Appendix

Contains proofs for "Moment Inequalities and Their Application" by Pakes, Porter, Ho, and Ishii

Let l denote a vector of ones.

Proofs

Proof of Theorem 1:

Let $\hat{\underline{\theta}} \in \hat{\underline{\Theta}}$. Given $\epsilon > 0$, show that $\Pr(\|\hat{\underline{\theta}} - \underline{\theta}\| < \epsilon) \longrightarrow 1$. By Lemma A2, there exists $0 < \delta < \epsilon$ such that $\{\theta \in \Theta_0 : |\theta_1 - \underline{\theta}_1| < \delta\} \subset \{\theta \in \Theta_0 : \|\theta - \underline{\theta}\| < \epsilon/2\}$. Consider $\theta \in \Theta_0^{\delta/2} \cap \{\theta' \in \Theta : \theta'_1 < \underline{\theta}_1 + \delta/2\}$. There exists $\theta' \in \Theta_0$ such that $\|\theta - \theta'\| \leq \delta/2$. But for $\theta' \in \Theta_0$, we must have $\theta'_1 \geq \underline{\theta}_1$. Hence

$$\underline{\theta}_1 \leq \theta'_1 \leq \theta_1 + \delta/2 < \underline{\theta}_1 + \delta.$$

So $\|\theta'_1 - \underline{\theta}_1\| < \delta$. Since $\theta' \in \Theta_0$, it follows by the definition of δ that $\|\theta' - \underline{\theta}\| < \epsilon/2$. So,

$$\|\theta - \underline{\theta}\| \leq \|\theta - \theta'\| + \|\theta' - \underline{\theta}\| < \delta/2 + \epsilon/2 < \epsilon.$$

Hence, $\Theta_0^{\delta/2} \cap \{\theta' \in \Theta : \theta'_1 < \underline{\theta}_1 + \delta/2\} \subset \{\theta : \|\theta - \underline{\theta}\| < \epsilon\}$ and

$$\Pr(\|\hat{\underline{\theta}} - \underline{\theta}\| < \epsilon) \geq \Pr(\{\hat{\underline{\theta}}_1 \leq \underline{\theta}_1 + \delta/2\} \cap \{\hat{\underline{\theta}} \in \Theta_0^{\delta/2}\}).$$

It then remains to show this last probability approaches one.

By Assumption A3, there exists $\delta' > 0$ such that $\inf_{\theta \in (\Theta_0^{\delta/2})^c} \|\mathcal{P}m(z,\theta)_-\| > \delta'$. Then, by Lemma A1 and Assumption A5, $\Pr\left(\inf_{\theta \in (\Theta_0^{\delta/2})^c} \|P_J m(z,\theta)_-\| \geq \delta'/2\right) \longrightarrow 1$. And, for any $\theta' \in \Theta_0$, $\Pr\left(\|P_J m(z,\theta')_-\| \leq \delta'/2\right) \longrightarrow 1$. Hence, $\Pr(\hat{\underline{\theta}} \in \Theta_0^{\delta/2}) \longrightarrow 1$.

By Assumption A4, there exists θ' with $\|\theta' - \underline{\theta}\| < \delta/4$ such that $\mathcal{P}m(z,\theta') > 0$. Since $\theta' \in \Theta_0$, $\underline{\theta}_1 \leq \theta_1{}' \leq \underline{\theta}_1 + \delta/4$. By Assumption A5, $\Pr(\theta' \in \hat{\Theta}) \geq \Pr(P_J m(z,\theta') \geq 0) \longrightarrow 1$.

Hence, with probability approaching one, $\hat{\underline{\theta}}_1 \leq {\theta_1}' + \delta/4 \leq \underline{\theta}_1 + \delta/2$.

The result follows as argued above. $\square$

For the proof of Theorem 2, define θ_1^* by $\theta_1^* = \min_{\{} \theta_1 : \underline{\Gamma}_0(\theta - \underline{\theta}) + P_J m_0(z, \underline{\theta}) \geq 0\}$ and take θ^* to be any vector satisfying $\underline{\Gamma}_0(\theta^* - \underline{\theta}) + P_J m_0(z, \underline{\theta}) \geq 0$ with first element θ_1^*. If the binding moments are unknown, then θ^* is infeasible.

Lemma 1 $\sqrt{J}(\hat{\underline{\theta}} - \underline{\theta}) = O_p(1)$.

Proof:

First, we will show that $P_J m(z, \underline{\theta} + c\lambda/\sqrt{J}) \geq 0$ with probability approaching one. Then, $\hat{\underline{\theta}}_1 \leq \underline{\theta}_1 + c\lambda_1/\sqrt{J} + o_p(1/\sqrt{J})$ yields $\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq c\lambda_1 + o_p(1/\sqrt{J})$. Second, show $\mathcal{P}m(z, \hat{\underline{\theta}} + c\lambda/\sqrt{J}) \geq 0$ with probability approaching one. Third, note that the second finding implies that $\hat{\underline{\theta}} + c\lambda/\sqrt{J} \in \Theta_0$ with probability approaching one, so by Lemma A4,

$$\begin{aligned} \|\hat{\underline{\theta}} - \underline{\theta} + c\lambda/\sqrt{J}\|\bar{\delta} &\leq \hat{\underline{\theta}}_1 - \underline{\theta}_1 + c\lambda_1/\sqrt{J} \\ &\leq 2c\lambda_1/\sqrt{J} + o_p(1/\sqrt{J}) \end{aligned}$$

So,

$$\|\sqrt{J}(\hat{\underline{\theta}} - \underline{\theta})\| \leq \|\sqrt{J}(\hat{\underline{\theta}} - \underline{\theta}) + c\lambda\| + \| - c\lambda\| \leq 2c\lambda/\bar{\delta} + \|c\lambda\| + o_p(1/\sqrt{J}).$$

By Assumption A7, we can choose c such that $c\underline{\Gamma}_0\lambda - \|\sqrt{J}P_J m_0(z, \underline{\theta})\|l > 0$ with probability as close to one as desired for large enough J.

$$\begin{aligned} P_J m_1(z, \underline{\theta} + c\lambda/\sqrt{J}) &\geq \mathcal{P}m_1(z, \underline{\theta}) - \|\mathcal{P}m_1(z, \underline{\theta} + c\lambda/\sqrt{J}) - \mathcal{P}m_1(z, \underline{\theta})\|l \\ &\quad - \|P_J m_1(z, \underline{\theta} + c\lambda/\sqrt{J}) - \mathcal{P}m_1(z, \underline{\theta} + c\lambda/\sqrt{J})\|l \end{aligned}$$

Then, by Assumptions A5 and A6(b), $\Pr(P_J m_1(z, \underline{\theta} + c\lambda/\sqrt{J}) > 0) \longrightarrow 1$. Also,

$$\begin{aligned} &\sqrt{J}P_J m_0(z, \underline{\theta} + c\lambda/\sqrt{J}) = c\underline{\Gamma}_0\lambda + \sqrt{J}P_J m_0(z, \underline{\theta}) \\ &\quad + \sqrt{J}[\mathcal{P}m_0(z, \underline{\theta} + c\lambda/\sqrt{J}) - \mathcal{P}m_0(z, \underline{\theta})] - c\underline{\Gamma}_0\lambda \\ &\quad + \sqrt{J}[P_J m_0(z, \underline{\theta} + c\lambda/\sqrt{J}) - \mathcal{P}m_0(z, \underline{\theta} + c\lambda/\sqrt{J}) - (P_J m_0(z, \underline{\theta}) - \mathcal{P}m_0(z, \underline{\theta}))] \end{aligned}$$

So by choice of c and Assumptions A6(a) and A8, $\Pr(\sqrt{J}P_J m_0(z,\underline{\theta}+c\lambda/\sqrt{J})\geq 0)$ can be as close to one as desired for large enough J. The same conclusion then follows for $\Pr(\sqrt{J}P_J m(z,\underline{\theta}+c\lambda/\sqrt{J})\geq 0)$, and note that $\sqrt{J}P_J m(z,\underline{\theta}+c\lambda/\sqrt{J})\geq 0$ implies $\hat{\underline{\theta}}_1 \leq \underline{\theta}_1 + c\lambda_1/\sqrt{J}+o_p(1/\sqrt{J})$.

By continuity of $\mathcal{P}m_1(z,\theta)$ at $\underline{\theta}$ and consistency of $\hat{\underline{\theta}}$, $\Pr(\mathcal{P}m_1(z,\hat{\underline{\theta}}+c\lambda/\sqrt{J})>0)\longrightarrow 1$. Also,

$$\begin{aligned}\sqrt{J}\mathcal{P}m_0(z,\hat{\underline{\theta}}+c\lambda/\sqrt{J}) &\geq \sqrt{J}P_J m_0(z,\hat{\underline{\theta}})+c\underline{\Gamma}_0\lambda - \|\sqrt{J}P_J m_0(z,\underline{\theta})\|l\\ &-\|\left[\frac{\partial}{\partial\theta}\mathcal{P}m_0(z,\hat{\underline{\theta}})-\frac{\partial}{\partial\theta}\mathcal{P}m_0(z,\underline{\theta})\right]c\lambda\|l\\ &-\|\sqrt{J}(\mathcal{P}m_0(z,\hat{\underline{\theta}}+c\lambda/\sqrt{J})-\mathcal{P}m_0(z,\hat{\underline{\theta}}))-\frac{\partial}{\partial\theta}\mathcal{P}m_0(z,\hat{\underline{\theta}})c\lambda\|l\\ &-\sqrt{J}\|P_J m_0(z,\hat{\underline{\theta}})-\mathcal{P}m_0(z,\hat{\underline{\theta}})-(P_J m_0(z,\underline{\theta})-\mathcal{P}m_0(z,\underline{\theta}))\|l.\end{aligned}$$

From the proof of Theorem 1, $\Pr(P_J m_0(z,\hat{\underline{\theta}})\geq 0)\longrightarrow 1$. By the choice of c, Assumption A6(a) and A8, $\Pr(\mathcal{P}m_0(z,\hat{\underline{\theta}}+c\lambda/\sqrt{J})\geq 0)$ can be as close to one as desired for large enough J, and the same conclusion then follows for $\Pr(\mathcal{P}m(z,\hat{\underline{\theta}}+c\lambda/\sqrt{J})\geq 0)$.

