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ON AN EXACT CONSTANT FOR THE ROSENTHAL INEQUALITY*

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Abstract. Let $\xi_1,\dots,\xi_n$ be independent random variables having symmetric distribution with finite pth moment, $2<p<\infty$. It is shown that the precise constant C_p^* in Rosenthal's inequality

$$\Big\|\sum_{i=1}^n \xi_i\Big\|_p \leqq C_p \max\bigg(\Big\|\sum_{i=1}^n \xi_i\Big\|_2,\ \bigg(\sum_{i=1}^n \|\xi_i\|_p^p\bigg)^{1/p}\bigg)$$

has the form

$$C_p^* = \bigg(1+\frac{2^{p/2}}{\pi^{1/2}}\,\Gamma\Big(\frac{p+1}{2}\Big)\bigg)^{1/p}, \quad 2<p<4,$$
$$C_p^* = \|\xi_1-\xi_2\|_p, \quad p\geqq 4,$$

where $\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}dx$ and ξ_1,ξ_2 are independent Poisson random variables with parameter 0.5. It is proved also that

$$\lim_{p\to\infty} C_p^*\frac{\log p}{p}=\frac{1}{e}.$$

Key words. Rosenthal's inequality, random variables with symmetric distribution, Poisson random variable, moment

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Let $\|\xi\|_s$ be the L_s-norm of the random variable ξ: $\|\xi\|_s=(\mathbf{E}\,|\xi|^s)^{1/s}$, where $\mathbf{E}(\cdot)$ means the expectation operator.

Rosenthal [1] has proved the inequality

$$\Big\|\sum_{i=1}^n \xi_i\Big\|_p \leqq C_p \max\bigg(\Big\|\sum_{i=1}^n \xi_i\Big\|_2,\ \bigg(\sum_{i=1}^n \|\xi_i\|_p^p\bigg)^{1/p}\bigg), \tag{1}$$

where C_p is a constant depending on p only, and $\xi_1,\dots,\xi_n$ are independent random variables (r.v's.) having symmetric distribution with finite pth moment, $2<p<\infty$. In particular, one can take $C_p=2^p$ as C_p.

Papers [2], [3], [5], and [9] deal with some refinements and generalizations of inequality (1).

Using Sazonov's estimate [2] one can obtain (1) with a constant of order $2^{p/4}$ in p.

It follows from the results of [3] and [4] that one can take $C_p=Lp$ as C_p where L is an absolute constant.

Let C_p^* be the exact constant in inequality (1), that is

$$C_p^*=\frac{\sup\|\sum_{i=1}^n\xi_i\|_p}{\max(\|\sum_{i=1}^n\xi_i\|_2,\ (\sum_{i=1}^n\|\xi_i\|_p^p)^{1/p})},$$

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where sup is taken over all nondegenerate independent r.v.'s $\xi_1, \dots, \xi_n$ with symmetric distributions having finite pth moment, $2 < p < \infty$.

As shown in [5],

$$\frac{p}{2^{1/2} e \max(1, \log p)} \leqq C_p^* \leqq \frac{7.35p}{\max(1, \log p)},$$

that is C_p^* behaves as $p/\log p$ if $p \to \infty$.

The problem of determining the exact value of C_p^* is closely related with the problem of calculating the extreme of a convex functional defined on the class of sums of independent random variables with symmetric distributions. This problem goes back to Prokhorov's paper [6]. It was also investigated in [7] and [8].

In the present paper we give a clear expression for C_p^*, $p > 2$, and find the asymptotics of C_p^* as $p \to \infty$.

THEOREM 1. *The exact value of C_p^* in Rosenthal's equation has the form*

$$C_p^* = \left(1 + \frac{2^{p/2}}{\pi^{1/2}} \Gamma\left(\frac{p+1}{2}\right)\right)^{1/p}, \quad 2 < p < 4,$$
$$C_p^* = \|\xi_1 - \xi_2\|_p, \quad p \geqq 4,$$

where

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad \xi_1, \xi_2$$

are independent Poisson random variables with parameter 0.5.

THEOREM 2. $C_{2s}^* = T_{2s}^{1/(2s)}$, *where T_{2s} is the number of partitions of a $(2s)$-set (a set consisting of $2s$ elements) into parts each of which includes an even number of elements*, $s \in \mathbf{N}$.

