K) = 2\,\frac{\delta F(K)[F(K)-KF'(K)]\sigma_P^2 - [BF'(K)-\delta]\sigma_E^2}{[\Phi(K)-K]^3}.$$

The function $\chi(K)$ is thus increasing on $(0,K^*]$ when $\sigma_E = 0$. Simulations show that $\chi(K)$ and $R_1(K)$ can be non-monotonic when $\sigma_E > 0$.

When $K \to 0$, the function $\chi(K)$ converges to $(\sigma_P/B)^2$ if $\sigma_E = 0$, and diverges to $+\infty$ if $\sigma_E > 0$. As a result, $R_1(K)$ converges to

$$\begin{cases} \psi^{-1}[(\sigma_P/B)^2] & \text{if } \sigma_E = 0, \\ 1 & \text{if } \sigma_E > 0. \end{cases}$$

We also note that $\chi'(K^*) > 0$ and thus $R_1'(K^*) < 0$.

As in the proof of Theorem 6, we note that for every $K \in (0, K^*]$, there exists a unique $R_2 \geq 1$ such that the second equation of (5.1) holds. This number, denoted $R_2(K)$, is decreasing with K and satisfies $R_2(K^*) = 1$. In order to analyze the behavior of the function R_2 as $K \to 0$, we rewrite the second equation of the system as: $R^2 - \eta(K)R - \gamma\sigma_P^2 F'(K)\zeta(K) = 0$, where $\zeta(K) \equiv F(K)/[\Phi(K) - K]$ and $\eta(K) \equiv \Phi'(K) - \gamma\sigma_P^2 F'(K)\zeta(K)$. This quadratic equation in R has one negative root, and one admissible root larger than 1. We thus infer that

$$R_2(K) = \frac{1}{2}\left\{\eta(K) + \left[\eta(K)^2 + 4\gamma\sigma_P^2 F'(K)\zeta(K)\right]^{1/2}\right\}.$$

When $K \to 0$, note that $\zeta(K) \to B^{-1}$ and $\eta(K) \sim BF'(K)[1 - \gamma(\sigma_P/B)^2]$. We thus distinguish several cases:

1. If $\gamma(\sigma_P/B)^2 \leq 1$, the function $R_2(K)$ diverges to $+\infty$ as $K \to 0$.
2. When $\gamma(\sigma_P/B)^2 > 1$, the quantity

$$R_2(K) = \frac{|\eta(K)|}{2}\left\{-1 + \left[1 + 4\gamma\sigma_P^2 F'(K)\zeta(K)\eta(K)^{-2}\right]^{1/2}\right\}$$

converges to the finite limit $\frac{\gamma(\sigma_P/B)^2}{\gamma(\sigma_P/B)^2 - 1}$.

We now consider the difference $\Delta(K) = R_2(K) - R_1(K)$, and observe that $\Delta(K^*) = 1 - R_1(K^*) < 0$. When $\gamma(\sigma_P/B)^2 \leq 1$, the function $\Delta(K) \to +\infty$ as $K \to 0$; continuity then imposes the existence of a steady state. Similarly when $\gamma(\sigma_P/B)^2 > 1$ and $\sigma_E > 0$, the function $\Delta(K) \to \frac{\gamma(\sigma_P/B)^2}{\gamma(\sigma_P/B)^2 - 1} - 1 > 0$ as $K \to 0$ and there exists a steady state. Finally when $\gamma(\sigma_P/B)^2 > 1$ and $\sigma_E = 0$, the limit $\Delta(0) = \frac{\gamma(\sigma_P/B)^2}{\gamma(\sigma_P/B)^2 - 1} - \psi^{-1}[(\sigma_P/B)^2]$ has an ambiguous sign; a steady state may not exist, as is the case in some simulations.

Like in the proof of Theorem 8, we can now analyze the comparative statics of a steady state. An increase in σ_E^2 or β leaves $R_2(K)$ unchanged and pushes down $R_1(K)$. When $|R_2'(K_\infty)| > |R_1'(K_\infty)|$, this leads to a higher capital stock K_∞ and a smaller interest rate R_∞. Similarly, an increase in B and $1 - \delta$ pushes up $R_2(K)$ and has no effect on $R_1(K)$, leading to a higher K_∞ and a smaller R_∞.