Note that $\mathcal{P}m(z,\hat{\underline{\theta}}+c\lambda/\sqrt{J})\geq 0$ implies that $\underline{\theta}+c\lambda/\sqrt{J}\in\Theta_0$. The latter event, by Lemma A4, implies $\|\hat{\underline{\theta}}-\underline{\theta}+c\lambda/\sqrt{J}\|\bar{\delta}\leq \hat{\underline{\theta}}_1-\underline{\theta}_1+c\lambda_1/\sqrt{J}$. The result follows by the triangle inequality as described above. □

Lemma 2 $\sqrt{J}(\theta^*-\underline{\theta})=O_p(1)$.

Proof:

It is straightforward to show that θ^* exists a.s. by Assumption A9. Recall the choice of c from the proof of Lemma 1. Note that $c\underline{\Gamma}_0\lambda+\sqrt{J}P_J m_0(z,\underline{\theta})\geq c\underline{\Gamma}_0\lambda-\|\sqrt{J}P_J m_0(z,\underline{\theta})\|l\geq 0$ implies that $\sqrt{J}(\theta_1^*-\underline{\theta}_1)\leq c\lambda_1$. Also, $c\underline{\Gamma}_0\lambda-\sqrt{J}P_J m_0(z,\underline{\theta})\geq c\underline{\Gamma}_0\lambda-\|\sqrt{J}P_J m_0(z,\underline{\theta})\|l\geq 0$ and $\underline{\Gamma}_0\sqrt{J}(\theta^*-\underline{\theta})+\sqrt{J}P_J m_0(z,\underline{\theta})\geq 0$ imply $\underline{\Gamma}_0[\sqrt{J}(\theta^*-\underline{\theta})+c\lambda]\geq 0$. By Lemma A3, this last event would imply $\|\sqrt{J}(\theta^*-\underline{\theta})+c\lambda\|\bar{\delta}\leq\sqrt{J}(\theta_1^*-\underline{\theta}_1)+c\lambda_1$. Then using the triangle inequality, for large enough J,

$$
\begin{aligned}
&\Pr\left(\|\sqrt{J}(\theta^* - \underline{\theta})\| \leq \frac{2c\lambda_1}{\bar{\delta}} + \|c\lambda\| \right) \\
&\geq \Pr(\{\bar{\delta}\|\sqrt{J}(\theta^* - \underline{\theta}) + c\lambda\| \leq \sqrt{J}(\theta_1^* - \underline{\theta}_1) + c\lambda_1\} \cap \{\sqrt{J}(\theta_1^* - \underline{\theta}_1) \leq c\lambda_1\}) \\
&\geq \Pr(c\underline{\Gamma}_0\lambda - \|\sqrt{J}P_J m_0(z, \underline{\theta})\| l \geq 0)
\end{aligned}
$$

By choice of c, the result follows. □

Lemma 3 $\sqrt{J}(\hat{\underline{\theta}}_1 - \theta_1^*) = o_p(1)$

Proof:

Let $L_J(\theta) = \underline{\Gamma}_0\sqrt{J}(\theta - \underline{\theta}) + \sqrt{J}P_J m_0(z, \underline{\theta})$.

First, show there exists $h_J = o(1)$ such that $L_J(\hat{\underline{\theta}} + h_J\lambda/\sqrt{J}) \geq \sqrt{J}P_J m_0(z, \hat{\underline{\theta}})$ with probability approaching one. Since $\sqrt{J}P_J m_0(z, \hat{\underline{\theta}}) \geq 0$ with probability approaching one (from the proof of consistency), we have $\theta_1^* \leq \hat{\underline{\theta}}_1 + h_J\lambda_1/\sqrt{J}$ with probability approaching one.

Second, show there exists $r_J = o(1)$ such that $\sqrt{J}P_J m_0(z, \theta^* + r_J\lambda/\sqrt{J}) \geq L_J(\theta^*)$ with probability approaching one. Since $L_J(\theta^*) \geq 0$ by the definition of θ^*, and $P_J m_1(z, \theta^* + r_J\lambda/\sqrt{J}) > 0$ with probability approaching one by consistency of θ^*, we have $\hat{\underline{\theta}}_1 \leq \theta_1^* + r_J\lambda_1/\sqrt{J} + o_p(1/\sqrt{J})$ with probability approaching one.

Then, $-h_J\lambda_1/\sqrt{J} \leq \hat{\underline{\theta}}_1 - \theta_1^* \leq r_J\lambda_1/\sqrt{J} + o_p(1/\sqrt{J})$ with probability approaching one. Since $r_J = o(1)$ and $h_J = o(1)$, it follows that $\sqrt{J}(\hat{\underline{\theta}}_1 - \theta_1^*) = o_p(1)$.

By Lemma 1, Lemma A5, and Assumption A8,

$$L_J(\hat{\underline{\theta}} + \lambda h_J/\sqrt{J}) - \sqrt{J}P_J m_0(z, \hat{\underline{\theta}}) = \underline{\Gamma}_0\lambda h_J + o_p(1).$$

Since $\sqrt{J}P_J m_0(z, \hat{\underline{\theta}}) \geq 0$ with probability approaching one, we can choose $h_J \downarrow 0$ such that $L_J(\hat{\underline{\theta}} + h_J\lambda/\sqrt{J}) \geq 0$ and hence $\theta_1^* \leq \hat{\underline{\theta}}_1 + h_J\lambda_1/\sqrt{J}$ with probability approaching one.

By Lemmas A6 and A7, we can choose $r_J \downarrow 0$ such that

$$\sqrt{J}P_J m_0(z, \theta^* + \frac{\lambda}{\sqrt{J}}r_J) - L_J(\theta^*) = \underline{\Gamma}_0\lambda r_J + o_p(1)$$

and $P_J m_0(z, \theta^* + \frac{\lambda}{\sqrt{J}} r_J) \geq 0$ with probability approaching one. By Lemma 1, $P_J m_1(z, \theta^* + \frac{\lambda}{\sqrt{J}} r_J) \geq 0$ and hence $P_J m(z, \theta^* + \frac{\lambda}{\sqrt{J}} r_J) \geq 0$ with probability approaching one. It follows that $\hat{\underline{\theta}}_1 \leq \theta_1^* + r_J \lambda_1 / \sqrt{J} + o_p(1)$ with probability approaching one. The choices of r_J and h_J then prove the result. □

Proof of Theorem 2:

By Lemma 3, it remains only to show that $\sqrt{J}(\theta_1^* - \underline{\theta}_1) \xrightarrow{d} \hat{\tau}_1$. Continuity of the optimal value of the linear program is shown in Lemma A8. The result then follows by the definition of θ_1^*, Assumption A7, and the continuous mapping theorem. □

For the proof of Theorem 3, we define the following infeasible simulation estimators

$$\tilde{\tau}_1 = \min\{\tau_1 : \tau \in \tilde{\mathfrak{T}}_J\} \text{ where } \tilde{\mathfrak{T}}_J = \arg\min_{\tau} \| \left(\hat{\underline{\Gamma}}_0 \tau + Z_0^*\right)_- \|$$

and

$$\tilde{\tilde{\tau}}_1 = \min\{\tau_1 : \tau \in \tilde{\tilde{\mathfrak{T}}}_J\} \text{ where } \tilde{\tilde{\mathfrak{T}}}_J = \arg\min_{\tau} \| \left(\hat{\underline{\Gamma}}_0 \tau + Z_0^* + \sqrt{J} P_J m_0(z, \hat{\underline{\theta}})\right)_- \|$$

with corresponding $\tilde{\tau}$ and $\tilde{\tilde{\tau}}$ and quantiles $\tilde{q}_{\alpha,J}$ and $\tilde{\tilde{q}}_{\alpha,J}$ in analogy to the notation in the paper. Also, let q_α denote the α^{th} quantile of the limit distribution $\hat{\tau}_1$.

Also, define $\tau_1^{lp}(\Gamma, Z) = \min_{\{} \tau_1 : 0 \leq \Gamma\tau + Z\}$. For matrix A and $\delta > 0$, let $\mathcal{N}_\delta^A = \{B : \|B - A\| < \delta\}$.

Proof of Theorem 3:

First, show that $\lim_{J \longrightarrow \infty} \Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{q}_{\alpha,J}) = \alpha$. Given any $\varepsilon > 0$, by Lemma A9, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q_{\alpha - \varepsilon/4}) \geq \alpha - \varepsilon/2$ for large enough J. Then by the continuity of quantiles in Lemma A10 and consistency of $(\hat{\underline{\Gamma}}_0, \hat{\underline{\Sigma}}_0)$, $\Pr(q_{\alpha-\varepsilon/4} \leq \tilde{q}_{\alpha,J}) \geq 1 - \varepsilon/2$ for large enough J. Hence, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{q}_{\alpha,J}) \geq \alpha - \varepsilon$ for large enough J. We can similarly bound the probability from above to find that $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{q}_{\alpha,J}) \longrightarrow \alpha$.