COROLLARY. $\lim_{p\to\infty} C_p^*(\log p)/p = 1/e$.

Remark 1. The next representation for the numbers T_{2s} follows from Theorems 1 and 2. It is an analogue of Dobinski's formula for Bell numbers:

$$T_{2s} = \frac{2}{e} \sum_{m=1}^{\infty} \sum_{k=0}^{\infty} \frac{m^{2s}}{k!(m+k)!2^{m+2k}}.$$

To prove Theorem 1 we need an auxiliary result which is of independent interest.

Our notation follows [8]. Let $\xi_1, \dots, \xi_n$ be independent r.v.'s with symmetric distribution having finite pth moment, $2 < p < \infty$, $a_i \geqq 0$, $b_i \geqq 0$, $a_i^p \leqq b_i$, $i = 1, \dots, n$, $A, B, D \geqq 0$. Set

$$(\xi, n) = (\xi_1, \dots, \xi_n), \quad (a, n) = (a_1, a_2, \dots, a_n),$$
$$(b, n) = (b_1, b_2, \dots, b_n),$$
$$M_1(n, a, b) = \bigl\{(\xi, n)\colon \mathbf{E}\xi_i^2 = a_i^2,\ \mathbf{E}\,|\xi_i|^p = b_i,\ i = 1, \dots, n\bigr\},$$
$$M_2(n, a, b) = \bigl\{(\xi, n)\colon \mathbf{E}\xi_i^2 = a_i^2,\ \mathbf{E}\,|\xi_i|^p = b_i,\ i = 1, \dots, n\bigr\},$$
$$U_1(A, B) = \left\{(\xi, n)\colon n \geqq 1,\ \sum_{i=1}^n \mathbf{E}\xi_i^2 = B^2,\ \sum_{i=1}^n \mathbf{E}\,|\xi_i|^p = A\right\},$$
$$U_2(A, B) = \left\{(\xi, n)\colon n \geqq 1,\ \sum_{i=1}^n \mathbf{E}\xi_i^2 \leqq B^2,\ \sum_{i=1}^n \mathbf{E}\,|\xi_i|^p \leqq A\right\},$$

$$U(D) = \left\{(\xi,\, n)\colon\ n \geqq 1,\ \max\left(\left(\sum_{i=1}^{n} \mathbf{E}\,|\xi_i|^2\right)^{p/2},\ \sum_{i=1}^{n} \mathbf{E}\,|\xi_i|^p\right) = D\right\}.$$

Let $\xi_1(A,B), \xi_2(A,B)$ be independent Poisson r.v.'s with parameter $\frac{1}{2}\,(B^p/A)^{2/(p-2)}$ and let $\varepsilon_1(a_1,b_1,p),\dots,\varepsilon_n(a_n,b_n,p)$ be independent r.v.'s with distribution $\mathbf{P}\{\varepsilon(a,b,p) = 0\} = 1-(a^p/b)^{2/(p-2)}$, $\mathbf{P}\{\varepsilon(a,b,p)\} = \pm(b/a^2)^{1/(p-2)} = \frac{1}{2}\,(a^p/b)^{2/(p-2)}$, $\varepsilon_1,\dots,\varepsilon_n$ be independent r.v.'s with distribution $\mathbf{P}\{\varepsilon_1 = 1\} = \mathbf{P}\{\varepsilon_1 = -1\} = \frac{1}{2}$, $i = 1,\dots,n$.

Proposition. *If* $2 < p < 4$, *then*

$$\sup_{(\xi,n)\in M_k(n,a,b)} \mathbf{E}\left|\sum_{i=1}^{n} \xi_i\right|^p = \sum_{i=1}^{n}(b_i - a_i^p) + \mathbf{E}\left|\sum_{i=1}^{n} a_i\varepsilon_i\right|^p, \quad k = 1,\,2, \tag{2}$$

$$\sup_{(\xi,n)\in U_k(A,B)} \mathbf{E}\left|\sum_{i=1}^{n} \xi_i\right|^p = A + \frac{2^{p/2}}{\pi^{1/2}}\,\Gamma\Big(\frac{p+1}{2}\Big)\,B^p, \quad k = 1,\,2. \tag{3}$$