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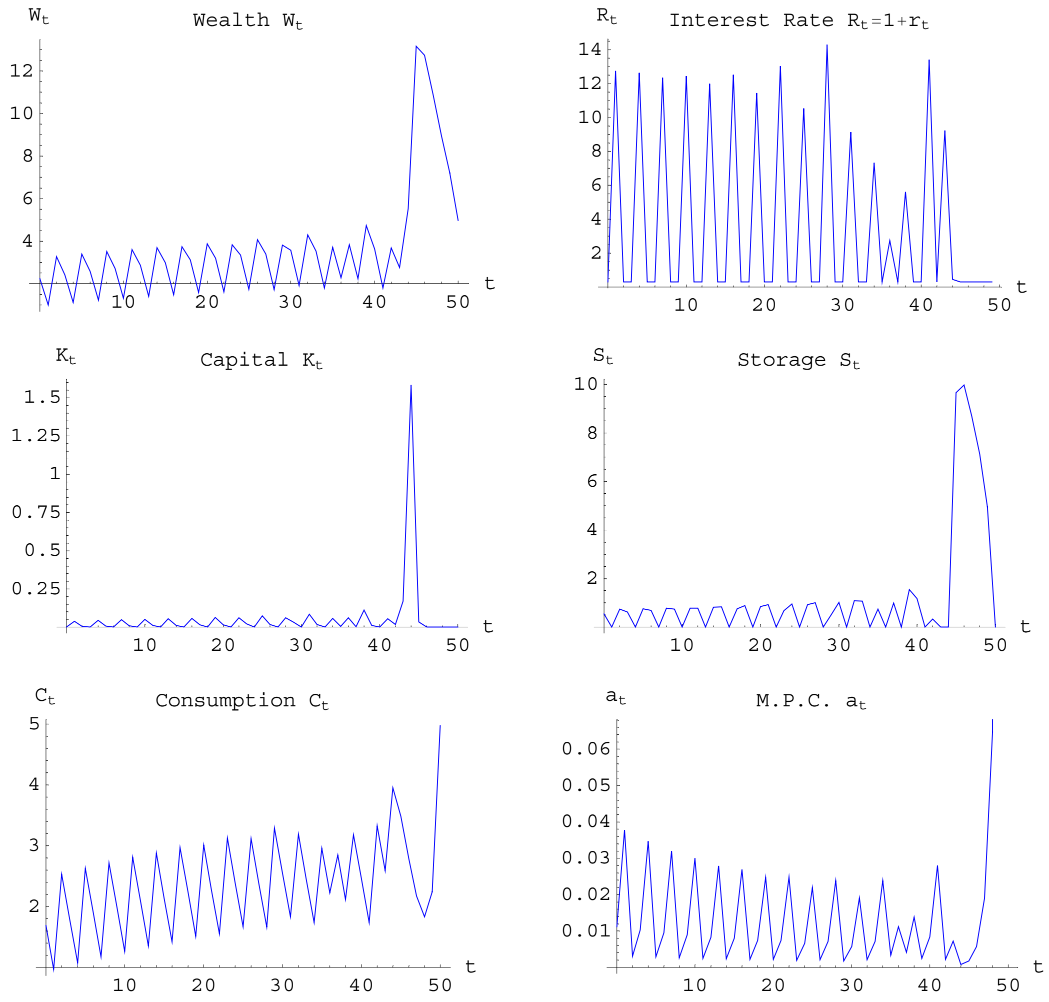


Wealth W_t

Interest Rate $R_t=1+r_t$

Capital K_t

Storage S_t

Consumption C_t

M.P.C. a_t

Figure 1: A=1.625, B=10, α=0.35, β=0.99, δ=0.1, ρ=0.3, e=0, σ_E=0, σ_P=60, T=50.

Figure 1

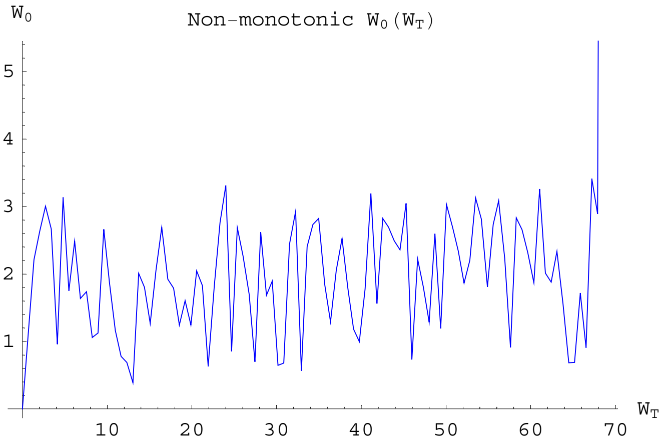


Figure 2: A=10, B=10, α=0.35, β=0.9, δ=0.5, ρ=0.5, e=0, σ_E=0, σ_P=100, T=40.