Since m_a contains m_0, the binding moments, $\tilde{\tau}_1|P_J$ stochastically dominates $\tau_1^{**}|P_J$ for each J. So, in the argument above, we also have $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q^{**}_{\alpha,J}) \geq 1 - \varepsilon/2$ for large enough J. The first conclusion of the theorem follows.

As an additional result, with an extra assumption we can show that the limit on the left of the inequality in the first conclusion of the theorem is nontrivial for $\alpha \in (0,1)$. That is, $\liminf_{J\longrightarrow\infty} \Pr\left(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q^{**}_{\alpha,J}\right) < 1$. The extra assumption is an expansion of Assumption A9: there exists some λ such that $\underline{\Gamma}_a\lambda > 0$. Fix α. Also, note that $\min\{\tau_1 : \underline{\Gamma}_a\tau \geq 0\} \geq \min\{\tau_1 : \underline{\Gamma}_0\tau \geq 0\}$, so the uniqueness of the minimum in Assumption A9 must also hold for Γ_a in a neighborhood of $\underline{\Gamma}_a$. The extra assumption assures existence of a solution to $\min\{\tau_1 : \Gamma_a\tau + Z_a^* \geq 0\}$. Then, by the argument given in the proof of Lemma A8, the solutions to this linear program, $\min\{\tau_1 : \Gamma_a\tau + Z_a^* \geq 0\}$, are uniformly bounded for $(\Gamma_a, \Sigma_a) \in \mathcal{N}_{\delta_b}^{\underline{\Gamma}_a,\underline{\Sigma}_a}$ and $\|Z_a^*\| \leq C_z$ for some $\delta_b > 0$ and any $C_z > 0$. Choose $\delta < \delta_b$ and C_z such that $\Pr^*(\|Z_a^*\| \leq C_z) \geq (1+\alpha)/2$ for all $Z_a^* \sim N(0,\Sigma)$ and $\Sigma \in \mathcal{N}_{\delta}^{\underline{\Sigma}_a}$. Let C be the uniform bound on the linear program solution on this set. So, if $(\Gamma_a, \Sigma_a) \in \mathcal{N}_{\delta}^{\underline{\Gamma}_a,\underline{\Sigma}_a}$, then $\Pr^*(\tau_1^{lp}(\Gamma_a, Z_a^*) \leq C) \geq \Pr^*(\|Z_a^*\| \leq C_z) \geq (1+\alpha)/2$ for $Z_a^* \sim N(0, \Sigma_a)$. It follows that $\Pr(q^{**}_{\alpha,J} > C) \leq \Pr((\hat{\Gamma}_a, \hat{\Sigma}_a) \notin \mathcal{N}_{\delta}^{\underline{\Gamma}_a,\underline{\Sigma}_a})$.

Now let $\beta > 0$ be such that for large enough J, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq C) \geq 1 - \beta$. Take J large enough that $\Pr((\hat{\Gamma}_a, \hat{\Sigma}_a) \in \mathcal{N}_{\delta}^{\underline{\Gamma}_a,\underline{\Sigma}_a}) > 1 - \beta/2$. So, for J large enough, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q^{**}_{\alpha,J}) \leq \Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq C) + \Pr(q^{**}_{\alpha,J} > C) \leq (1-\beta) + \beta/2 = 1 - \beta/2$.

Next, consider $\tilde{\tau}_1$ and $\tilde{\tilde{\tau}}_1$. If $P_J m_0(z, \hat{\underline{\theta}}) \geq 0$, then $\tilde{\tilde{\tau}}_1|P_J$ stochastically dominates $\tilde{\tau}_1|P_J$. From the proof of Theorem 1, $\Pr(P_J m(z, \hat{\underline{\theta}}) \geq 0) \longrightarrow 1$. Hence, for large enough J, $\Pr(\tilde{\tilde{q}}_{\alpha,J} \leq \tilde{q}_{\alpha,J}) \geq \Pr(P_J m_0(z, \hat{\underline{\theta}}) \geq 0) \geq \Pr(P_J m(z, \hat{\underline{\theta}}) \geq 0) > 1 - \varepsilon$. So for J large enough, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{q}_{\alpha,J}) + \Pr(\tilde{\tilde{q}}_{\alpha,J} > \tilde{q}_{\alpha,J}) \leq \alpha + 2\varepsilon$.

Next, consider $\tilde{\tilde{\tau}}_1$ and τ_1^* and show that for given $\varepsilon > 0$, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q^*_{\alpha,J}) \leq \alpha + 6\varepsilon$ for large enough J.

Note that

$$\begin{aligned}
\sqrt{J}P_J m_0(z, \hat{\underline{\theta}}) \quad = \quad & \underline{\Gamma}_0\sqrt{J}(\hat{\underline{\theta}} - \underline{\theta}) + \sqrt{J}P_J m_0(z, \underline{\theta}) \\
& + \sqrt{J}[\mathcal{P}m_0(z, \hat{\underline{\theta}}) - \mathcal{P}m_0(z, \underline{\theta})] - \underline{\Gamma}_0\sqrt{J}(\hat{\underline{\theta}} - \underline{\theta}) \\
& + \sqrt{J}[P_J m_0(z, \hat{\underline{\theta}}) - \mathcal{P}m_0(z, \hat{\underline{\theta}}) - (P_J m_0(z, \underline{\theta}) - \mathcal{P}m_0(z, \underline{\theta}))].
\end{aligned}$$

The expressions in the last two lines are $o_p(1)$ from the proof of Lemma 3. By Assumption A7 and Lemma 1, $\sqrt{J}P_J m_0(z,\hat{\underline{\theta}}) = O_p(1)$. Similarly, using Assumptions A6′-A8′, $\sqrt{J}P_J m_1(z,\hat{\underline{\theta}}) - \sqrt{J}\mathcal{P}m_1(z,\underline{\theta}) = O_p(1)$.

Take C_z large enough and $\delta_\Sigma > 0$ small enough that for all $\Sigma \in \mathcal{N}^{\Sigma}_{\delta_\Sigma}$ and $Z^* \sim N(0,\Sigma)$, $\Pr(\|Z^*\| \leq C_z) \geq 1-\varepsilon$. Choose J_a,C_0, $C_1 > 0$ such that for $J \geq J_a$, $\Pr(\|\sqrt{J}P_J m_0(z,\hat{\underline{\theta}})\| \leq C_0) \geq 1-\varepsilon$ and $\Pr(\|\sqrt{J}P_J m_1(z,\hat{\underline{\theta}}) - \sqrt{J}\mathcal{P}m_1(z,\underline{\theta})\| \leq C_1) \geq 1-\varepsilon$.

From the proof of Lemma A8, there exists $\delta_\Gamma > 0$ such that the solution $\bar{\tau}$ to the linear program $\min\{\tau_1 : 0 \leq \Gamma_0\tau + Z_0\}$ is uniformly bounded if $\Gamma_0 \in \mathcal{N}^{\Gamma_0}_{\delta_\Gamma}$ and $\|Z_0\| \leq C_z + C_0$. Let C_τ denote that uniform bound on $\bar{\tau}$ and let $C_2 = \sup\{\|\Gamma_1\tau\| : \|\tau\| \leq C_\tau, \Gamma_1 \in \mathcal{N}^{\Gamma_1}_{\delta_\Gamma}\}$, which is finite.

Take J_b large enough that $\sqrt{J}\mathcal{P}m_1(z,\underline{\theta}) \geq (C_z + C_1 + C_2)l$ for $J \geq J_b$. Take J_c such that $\Pr((\hat{\Gamma},\hat{\Sigma}) \in \mathcal{N}^{\Gamma}_{\delta_\Gamma} \times \mathcal{N}^{\Sigma}_{\delta_\Sigma}) \geq 1-\varepsilon$ for $J \geq J_c$. From the argument above, there exists J_d such that for $J \geq J_d$, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{\tilde{q}}_{\alpha+\varepsilon,J}) \leq \alpha + 3\varepsilon$. Let $\bar{J} = \max\{J_a, J_b, J_c, J_d\}$. Also let $\mathcal{A} = \{\|\sqrt{J}P_J m_0(z,\hat{\underline{\theta}})\| \leq C_0\} \cap \{\|\sqrt{J}P_J m_1(z,\hat{\underline{\theta}}) - \sqrt{J}\mathcal{P}m_1(z,\underline{\theta})\| \leq C_1\} \cap \{(\hat{\Gamma},\hat{\Sigma}) \in \mathcal{N}^{\Gamma}_{\delta_\Gamma} \times \mathcal{N}^{\Sigma}_{\delta_\Sigma}\}$. Note that $\Pr(\mathcal{A}) \geq 1-3\varepsilon$ for $J \geq \bar{J}$.