If $3 \leqq p < 4$, *then*

$$\inf_{(\xi,n)\in M_1(n,a,b)} \mathbf{E}\left|\sum_{i=1}^{n} \xi_i\right|^p = \mathbf{E}\left|\sum_{i=1}^{n} \varepsilon_i(a_i,b_i,p)\right|^p, \tag{4}$$

If $p \geqq 4$, *then*

$$\sup_{(\xi,n)\in M_k(n,a,b)} \mathbf{E}\left|\sum_{i=1}^{n} \xi_i\right|^p = \mathbf{E}\left|\sum_{i=1}^{n} \varepsilon_i(a_i,b_i,p)\right|^p, \quad k = 1,\,2, \tag{5}$$

$$\sup_{(\xi,n)\in U_k(A,B)} \mathbf{E}\left|\sum_{i=1}^{n} \xi_i\right|^p = \Big(\frac{A}{B^2}\Big)^{p/(p-2)} \mathbf{E}\,\big|\xi_1(A,\,B) - \xi_2(A,\,B)\big|^p, \quad k = 1,\,2, \tag{6}$$

$$\inf_{(\xi,n)\in M_1(n,a,b)} \mathbf{E}\left|\sum_{i=1}^{n} \xi_i\right|^p = \sum_{i=1}^{n}(b_i - a_i^p) + \mathbf{E}\left|\sum_{i=1}^{n} a_i\varepsilon_i\right|^p. \tag{7}$$

Remark 2. Relations (5)–(7) constitute the statement of Theorem 5 in [8].

Remark 3. Relations (5) and (7) mean that for $p \geqq 4$

$$\sup_{(\xi,n)\in M_k(n,a,b)} \mathbf{E}\left|\sum_{i=1}^{n} \xi_i\right|^p, \quad k = 1,\,2,$$

is attained at three-point and

$$\inf_{(\xi,n)\in M_1(n,a,b)} \mathbf{E}\left|\sum_{i=1}^{n} \xi_i\right|^p$$

at two-point distributions. Relations (2) and (4) show that in the case $3 \leqq p < 4$ one should permute the answers above.

Remark 4. Letting $k = 1$, $n = 2$, $p = 3$ in (2), we see that for independent random variables X and Y with common symmetric distribution and finite third moment the following inequalities hold true:

$$\mathbf{E}\,|X+Y|^3 \leqq 2\mathbf{E}\,|X|^3 + 2(\mathbf{E}X^2)^{3/2},$$

$$\mathbf{E}\,|X-Y|^3 \leqq 2\mathbf{E}\,|X|^3 + 2(\mathbf{E}X^2)^{3/2},$$

in which the constants are optimal. Provided $\mathbf{E}X = 0$ these inequalities have been proved by Esseen [9].

Before proceeding to demonstrate relations (2)–(4), we establish several lemmas.

Let ε be a r.v. with distribution $\mathbf{P}\{\varepsilon = 1\} = \mathbf{P}\{\varepsilon = -1\} = \frac{1}{2}$. Following [8] we set

$$H(x,\,p,\,z) = \mathbf{E}\,|\varepsilon x^{1/2} + z|^p - x^{p/2} - z^p,$$
$$R(x,\,p,\,z) = x^{-2/(p-2)}\Big(\mathbf{E}\,\big|\varepsilon x^{1/(p-2)} + z\big|^p - z^p\Big), \quad x > 0,\ z > 0.$$

Lemma 1. *The function $H(x,p,z)$ is non-negative and concave in x for $p \in (2,4)$.*

Proof. Let $p \in (2,4)$ and $f(x) = \mathbf{E}\,|\varepsilon x + z|^p - x^p - z^p$. It is not difficult to check that

$$f'(x) = p\Big(\frac{1}{2}\,\big((x+z)^{p-1} + |x-z|^{p-2}(x-z)\big) - x^{p-1}\Big) \geqq 0$$

for all $x \geqq 0$, $z \geqq 0$. Since $f(0) = 0$, the function $H(x,p,z)$ is non-negative.