Figure 2

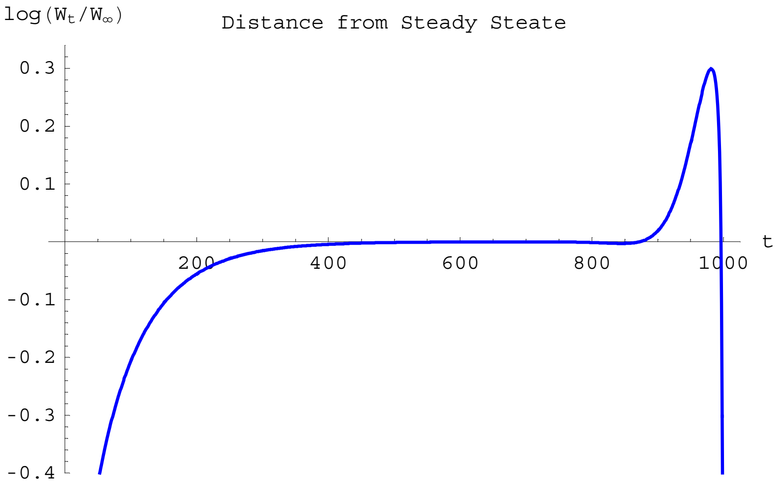
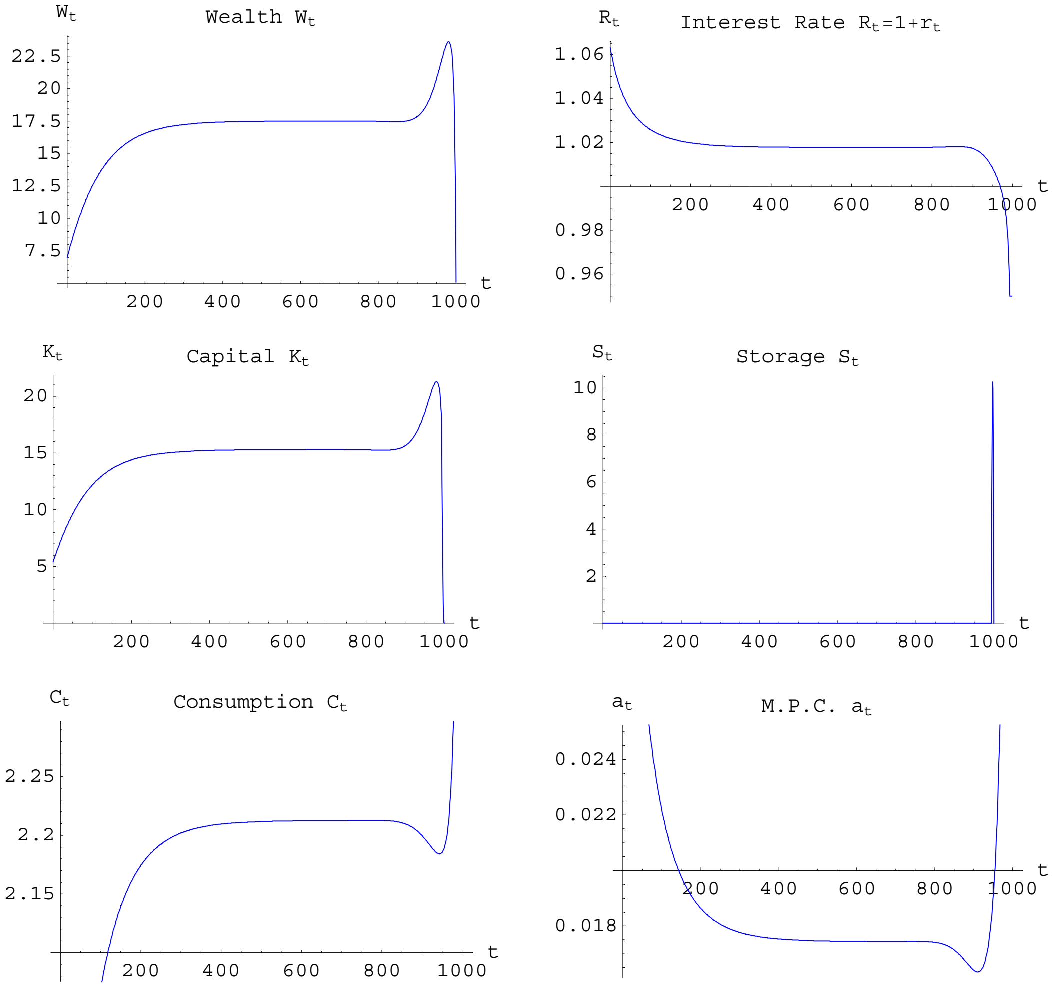