Suppose $\mathcal{A}$ holds and $J \geq \bar{J}$. If $\|Z_0^*\| \leq C_z$, then $\|\tilde{\tilde{\tau}}\| \leq C_\tau$. If $\|Z_1^*\| \leq C_z$, then $\hat{\Gamma}_1\tilde{\tilde{\tau}} + Z_1^* + \sqrt{J}P_J m_1(z,\hat{\underline{\theta}}) \geq \sqrt{J}\mathcal{P}m_1(z,\underline{\theta}) - (\|\hat{\Gamma}_1\tilde{\tilde{\tau}}\| + \|Z_1^*\| + \|\sqrt{J}P_J m_1(z,\hat{\underline{\theta}}) - \sqrt{J}\mathcal{P}m_1(z,\underline{\theta}))\|)l \geq 0$. Hence $\tau_1^* = \tilde{\tilde{\tau}}_1$. Also, if $\mathcal{A}$ holds and $J \geq \bar{J}$, then

$$\begin{aligned}
\Pr{}^*(\tau_1^* \leq \tilde{\tilde{q}}_{\alpha+\varepsilon,J}) &\geq \Pr{}^*(\{\tau_1^* \leq \tilde{\tilde{q}}_{\alpha+\varepsilon,J}\} \cap \{\|Z_0^*\| \leq C_z\} \cap \{\|Z_1^*\| \leq C_z\}) \\
&= \Pr{}^*(\{\tilde{\tilde{\tau}}_1 \leq \tilde{\tilde{q}}_{\alpha+\varepsilon,J}\} \cap \{\|Z_0^*\| \leq C_z\} \cap \{\|Z_1^*\| \leq C_z\}) \\
&\geq \Pr{}^*(\{\tilde{\tilde{\tau}}_1 \leq \tilde{\tilde{q}}_{\alpha+\varepsilon,J}\} \cap \{\|Z^*\| \leq C_z\}) \\
&\geq (\alpha+\varepsilon) + (1-\varepsilon) - 1 = \alpha.
\end{aligned}$$

So, $\tilde{\tilde{q}}_{\alpha+\varepsilon,J} \geq q^*_{\alpha,J}$. Finally, for $J \geq \bar{J}$,

$$\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q^*_{\alpha,J}) \leq \Pr(\{\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{\tilde{q}}_{\alpha+\varepsilon,J}\}) + \Pr(\mathcal{A}^c) \leq \alpha + 6\epsilon.$$

Result (b) follows. □

And define infeasible estimators for the linear case,

$$\tilde{\tau}_1^L = \min\{\tau_1 : \tau \in \tilde{\mathfrak{T}}_J^L\} \quad \text{where} \quad \tilde{\mathfrak{T}}_J^L = \arg\min_\tau \| \left(\bar{w}_{1,0} + y^*_{1,0}/\sqrt{J})\tau + (y^*_{1,0}\hat{\underline{\theta}} + y^*_{2,0} \right) \|$$

and

$$\tilde{\tilde{\tau}}_1^L = \min\{\tau_1 : \tau \in \tilde{\tilde{\mathfrak{T}}}_J^L\}$$
$$\text{where} \quad \tilde{\tilde{\mathfrak{T}}}_J^L = \arg\min_\tau \| \Big((\bar{w}_{1,0} + y_{1,0}^*/\sqrt{J})\tau + (y_{1,0}^*\underline{\hat{\theta}} + y_{2,0}^*) + \sqrt{J}(\bar{w}_{1,0}\underline{\hat{\theta}} + \bar{w}_{2,0}) \Big)_- \|$$

with quantiles defined analogously to the notation in the paper.

Proof of Theorem 5:

First show that given $\varepsilon > 0$, for large enough J, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{q}_{\alpha,J}^L) \geq \alpha - 6\varepsilon$. The proof is similar to the corresponding nonlinear result. For large enough J, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q_{\alpha-3\varepsilon}) \geq \alpha - 4\varepsilon$. Also, by continuity of the quantiles as given in Lemma A10 and consistency of $(\hat{\Gamma}_0, \hat{\Sigma}_0)$, for J large enough, $\Pr(q_{\alpha-2\varepsilon} \leq \tilde{q}_{\alpha-\varepsilon,J}) \geq 1 - \varepsilon$. Now let $\eta = q_{\alpha-2\varepsilon} - q_{\alpha-3\varepsilon} > 0$. Below we will show that for large enough J,

$$\Pr(\tilde{q}_{\alpha-\varepsilon,J} \leq \tilde{q}_{\alpha,J}^L + \eta) \geq 1 - \varepsilon. \tag{1}$$

The desired first result will then follow, for large enough J,

$$\begin{aligned}
&\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{q}_{\alpha,J}^L) \\
&\geq \Pr(\{\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q_{\alpha-3\varepsilon}\} \cap \{q_{\alpha-2\varepsilon} \leq \tilde{q}_{\alpha-\varepsilon,J}\} \cap \{\tilde{q}_{\alpha-\varepsilon,J} \leq \tilde{q}_{\alpha,J}^L + \eta\}) \\
&\geq (\alpha - 4\varepsilon) + (1 - \varepsilon) + (1 - \varepsilon) - 2 = \alpha - 6\varepsilon.
\end{aligned}$$

Now we show that equation (1) holds. Take $\delta_\Sigma > 0$ and C_z large enough that $\Pr^*(\|Z_0^*\| \leq C_z) \geq 1 - \varepsilon/2$ for all $Z_0^* \sim N(0, \Sigma)$ with $\Sigma \in \mathcal{N}_{\delta_\Sigma}^{\Sigma_0}$. Now by Lemma A8 (and its proof), there exists $\delta_\Gamma > 0$ such that $\tau^{lp}(\Gamma, Z)$ exists and is uniformly continuous on $\Gamma \in \mathcal{N}_{\delta_\Gamma}^{\Gamma_0}$ and $\|Z\| \leq C_z$. By the uniform continuity there exists $\delta_\tau > 0$ such that for $\Gamma \in \mathcal{N}_{\delta_\Gamma}^{\Gamma_0}$, $\|Z\| \leq C_z$, and $\|(\Gamma, Z) - (\bar{\Gamma}, \bar{Z})\| \leq \delta_\tau$, $|\tau_1^{lp}(\Gamma, Z) - \tau_1^{lp}(\bar{\Gamma}, \bar{Z})| \leq \eta$. Now take $\delta_1 > 0$ and C_1 such that $\Pr^*(\|y_{1,0}^*\| \leq C_1) \geq 1 - \varepsilon/2$ for $y^* \sim N(0, \Sigma)$ with $\Sigma \in \mathcal{N}_{\delta_1}^{\Sigma_w}$. Let $\mathcal{G} = \{\hat{\underline{\Sigma}}_0 \in \mathcal{N}_{\delta_\Sigma}^{\Sigma_0}\} \cap \{\hat{\underline{\Gamma}}_0 \in \mathcal{N}_{\delta_\Gamma}^{\Gamma_0}\} \cap \{\hat{\Sigma}_w \in \mathcal{N}_{\delta_1}^{\Sigma_w}\}$. Then take J large enough that $\Pr(\mathcal{G}) \geq 1 - \varepsilon$ and $C_1/\sqrt{J} \leq \delta_\tau$.

Now suppose that J is large, $\mathcal{G}$ holds, $\|Z_0^*\| \leq C_z$, and $\|y_{1,0}^*\| \leq C_1$. Then $\|(\bar{w}_{1,0}, Z_0^*) - (\bar{w}_{1,0} + y_{1,0}^*/\sqrt{J}, Z_0^*)\| \leq \|y_{1,0}^*/\sqrt{J}\| \leq \delta_\tau$. Hence, $|\tilde{\tau}_1 - \tilde{\tau}_1^L| \leq \eta$. Then, if $\mathcal{G}$ holds and J is large enough,

$$\Pr^*(|\tilde{\tau}_1 - \tilde{\tau}_1^L| \le \eta) \ge \Pr^*(\{\|Z_0^*\| \le C_z\} \cap \{\|y_{1,0}^*\| \le C_1\}) \ge 1 - \varepsilon.$$

Further,

$$\begin{aligned}
\Pr^*(\tilde{\tau}_1^L \le \tilde{q}_{\alpha-\varepsilon,J} - \eta) &\le \Pr^*(\{\tilde{\tau}_1^L + \eta \le \tilde{q}_{\alpha-\varepsilon,J}\} \cap \{|\tilde{\tau}_1 - \tilde{\tau}_1^L| \le \eta\}) + \Pr^*(|\tilde{\tau}_1 - \tilde{\tau}_1^L| > \eta) \\
&\le \Pr^*(\{\tilde{\tau}_1^L + \eta \le \tilde{q}_{\alpha-\varepsilon,J}\} \cap \{\tilde{\tau}_1 \le \tilde{\tau}_1^L + \eta\}) + \varepsilon \\
&\le \Pr^*(\tilde{\tau}_1 \le \tilde{q}_{\alpha-\varepsilon,J}) + \varepsilon \\
&\le \alpha - \varepsilon + \varepsilon = \alpha,
\end{aligned}$$

so that $\tilde{q}_{\alpha,J}^L \ge \tilde{q}_{\alpha-\varepsilon,J} - \eta$. Finally, then, for J large enough, $\Pr(\tilde{q}_{\alpha-\varepsilon,J} \le \tilde{q}_{\alpha,J}^L + \eta) \ge \Pr(\mathcal{G}) \ge 1 - \varepsilon$.

Now using the same argument as in the analogous nonlinear case, if m_a contains m_0, the binding moments, then $\tilde{\tau}_1^L|P_J$ stochastically dominates $\tau_1^O|P_J$. So, in the argument above, we can substitute $\Pr(q_{\alpha-3\varepsilon} \le q_{\alpha,J}^O) \ge 1 - 2\varepsilon$, and the first conclusion of the result follows.

Using the same argument that compared $\tilde{\tau}_1$ and $\tilde{\tilde{\tau}}_1$ in the general nonlinear case, we can compare $\tilde{\tau}_1^L$ and $\tilde{\tilde{\tau}}_1^L$, yielding the analogous conclusions. For large enough J, $\Pr(\tilde{\tilde{q}}_{\alpha,J}^L \le \tilde{q}_{\alpha,J}^L) > 1 - \varepsilon$, and hence $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \le \tilde{\tilde{q}}_{\alpha,J}^L) \le \alpha + 2\varepsilon$.