We show that $H(x,p,z)$ is concave in x for $p \in (2,4)$. Since

$$\frac{\partial^2 H}{\partial x^2} = 0.25\,p x^{p/2-2}\big((p-2)\,\mathbf{E}\,|\varepsilon + u|^{p-2} - \mathbf{E}\,|\varepsilon + u|^{p-2}\varepsilon u - (p-2)\big),$$

where $u = x^{-1/2}z$, it suffices to demonstrate that

$$g(u) = (p-2)\,\mathbf{E}\,|\varepsilon + u|^{p-2} - \mathbf{E}\,|\varepsilon + u|^{p-2}\varepsilon u - (p-2) \leqq 0$$

for $u \geqq 0$ and $p \in (2,4)$.

For $u \geqq 1$, $0 \leqq \alpha < 1$ we have

$$\begin{aligned}(u+\alpha)\,(u-1)^{\alpha} &\leqq (u+\alpha)\left(1 - \frac{\alpha}{u} + \frac{\alpha(\alpha-1)}{2u^2}\right)u^{\alpha}\\ &\leqq (u-\alpha)\left(1 - \frac{\alpha}{u} + \frac{\alpha(\alpha-1)}{2u^2}\right)u^{\alpha} \leqq (u-\alpha)\,(u+1)^{\alpha},\end{aligned}$$

that is $\alpha\mathbf{E}\,|\varepsilon + u|^{\alpha} \leqq \mathbf{E}\,|\varepsilon + u|^{\alpha}\varepsilon u$. Consequently, $g(u) \leqq 0$ for $p \in (2,3)$, $u \geqq 1$ and $g'(u) \leqq 0$ for $p \in [3,4)$, $u \geqq 1$.

We know that for $0 \leqq u < 1$ and $0 \leqq \alpha < 1$

$$\mathbf{E}\,|\varepsilon + u|^{\alpha} \leqq 1, \qquad (\alpha - u)\,(1+u)^{\alpha} \leqq (u+\alpha)\,(1-u)^{\alpha};$$

hence, $g(u) \leqq 0$ for $p \in (2,3)$, $0 \leqq u < 1$ and $g'(u) \leqq 0$ for $p \in [3,4)$, $0 \leqq u < 1$.

Combining these estimates and recalling that $g(0) = 0$, we conclude that $g(u) \leqq 0$ for $p \in (2,4)$ and $u \geqq 0$. Lemma 1 is proved.

Lemma 2. *For all $p \in (2,4)$, $0 \leqq a_1 \leqq a_2$, $0 \leqq b_1 \leqq b_2$, $z \geqq 0$ the following inequality is valid*:

$$b_1 - a_1^p + \mathbf{E}\,|a_1\varepsilon + z|^p \leqq b_2 - a_2^p + \mathbf{E}\,|a_2\varepsilon + z|^p.$$

Proof. While checking the non-negativity of $H(x,p,z)$ we have established that the function $\mathbf{E}\,|\varepsilon x + z|^p - x^p$ is nondecreasing in $x \geqq 0$ for $z \geqq 0$. Hence, the statement of Lemma 2 follows easily.

Lemma 3. *The function $R(x,p,z)$ is non-negative and convex in x for $p \in [3,4)$.*

Proof. By Jensen's inequality $R(x,p,z)$ is non-negative for $p \geqq 1$. To demonstrate the convexity of $R(x,p,z)$ in x for $p \in [3,4)$ it suffices to establish the validity of the inequality (see [8, the proof of Lemma 3.3]):

$$M(u) = (p-3)\,\mathbf{E}\,|\varepsilon u+1|^{p-3}(\varepsilon u+1) - (p-1)\,\mathbf{E}\,|\varepsilon u+1|^{p-2} + 2 \leqq 0, \quad u \geqq 0.$$

It is not difficult to show that the following inequalities are valid:

$$(\alpha u - 1)\,(u+1)^{\alpha} - (\alpha u+1)\,(u-1)^{\alpha} \leqq 0$$

for $u \geqq 1$, $0 \leqq \alpha < 1$,

$$\begin{aligned}(\alpha u+1)\,(1-u)^{\alpha} &\leqq (\alpha u+1)\Big(1-\alpha u+\frac{\alpha(\alpha-1)u^2}{2}\Big)\\ &\leqq (1-\alpha u)\Big(1+\alpha u+\frac{\alpha(\alpha-1)u^2}{2}\Big) \leqq (1-\alpha u)\,(u+1)^{\alpha}\end{aligned}$$