Figure 3: A=10, B=1, α=0.4, β=0.95, δ=0.05, ρ=0.95, e=0, σ_E=0, σ_P=0.5, T=1000.

Figure 3


Wealth W_t

Interest Rate $R_t=1+r_t$

Capital K_t

Storage S_t

Consumption C_t

M.P.C. a_t

Figure 4: A=10, B=1, α=0.4, β=0.95, δ=0.05, ρ=0.95, e=0, σ_E=0, σ_P=0.5, T=1000.

Figure 4

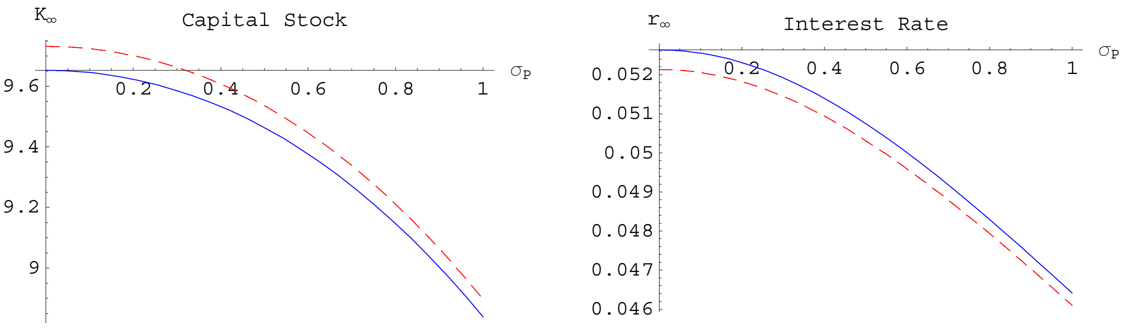


Figure 5: γ=2, β=0.95, α=0.4, δ=0.05, and σ_E=0 (solid) or σ_E=0.25 (dashed)

Figure 5

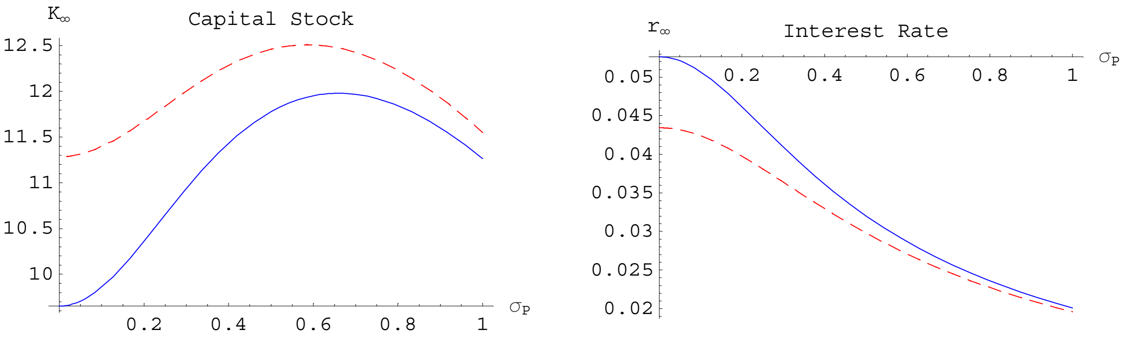


Figure 6: γ=10, β=0.95, α=0.4, δ=0.05, and σ_E=0 (solid) or σ_E=0.25 (dashed)