Next, consider $\tilde{\tilde{\tau}}_1^L$ and τ_1^I. With some modification, this part of the proof follows the analogous result for the nonlinear case. We show that for given $\varepsilon > 0$, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \le q_{\alpha,J}^I) \le \alpha + 7\varepsilon$ for large enough J.

Note that $\sqrt{J}P_J m_0(w, \hat{\underline{\theta}}) = \sqrt{J}(\bar{w}_1\hat{\underline{\theta}} + \bar{w}_2)$. So, $\sqrt{J}(\bar{w}_{1,0}\hat{\underline{\theta}} + \bar{w}_{2,0}) = O_p(1)$ and $\sqrt{J}[(\bar{w}_{1,1}\hat{\underline{\theta}} + \bar{w}_{2,1}) - (\mathcal{P}w_{1,1}\underline{\theta} + \mathcal{P}w_{2,1})] = O_p(1)$ follow as in the nonlinear case, but without need for the stochastic equicontinuity and differentiability assumptions (due to the linearity).

Take C_z large enough and $\delta_\Sigma > 0$ small enough that for all $\Sigma \in \mathcal{N}_{\delta_\Sigma}^{\Sigma}$ and $Z^* \sim N(0, \Sigma)$, $\Pr(\|Z^*\| \le C_z) \ge 1 - \varepsilon/2$. Then, take J_a,C_0, C_1, $\delta_\Gamma > 0$, C_τ, and C_2 as defined in the analogous nonlinear proof. Also take $\delta_y > 0$ and C_y such that $\Pr^*(\|y_1^*\| \le C_y) \ge 1 - \varepsilon/2$ for $y^* \sim N(0, \Sigma)$ with $\Sigma \in \mathcal{N}_{\delta_y}^{\Sigma_w}$.

Take J_b large enough that $\sqrt{J}\mathcal{P}m_1(w, \underline{\theta}) \ge (C_z + C_1 + C_2)l$ for $J \ge J_b$. Take J_c such that

$\Pr((\hat{\Gamma},\hat{\Sigma}) \in \mathcal{N}^{\Gamma}_{\delta_\Gamma/2} \times \mathcal{N}^{\Sigma}_{\delta_\Sigma}) \geq 1-\varepsilon$ for $J \geq J_c$. Take J_d such that $C_y/\sqrt{J_d} \leq \delta_\Gamma/2$. From the argument above, there exists J_e such that for $J \geq J_e$, $\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{\tilde{q}}^L_{\alpha+\varepsilon,J}) \leq \alpha + 3\varepsilon$.

Let $\bar{J} = \max\{J_a, J_b, J_c, J_d, J_e\}$. Also let $\mathcal{A} = \{\|\sqrt{J}P_J m_0(w,\hat{\underline{\theta}})\| \leq C_0\} \cap \{\|\sqrt{J}P_J m_1(w,\hat{\underline{\theta}}) - \sqrt{J}Qm_1(w,\underline{\theta})\| \leq C_1\} \cap \{(\hat{\Gamma},\hat{\Sigma}) \in \mathcal{N}^{\Gamma}_{\delta_\Gamma/2} \times \mathcal{N}^{\Sigma}_{\delta_\Sigma}\} \cap \{\hat{\Sigma}_w \in \mathcal{N}^{\Sigma_w}_{\delta_y}\}$. Note that $\Pr(\mathcal{A}) \geq 1-4\varepsilon$ for $J \geq \bar{J}$.

Suppose $\mathcal{A}$ holds and $J \geq \bar{J}$. If $\|Z_0^*\| \leq C_z$ and $\|y^*_{1,0}/\sqrt{J}\| \leq \delta_\Gamma/2$, then $\bar{w}_{1,0} + y^*_{1,0}/\sqrt{J} \in \mathcal{N}^{\Gamma_0}_{\delta_\Gamma}$ and so $\|\tilde{\tilde{\tau}}^L\| \leq C_\tau$. If $\|Z_1^*\| \leq C_z$ and $\|y^*_{1,1}/\sqrt{J}\| \leq \delta_\Gamma/2$, then

$$\begin{aligned}
&(\bar{w}_{1,1} + y^*_{1,1}/\sqrt{J})\tilde{\tilde{\tau}}^L + (y^*_{1,1}\hat{\underline{\theta}} + y^*_{2,1}) + \sqrt{J}(\bar{w}_{1,1}\hat{\underline{\theta}} + \bar{w}_{2,1}) \\
&\quad \geq \sqrt{J}(\mathcal{P}w_{1,1}\underline{\theta} + \mathcal{P}w_{2,1}) \\
&\qquad -(\|(\bar{w}_{1,1} + y^*_{1,1}/\sqrt{J})\tilde{\tilde{\tau}}^L\| + \|Z_1^*\| + \|\sqrt{J}[(\bar{w}_{1,1}\hat{\underline{\theta}} + \bar{w}_{2,1}) - (\mathcal{P}w_{1,1}\underline{\theta} + \mathcal{P}w_{2,1})]\|)l \\
&\quad \geq 0
\end{aligned}$$

Hence $\tau_1^I = \tilde{\tilde{\tau}}_1^L$. Also, if $\mathcal{A}$ holds and $J \geq \bar{J}$, then

$$\begin{aligned}
&\Pr{}^*(\tau_1^I \leq \tilde{\tilde{q}}^L_{\alpha+\varepsilon,J}) \\
&\geq \Pr{}^*(\{\tau_1^I \leq \tilde{\tilde{q}}^L_{\alpha+\varepsilon,J}\} \cap \{\|Z_0^*\| \leq C_z\} \cap \{\|y^*_{1,0}/\sqrt{J}\| \leq \delta_\Gamma/2\} \cap \{\|Z_1^*\| \leq C_z\} \cap \{\|y^*_{1,1}/\sqrt{J}\| \leq \delta_\Gamma/2\}) \\
&= \Pr{}^*(\{\tilde{\tilde{\tau}}_1^L \leq \tilde{\tilde{q}}^L_{\alpha+\varepsilon,J}\} \cap \{\|Z_0^*\| \leq C_z\} \cap \{\|y^*_{1,0}/\sqrt{J}\| \leq \delta_\Gamma/2\} \cap \{\|Z_1^*\| \leq C_z\} \cap \{\|y^*_{1,1}/\sqrt{J}\| \leq \delta_\Gamma/2\}) \\
&\geq \Pr{}^*(\{\tilde{\tilde{\tau}}_1 \leq \tilde{\tilde{q}}^L_{\alpha+\varepsilon,J}\} \cap \{\|Z^*\| \leq C_z\} \cap \{\|y_1^*/\sqrt{J}\| \leq \delta_\Gamma/2\}) \\
&\geq \alpha + \varepsilon + (1-\varepsilon/2) + (1-\varepsilon/2) - 2 = \alpha.
\end{aligned}$$

So, $\tilde{\tilde{q}}^L_{\alpha+\varepsilon,J} \geq q^I_{\alpha,J}$.

Finally, for $J \geq \bar{J}$,

$$\Pr(\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq q^I_{\alpha,J}) \leq \Pr(\{\sqrt{J}(\hat{\underline{\theta}}_1 - \underline{\theta}_1) \leq \tilde{\tilde{q}}^L_{\alpha+\varepsilon,J}\}) + \Pr(\mathcal{A}^c) \leq \alpha + 7\varepsilon.$$

Result (b) follows. □

Proof of Theorem 4:

Define $c_{\alpha/2}$ by $\Pr^*(\|Z_J^*(\theta_0)_-\| \geq c_{\alpha/2}) = \alpha/2$. Now note that

$$\begin{aligned}
&\Pr(\inf_\theta \|(\sqrt{J}P_J m(z,\theta))_-\| \geq \bar{z}_{\alpha,J}) \\
&\leq \Pr(\|(\sqrt{J}[P_J m(z,\theta_0) - \mathcal{P}m(z,\theta_0)])_-\| \geq \bar{z}_{\alpha,J}) \\
&\leq \Pr(\|(\sqrt{J}[P_J m(z,\theta_0) - \mathcal{P}m(z,\theta_0)])_-\| \geq c_{\alpha/2}) + \Pr(\bar{z}_{\alpha,J} < c_{\alpha/2}) \\
&\leq \Pr(\|(\sqrt{J}[P_J m(z,\theta_0) - \mathcal{P}m(z,\theta_0)])_-\| \geq c_{\alpha/2}) + \Pr(\theta_0 \notin CI_{1-\alpha/2,J})
\end{aligned}$$

By assumption (c), $\overline{\lim}_{J\longrightarrow\infty} \Pr(\theta_0 \notin CI_{1-\alpha/2,J}) \leq \alpha/2$. Also, it is straightforward to show that $F(s)$ is continuous at $s > 0$.

Let s_β be such that $F(s_\beta) = 1 - \beta$ for $\beta < 1 - F(0)$. Given $\varepsilon > 0$, choose $\delta > 0$ small enough that $F(s_{\alpha/2}) - F(s_{\alpha/2-\delta}) \leq \varepsilon$. So, for J large enough, $\Pr(\|(\sqrt{J}[P_J m(z,\theta_0) - \mathcal{P}m(z,\theta_0)])_-\| \geq s_{\alpha/2-\delta}) < \alpha/2 + 2\varepsilon$.

Let $\mathcal{B} = \{z : \|z_-\| \geq s_{\alpha/2-\delta}\|\}$. For $\mathcal{Z}^* \sim N(0, \Sigma(\theta_0))$, $\Pr^*(\mathcal{Z}^* \in \mathcal{B}) = \alpha/2 - \delta$. Choose $\eta > 0$ small enough that for any $\Sigma \in \mathcal{N}^\eta_{\Sigma(\theta_0)}$ and $\mathcal{Z}^* \sim N(0,\Sigma)$, $\Pr^*(\|\mathcal{Z}^*_-\| \geq s_{\alpha/2-\delta}) \leq \alpha/2$. For large enough J, $\Pr(\hat{\Sigma} \in \mathcal{N}^\eta_{\Sigma(\theta_0)}) \geq 1 - \varepsilon$. If $\hat{\Sigma} \in \mathcal{N}^\eta_{\Sigma(\theta_0)}$ and $\mathcal{Z}^* \sim N(0, \hat{\Sigma})$, then $\Pr^*(\|\mathcal{Z}^*_-\| \geq s_{\alpha/2-\delta}) \leq \alpha/2$ and so $c_{\alpha/2} \geq s_{\alpha/2-\delta}$. Hence for large enough J, $\Pr(c_{\alpha/2} \geq s_{\alpha/2-\delta}) \geq 1 - \varepsilon$.

Hence, for large enough J,

$$\begin{aligned}
&\Pr(\|(\sqrt{J}[P_J m(z,\theta_0) - \mathcal{P}m(z,\theta_0)])_-\| \geq c_{\alpha/2}) \\
&\leq \Pr(\|(\sqrt{J}[P_J m(z,\theta_0) - \mathcal{P}m(z,\theta_0)])_-\| \geq s_{\alpha/2-\delta}) + \Pr(s_{\alpha/2-\delta} \geq c_{\alpha/2}) \\
&\leq \alpha/2 + 2\varepsilon + \varepsilon = \alpha/2 + 3\varepsilon.
\end{aligned}$$

So, $\overline{\lim}_{J\longrightarrow\infty} \Pr(\|(\sqrt{J}[P_J m(z,\theta_0) - \mathcal{P}m(z,\theta_0)])_-\| \geq c_{\alpha/2}) \leq \alpha/2$. The result follows. □

Lemmas

Lemma A1 $\|a_-\| \geq \|b_-\| - \|a - b\|$.

Lemma A2 *For any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $\{\theta \in \Theta_0 : |\theta_1 - \underline{\theta}_1| < \delta\} \subset \{\theta \in \Theta_0 : \|\theta - \underline{\theta}\| < \epsilon\}$.

PROOF: Suppose not. Then there exists a sequence $\delta_n > 0$ such that $\delta_n \longrightarrow 0$ and some $\theta_n \in \Theta_0$ such that $|\theta_{n,1} - \underline{\theta}_1| < \delta_n$ but $\|\theta_n - \underline{\theta}\| \geq \epsilon$. Clearly, $\theta_{n,1} \longrightarrow \underline{\theta}_1$. By Assumption A1, there exists a convergent subsequence $\theta_{n'} \longrightarrow \theta_s \in \Theta_0$. Now, $\theta_{n',1} \longrightarrow \underline{\theta}_1$ and $\theta_{n',1} \longrightarrow \theta_{s,1}$, so we must have $\theta_{s,1} = \underline{\theta}_1$. Since $\underline{\theta}$ is the unique minimizing point of $min_{\theta \in \Theta_0} \theta_1$, we must have $\theta_s = \underline{\theta}$, which contradicts the supposition. $\square$

Lemma A3 *There exists $\bar{\delta} > 0$ such that for any τ with $\underline{\Gamma}_0 \tau \geq 0$, $\tau_1 \geq \bar{\delta} \|\tau\|$.*

PROOF: Let $\mathcal{T} = \{\tau : \|\tau\| = 1, \underline{\Gamma}_0 \tau \geq 0\}$, and $\bar{\delta} = \inf\{\tau_1 : \tau \in \mathcal{T}\}$. By Assumption A9, there exists $\lambda \neq 0$ such that $\underline{\Gamma}_0 \frac{\lambda}{\|\lambda\|} > 0$. So $\mathcal{T}$ is non-empty and the infimum, $\bar{\delta}$, exists. Moreover, since zero is the unique solution to $\min\{\tau_1 : \underline{\Gamma}_0 \tau \geq 0\}$, $\tau \in \mathcal{T}$ implies $\tau_1 > 0$. Hence, $\bar{\delta} \geq 0$. Suppose $\bar{\delta} = 0$. Then there exists $\tau_n \in \mathcal{T}$ such that $\tau_{n,1} \downarrow 0$. The set $\{\tau : \|\tau\| = 1\}$ is compact so there exists a convergent subsequence $\tau_{n'}$ with $\tau_{n'} \longrightarrow \tau_0$ where $\tau_{n',1} \downarrow 0$, so $\tau_{0,1} = 0$ and $\|\tau_0\| = 1$, i.e. $\tau_0 \neq 0$. But $\underline{\Gamma}_0 \tau_{n'} \geq 0$, so $\underline{\Gamma}_0 \tau_0 \geq 0$, which contradicts the uniqueness of zero as a solution $\min_{\tau : \underline{\Gamma}_0 \tau \geq 0} \tau_1$. It follows that $\bar{\delta} > 0$. Now take any $\tau \neq 0$ with $\underline{\Gamma}_0 \tau \geq 0$. Then $\tau_1 > 0$ and so $\|\tau\| > 0$. Hence $\underline{\Gamma}_0 \frac{\tau}{\|\tau\|} \geq 0$, which implies $\frac{\tau}{\|\tau\|} \geq \bar{\delta}$, ie $\tau_1 \geq \bar{\delta} \|\tau\|$. Since this inequality also holds for $\tau = 0$, the lemma follows. $\square$

Lemma A4 *For $\bar{\delta}$ from Lemma A3, there exists $\bar{\eta}$ such that if $\theta \in \Theta_0$ and $\|\theta - \underline{\theta}\| \leq \bar{\eta}$, then $\theta_1 - \underline{\theta}_1 \geq \frac{\bar{\delta}}{2} \|\theta - \underline{\theta}\|$.*

PROOF: Suppose not. Then, there exists a sequence $\theta_n \longrightarrow \underline{\theta}$ such that $\theta_n \in \Theta_0$ and $\theta_{n,1} - \underline{\theta}_1 < \frac{\bar{\delta}}{2} \|\theta_n - \underline{\theta}\|$. Let $\tau_n = \frac{\theta_n - \underline{\theta}}{\|\theta_n - \underline{\theta}\|}$, so $\|\tau_n\| = 1$. $\{\tau : \|\tau\| = 1\}$ is compact so τ_n has a convergent subsequence $\tau_{n'} \longrightarrow \tau_0$ and $\|\tau_0\| = 1$.

$$\begin{aligned}
\underline{\Gamma}_0 \tau_0 &= lim_{n' \longrightarrow \infty} \underline{\Gamma}_0 \frac{\theta_{n'} - \underline{\theta}}{\|\theta_{n'} - \underline{\theta}\|} = lim_{n' \longrightarrow \infty} \frac{1}{\|\theta_{n'} - \underline{\theta}\|} [\mathcal{P} m_0(z, \theta_{n'}) - \mathcal{P} m_0(z, \underline{\theta})] \\
&= lim_{n' \longrightarrow \infty} \frac{1}{\|\theta_{n'} - \underline{\theta}\|} \mathcal{P} m_0(z, \theta_{n'}) \geq 0
\end{aligned}$$

where $\mathcal{P} m_0(z, \theta_{n'}) \geq 0$ and $\mathcal{P} m_0(z, \underline{\theta}) = 0$. Hence, $\underline{\Gamma}_0 \tau_0 \geq 0$, and by Lemma A3, $\tau_{0,1} \geq$

$\bar{\delta}\|\tau_0\| = \bar{\delta}$. But, $\theta_{n',1} - \underline{\theta} \le \frac{\bar{\delta}}{2}\|\theta_{n'} - \underline{\theta}\|$, i.e.

$$\frac{\bar{\delta}}{2} \ge \frac{\theta_{n',1} - \underline{\theta}}{\|\theta_{n'} - \underline{\theta}\|} = \tau_{n',1} \longrightarrow \tau_{0,1}$$

which yields a contradiction and the lemma follows. □

Lemma A5 *If $\mathcal{P}m_0(z,\theta)$ is continuously differentiable in an open neighborhood of $\underline{\theta}$ and $C > 0$, then for large enough J and some sequence $\zeta_J \downarrow 0$,*

$$\sup_{\theta:\|\theta-\underline{\theta}\|\le \frac{C}{\sqrt{J}}} \left\| \underline{\Gamma}_0(\theta - \underline{\theta}) - [\mathcal{P}m_0(z,\theta) - \mathcal{P}m_0(z,\underline{\theta})] \right\| \le \zeta_J \frac{C}{\sqrt{J}}.$$

PROOF: By continuous differentiability,

$$\sup_{\theta:\|\theta-\underline{\theta}\|\le \frac{C}{\sqrt{J}}} \left\| \underline{\Gamma}_0 \left(\frac{\theta - \underline{\theta}}{\|\theta - \underline{\theta}\|} \right) - \frac{\mathcal{P}m_0(z,\theta) - \mathcal{P}m_0(z,\underline{\theta})}{\|\theta - \underline{\theta}\|} \right\| \longrightarrow 0.$$