Figure 6

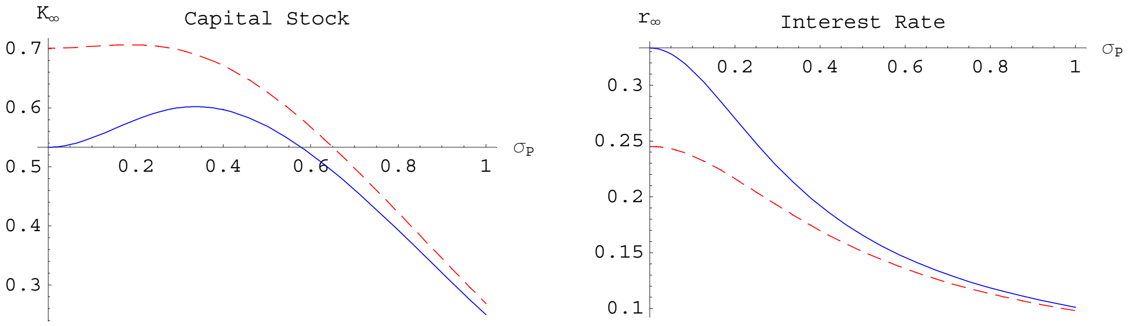


Figure 7: γ=6, β=0.75, α=0.4, δ=0.25, and σ_E=0 (solid) or σ_E=0.25 (dashed)

Figure 7

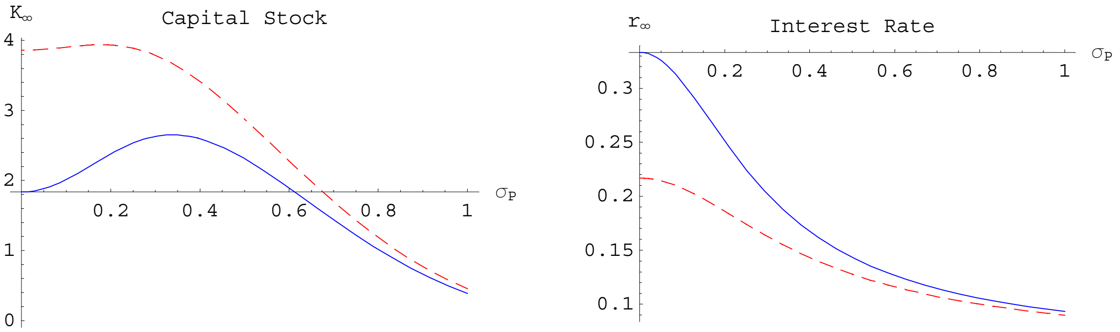


Figure 8: γ=6, β=0.75, α=0.7, δ=0.25, and σ_E=0 (solid) or σ_E=0.25 (dashed)

Figure 8

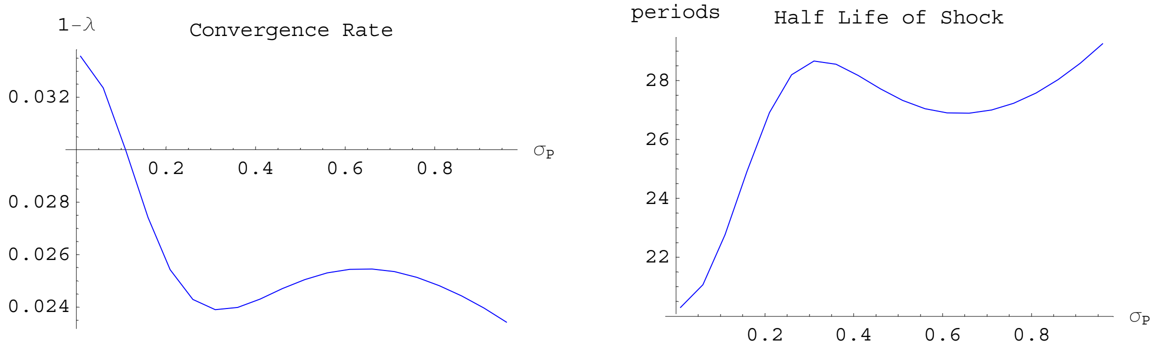


Figure 9: γ=6, β=0.75, α=0.7, δ=0.25, and σ_E=0.

Figure 9