So, there exists $\zeta_J \downarrow 0$ such that $\sup_{\theta:\|\theta-\underline{\theta}\|\le \frac{C}{\sqrt{J}}} \left\| \underline{\Gamma}_0 \left(\frac{\theta-\underline{\theta}}{\|\theta-\underline{\theta}\|} \right) - \frac{\mathcal{P}m_0(z,\theta)-\mathcal{P}m_0(z,\underline{\theta})}{\|\theta-\underline{\theta}\|} \right\| \le \zeta_J$. Hence, $\sup_{\theta:\|\theta-\underline{\theta}\|\le \frac{C}{\sqrt{J}}} \left\| \underline{\Gamma}_0(\theta - \underline{\theta}) - [\mathcal{P}m_0(z,\theta) - \mathcal{P}m_0(z,\underline{\theta})] \right\| \le \zeta_J \frac{C}{\sqrt{J}}$ for J large enough. □

Lemma A6

$$\sup_{\omega\in[0,1]} \left\| \underline{\Gamma}_0 \left(\theta^* + \frac{\lambda}{\sqrt{J}}\omega - \underline{\theta} \right) - \left[\mathcal{P}m_0 \left(Z, \theta^* + \frac{\lambda}{\sqrt{J}}\omega \right) - \mathcal{P}m_0(z,\underline{\theta})\right] \right\| = o_p(1/\sqrt{J})$$

PROOF: Since $\sqrt{J}(\theta^* - \underline{\theta}) = O_p(1)$, by choosing C' large enough, $\sup_{\omega\in[0,1]} \|\theta^* + \frac{\lambda}{\sqrt{J}}\omega - \underline{\theta}\| \le \frac{C'+\|\lambda\|}{\sqrt{J}}$ with probability as close to one as desired. The result then follows by Lemma A5 setting $C = C' + \|\lambda\|$. □

Lemma A7

$$\sup_{\omega\in[0,1]} \|P_J m_0(z,\theta^* + \frac{\lambda}{\sqrt{J}}\omega) - \mathcal{P} m_0(z,\theta^* + \frac{\lambda}{\sqrt{J}}\omega) - (P_J m_0(z,\underline{\theta}) - \mathcal{P} m_0(z,\underline{\theta}))\| = o_p(1/\sqrt{J})$$

PROOF: Note that $\sup_{\omega\in[0,1]} \|\theta^* + \frac{\lambda}{\sqrt{J}}\omega - \underline{\theta}\| = o_p(1)$ implies there exists $\delta_J \downarrow 0$ such that $\sup_{\omega\in[0,1]} \|\theta^* + \frac{\lambda}{\sqrt{J}}\omega - \underline{\theta}\| \leq \delta_J$ with probability approaching one. The result then follows by Assumption A8. $\square$

Lemma A8 *There exists $\eta > 0$ such that $\tau_1^{lp}(\Gamma, Z)$ is continuous at each $\Gamma \in \mathcal{N}_\eta^{\Gamma_0}$ and $Z \in \mathbb{R}^m$.*

PROOF: Find $\eta > 0$ such that given $\bar{\Gamma} \in \mathcal{N}_\eta^{\Gamma_0}$, any $\bar{Z}$, and $\varepsilon > 0$, we can find a $\delta > 0$ such that $|\tau_1^{lp}(\Gamma, Z) - \tau_1^{lp}(\bar{\Gamma}, \bar{Z})| < \varepsilon$ for all $(\Gamma, Z) \in \mathcal{N}_\delta^{\bar{\Gamma},\bar{Z}}$.

There exists $\delta_a > 0$ such that $\mathcal{N}_{\delta_a}^{\Gamma_0}$ is contained in the neighborhood given in Assumption A9 and $\Gamma\lambda > 0$ for all $\Gamma \in \mathcal{N}_{\delta_a}^{\Gamma_0}$.

We first note that by Kall (1970) Theorem 4, for all $\Gamma \in \mathcal{N}_{\delta_a}^{\Gamma_0}$ and any Z, $\tau_1^{lp}(\Gamma, Z)$ is well defined and a solution to the corresponding linear program exists.

Next show (uniform) boundedness of the solutions of the linear programs on some neighborhood contained in $\mathcal{N}_{\delta_a}^{\Gamma_0,\bar{Z}}$, following the approach in the proof of Bereanu (1976) Lemma 2.1. In fact, it will be useful (for other results in the paper) to show a stronger result. Take any constant $C_z > 0$ we will show uniform boundedness for $(\Gamma, Z) \in \mathcal{N}_\delta^{\Gamma_0} \times \{Z : \|Z\| \leq C_z\}$, for some $\delta \leq \delta_a$ (take C_z large enough that $\|\bar{Z}\| + \delta_a \leq C_z$). If solutions are not bounded on any such set, then there exists a sequence $\delta_n \longrightarrow 0$ and $(\Gamma_n, Z_n) \in \mathcal{N}_{\delta_n}^{\Gamma_0} \times \{Z : \|Z\| \leq C_z\}$ such that for some solution τ_n^* to the linear program $\min\{\tau_1 : \Gamma_n \tau + Z_n \geq 0\}$, $\|\tau_n^*\| \longrightarrow \infty$. By the Duality Theorem (of Linear Programming), for any solution τ_n^* to the primal linear program, there exists a corresponding solution β_n^* to the dual linear program. The compactness of $\{Z : \|Z\| \leq C_z\}$ implies that there exists a convergent subsequence $\{n'\}$ such that $Z_{n'} \longrightarrow \tilde{Z} \in \{Z : \|Z\| \leq C_z\}$. Let $A_n = [\Gamma_n \quad -\Gamma_n]$, $A_0 = [\underline{\Gamma}_0 \quad -\underline{\Gamma}_0]$, $e_1' = (1, 0, \ldots, 0)$, $c' = (e_1', -e_1')$, and

$$B_n = \begin{bmatrix} A_n & 0 \\ 0 & -A_n' \\ -c' & Z_n' \end{bmatrix}, \quad B_0 = \begin{bmatrix} A_0 & 0 \\ 0 & -A_0' \\ -c' & \tilde{Z}' \end{bmatrix} \quad d_n = \begin{pmatrix} -Z_n \\ -c \\ 0 \end{pmatrix}.$$

Then there is $u_n^* = (\tau_n^+, \tau_n^-, \beta_n^*) \geq 0$ such that $B_n u_n^* \geq d_n$ and $\tau_n^* = \tau_n^+ - \tau_n^-$. Also, $B_n \frac{u_n^*}{\|u_n^*\|} \geq \frac{1}{\|u_n^*\|} d_n$ (and $\|u_n^*\| \longrightarrow \infty$). Since $u_{n'}^*/\|u_{n'}^*\|$ is a sequence on the compact unit ball, it has a convergent subsequence $\{n''\}$ such that $u_{n''}^*/\|u_{n''}^*\| \longrightarrow u^* = (\tau^{*+}, \tau^{*-}, \beta^*)$, $\|u^*\| = 1$, $u^* \geq 0$ and $B_0 u^* \geq 0$. The last conclusion implies $\underline{\Gamma}_0' \beta^* = 0$. Note that if $\beta^* \neq 0$ (recalling $\beta^* \geq 0$), then $\beta^{*\prime} \underline{\Gamma}_0 \lambda > 0$ and hence $\beta^{*\prime} \underline{\Gamma}_0 \neq 0$. So, $\beta^* = 0$. So we have $\underline{\Gamma}_0 \tau^* \geq 0$ and $\tau_1^* \leq 0$ (with $\tau^* \neq 0$), which contradicts Assumption A9. Hence, for some $\delta_b \leq \delta_a$, there exists C such that for any solution, $\bar{\tau}$, of the linear program $\min\{\tau_1 : \Gamma\tau + Z \geq 0\}$ with $(\Gamma, Z) \in \mathcal{N}_{\delta_b}^{\Gamma_0} \times \{Z : \|Z\| \leq C_z\}$, $\|\bar{\tau}\| \leq C$.

Let $\gamma = \min_{\Gamma \in \mathcal{N}_{\delta_b}^{\Gamma_0}} \min_j [\Gamma\lambda]_j > 0$. Take any $\bar{\Gamma} \in \mathcal{N}_{\delta_b/2}^{\Gamma_0}$. Choose $\rho > 0$ small enough that $\rho\lambda_1 < \varepsilon$. And take $\delta > 0$ such that $\delta \leq \delta_b/2$ and $\|(\bar{\Gamma} - \Gamma)\tau\| + \|\bar{Z} - Z\| \leq \rho\gamma$ for all $\|\tau\| \leq C$ and $(\Gamma, Z) \in \mathcal{N}_\delta^{\bar{\Gamma}, \bar{Z}}$.

Take any $(\Gamma, Z) \in \mathcal{N}_\delta^{\bar{\Gamma}, \bar{Z}}$ and let $\bar{\tau}$ be any solution to the linear program $\min\{\tau_1 : \Gamma\tau + Z \geq 0\}$. Then,

$$\bar{\Gamma}(\bar{\tau} + \rho\lambda) + \bar{Z} = (\bar{\Gamma} - \Gamma)\bar{\tau} + (\bar{Z} - Z) + \bar{\Gamma}\rho\lambda + (\Gamma\bar{\tau} + Z) \geq 0,$$

so $\tau_1^{lp}(\Gamma, Z) + \varepsilon \geq \tau_1^{lp}(\bar{\Gamma}, \bar{Z})$.

Take any $(\Gamma, Z) \in \mathcal{N}_\delta^{\bar{\Gamma}, \bar{Z}}$ and let $\bar{\tau}_0$ be any solution to the linear program $\min\{\tau_1 : \bar{\Gamma}\tau + \bar{Z} \geq 0\}$. Then,

$$\Gamma(\bar{\tau}_0 + \rho\lambda) + Z = (\Gamma - \bar{\Gamma})\bar{\tau}_0 + (Z - \bar{Z}) + \Gamma\rho\lambda + (\bar{\Gamma}\bar{\tau}_0 + \bar{Z}) \geq 0,$$

so $\tau_1^{lp}(\bar{\Gamma}, \bar{Z}) + \varepsilon \geq \tau_1^{lp}(\Gamma, Z)$.

Hence, $|\tau_1^{lp}(\Gamma, Z) - \tau_1^{lp}(\bar{\Gamma}, \bar{Z})| \leq \varepsilon$, and the result follows with $\eta = \delta_b/2$. □

Lemma A9 *$\hat{\tau}_1 = \min\{\tau_1 : \underline{\Gamma}_0\tau + Z\}$ where $Z \sim N(0, \underline{\Sigma}_0)$. $\hat{\tau}_1$ has a continuous distribution (continuous c.d.f.). Also, $\mathbb{R}$ is the support of $\hat{\tau}_1$; in particular, for any $q \in \mathbb{R}$, $\Pr(\hat{\tau}_1 \leq q) \in (0, 1)$.*

PROOF: Let F denote the c.d.f. of $\hat{\tau}_1$. Suppose F is discontinuous at some t. Let $\epsilon = \Pr(\hat{\tau}_1 = t)$. Then $\epsilon > 0$. Let $A_t = \{Z : \tau_1^{lp}(\underline{\Gamma}_1, Z) = \hat{\tau}_1\}$. Then, $\epsilon = \Pr(Z \in A_t)$. For $c > 0$, define $A_c = \{Z : Z = Z' + \underline{\Gamma}_0 c\lambda \text{ for } Z' \in A_t\}$. For c small enough, $\Pr(Z \in A_c) \geq \epsilon/2$.

Take any $Z \in A_t$. Let $\bar{\tau}$ be a solution to $\min\{\tau_1 : \underline{\Gamma}_0\tau + Z \geq 0\}$, so $t = \bar{\tau}_1$. Let $\bar{\bar{\tau}}$ be a solution to $\min\{\tau_1 : \underline{\Gamma}_0\tau + (Z + \underline{\Gamma}_0 c\lambda) \geq 0\}$. Since $0 \leq \underline{\Gamma}_0\bar{\tau} + Z = \underline{\Gamma}_0(\bar{\tau} - c\lambda) + (Z + \underline{\Gamma}_0 c\lambda)$,

$\bar{\bar{\tau}}_1 \leq \bar{\tau}_1 - c\lambda_1$. Also, $0 \leq \underline{\Gamma}_0 \bar{\bar{\tau}} + (Z + \underline{\Gamma}_0 c\lambda) = \underline{\Gamma}_0(\bar{\bar{\tau}} + c\lambda) + Z$, so $\bar{\tau}_1 \leq \bar{\bar{\tau}}_1 + c\lambda_1$. Hence, $\bar{\bar{\tau}}_1 = t - c\lambda_1$. So, $\Pr(\hat{\tau}_1 = t - c\lambda_1) \geq \Pr(Z \in A_c) \geq \epsilon/2$. We can pick an infinite number of small c yielding such mass points, which yields a contradiction. Hence, F is continuous everywhere.

Now take $q \in \mathbb{R}$. Choose c such that $c\lambda_1 = q$. Then,

$$0 < \Pr(Z \geq -\underline{\Gamma}_0 c\lambda) = \Pr(\underline{\Gamma}_0 c\lambda + Z \geq 0) \leq \Pr(\hat{\tau}_1 \leq c\lambda_1) = \Pr(\hat{\tau}_1 \leq q).$$

Now, take c such that $c\lambda_1 = -q$. Take the solution $\tau^{lp}(\underline{\Gamma}_0, Z)$ to $\min\{\tau_1 : \underline{\Gamma}_0 \tau + Z \geq 0\}$ for $Z < \underline{\Gamma}_0 c\lambda$. Then,

$$\underline{\Gamma}_0(\tau^{lp}(\underline{\Gamma}_0, Z) + c\lambda) > \underline{\Gamma}_0 \tau^{lp}(\underline{\Gamma}_0, Z) + Z \geq 0.$$

So, $\tau_1^{lp}(\underline{\Gamma}_0, Z) + c\lambda_1 \geq 0$, ie $\tau_1^{lp}(\underline{\Gamma}_0, Z) \geq q$. Hence,

$$0 < \Pr(Z < \underline{\Gamma}_0 c\lambda) \leq \Pr(q \leq \hat{\tau}_1).$$

$\square$

Define $q_\alpha^{\Gamma,\Sigma} = \inf\{q : \Pr^*(\tau_1^{lp}(\Gamma, Z^*) \leq q) \geq \alpha\}$ where $Z^* \sim N(0, \Sigma)$.

Lemma A10 *Given $\alpha \in (0,1)$, $q_\alpha^{\Gamma,\Sigma}$ is continuous in (Γ, Σ) at $(\underline{\Gamma}_0, \underline{\Sigma}_0)$.*

PROOF: Given $\eta > 0$, show that there exists $\delta > 0$ such that $|q_\alpha^{\Sigma,\Gamma} - q_\alpha^{\underline{\Gamma}_0,\underline{\Sigma}_0}| \leq \eta$ for $(\Gamma, \Sigma) \in \mathcal{N}_\delta^{\underline{\Gamma}_0,\underline{\Sigma}_0}$ (defined analogously to the neighborhoods defined prior to Lemma A8).

Take $\eta_a > 0$ such that $\eta_a \leq \eta/2$, $\min\{q_{\alpha+2\eta_a}^{\underline{\Gamma}_0,\underline{\Sigma}_0} - q_\alpha^{\underline{\Gamma}_0,\underline{\Sigma}_0}, q_\alpha^{\underline{\Gamma}_0,\underline{\Sigma}_0} - q_{\alpha-2\eta_a}^{\underline{\Gamma}_0,\underline{\Sigma}_0}\} \leq \eta/2$, and $\alpha \in (2\eta_a, 1 - 2\eta_a)$.

Take $\delta_a > 0$ to be the "η" in the statement of Lemma A8. Choose C_z such that $\Pr^*(\|Z^*\| \leq C_z) \geq 1 - \eta_a$ for all $\Sigma \in \mathcal{N}_{\delta_a}^{\underline{\Sigma}_0}$ and $Z^* \sim N(0, \Sigma)$. Then τ_1^{lp} is uniformly continuous on $\mathcal{N}_{\delta_a}^{\underline{\Gamma}_0} \times \{Z : \|Z\| \leq C_z\}$. Choose δ_b such that $\|\tau_1^{lp}(\Gamma, Z) - \tau_1^{lp}(\underline{\Gamma}_0, Z)\| \leq \eta_a$ for all $\Gamma \in \mathcal{N}_{\delta_b}^{\underline{\Gamma}_0}$ and all $\|Z\| \leq C_z$.

Define $\mathcal{A}_\beta = \{Z : \tau_1^{lp}(\underline{\Gamma}_0, Z) \leq q_\beta^{\underline{\Gamma}_0,\underline{\Sigma}_0}\}$. Now take $0 < \delta \leq \delta_b$ such that $\min\{|\Pr^*(Z^* \in A_{\alpha+2\eta_a}) - \Pr(Z_0 \in A_{\alpha+2\eta_a})|, |\Pr^*(Z^* \in A_{\alpha-2\eta_a}^c) - \Pr(Z_0 \in A_{\alpha-2\eta_a})|\} \leq \eta_a$ for $Z_0 \sim N(0, \underline{\Sigma}_0)$, $Z^* \sim N(0, \Sigma)$, and all $\Sigma \in \mathcal{N}_\delta^{\underline{\Sigma}_0}$.

Then, for $Z_0 \sim N(0, \underline{\Sigma}_0)$ and $Z^* \sim N(0, \Sigma)$ with $(\Gamma, \Sigma) \in \mathcal{N}_\delta^{\Gamma_0, \underline{\Sigma}_0}$,

$$\begin{aligned}
\Pr{}^*(\tau_1^{lp}(\Gamma, Z^*) \leq q_{\alpha+2\eta_a}^{\Gamma_0, \underline{\Sigma}_0} + \eta_a) &\geq \Pr{}^*(\{Z^* \in A_{\alpha+2\eta_a}\} \cap \{\|Z^*\| \leq C_z\}) \\
&\geq \Pr{}^*(Z^* \in A_{\alpha+2\eta_a}) + \Pr{}^*(\|Z^*\| \leq C_z) - 1 \\
&\geq (\alpha + 2\eta_a) - \eta_a + (1 - \eta_a) - 1 = \alpha.
\end{aligned}$$

Similarly, $\Pr^*(\tau_1^{lp}(\Gamma, Z^*) \geq q_{\alpha-2\eta_a}^{\Gamma_0, \underline{\Sigma}_0} + \eta_a) \geq 1 - \alpha$, so $\Pr^*(\tau_1^{lp}(\Gamma, Z^*) < q_{\alpha-2\eta_a}^{\Gamma_0, \underline{\Sigma}_0} + \eta_a) \leq \alpha$. So, $q_\alpha^{\Sigma, \Gamma} \leq q_{\alpha+2\eta_a}^{\underline{\Sigma}_0, \underline{\Gamma}_0} + \eta_a \leq q_\alpha^{\underline{\Sigma}_0, \underline{\Gamma}_0} + \eta$ by the definition of η_a. Similarly, $q_\alpha^{\Sigma, \Gamma} \geq q_\alpha^{\underline{\Sigma}_0, \underline{\Gamma}_0} - \eta$. The result follows. $\square$