for $0 \leqq u < 1$, $0 \leqq \alpha < 1$. Since

$$M'(u) = \frac{p-1}{2}\Big[\big((p-3)\,u-1\big)\,(u+1)^{p-3} - \big((p-3)\,u+1\big)\,(u-1)^{p-3}\Big]$$

for $u \geqq 1$ and

$$M'(u) = \frac{p-1}{2}\Big[\big((p-3)\,u+1\big)\,(1-u)^{p-3} - \big(1-(p-3)\,u\big)\,(u+1)^{p-3}\Big]$$

for $0 \leqq u < 1$, it follows that $M'(u) \leqq 0$ for $u \geqq 0$. On account of $M(0)=0$ we obtain the statement of Lemma 3.

Lemmas 4–7 below are analogues of the lemmas proved in [8].

LEMMA 4. *Let ξ be a random variable with symmetric distribution and let ξ,η,ε be independent random variables with*, $\mathbf{E}\xi^2 = a^2$, $\mathbf{E}\,|\xi|^p < \infty$, $\mathbf{E}\,|\eta|^p < \infty$, $p \in (2,4)$. *Then*

$$\mathbf{E}\,|\xi+\eta|^p \leqq \mathbf{E}\,|\xi|^p + \mathbf{E}\,|\varepsilon\alpha+\eta|^p - a^p. \tag{8}$$

Proof. From the proof of Lemma 7.4 in [8] it follows that to demonstrate (8) it suffices to check that

$$H(x^2,p,z)\,\lambda + H(y^2,p,z)\,\mu \leqq H(a^2)$$

for $\lambda+\mu \leqq 1$, λ, $\mu \geqq 0$, $\lambda x^2 + \mu y^2 = a^2$. By Lemma 1

$$\begin{aligned}H(x^2,\,p,\,z)\,\lambda + H(y^2,\,p,\,z)\,\mu &= (\lambda+\mu)\Big(H(x^2,\,p,\,z)\,\frac{\lambda}{\lambda+\mu} + H(y^2,\,p,\,z)\,\frac{\mu}{\lambda+\mu}\Big)\\ &\leqq (\lambda+\mu)\,H\Big(\frac{a^2}{\lambda+\mu}\Big) = (\lambda+\mu)\,H\Big(\frac{a^2}{\lambda+\mu}\Big) + (1-\lambda-\mu)\,H(0) \leqq H(a^2).\end{aligned}$$

Lemma 4 is proved.

LEMMA 5. *Let ξ be a random variable with symmetric distribution and let ξ,η,ε be independent random variables with $\mathbf{E}\,|\eta|^p < \infty$, $\mathbf{E}\xi^2 \leqq a^2$, $\mathbf{E}\,|\xi|^p \leqq b$ and $a^p \leqq b$, $a,b \geqq 0$, $p \in (2,4)$. Then*

$$\mathbf{E}\,|\xi+\eta|^p \leqq b + \mathbf{E}\,|\varepsilon\alpha+\eta|^p - a^p.$$

Proof. Lemma 5 follows from Lemmas 2 and 4.

Lemma 6. *If η and ε are independent random variables with $\mathbf{E}\,|\eta|^p < \infty$ and $a^p \leqq b$, $a, b \geqq 0$, $p \in (2,4)$, then*

$$\sup_{\xi} \mathbf{E}\,|\xi + \eta|^p = b + \mathbf{E}\,|\varepsilon\alpha + \eta|^p - a^p,$$

where sup *is taken over all random variables ξ with symmetric distributions being independent of η and satisfying the conditions $\mathbf{E}\xi^2 \leqq a^2$, $\mathbf{E}\,|\xi|^p \leqq b$.*

Proof. Lemma 6 easily follows from Lemma 5 and the proof of Lemma 7.5 in [8].

Lemma 7. *Let ξ be a random variable with symmetric distribution, and let $\xi, \eta, \varepsilon(a,b,p)$ be independent random variables with*

$$\mathbf{E}\,|\eta|^p < \infty, \qquad \mathbf{E}\xi^2 = a^2, \qquad \mathbf{E}\,|\xi|^p = b, \quad a^p \leqq b,\ a,b \geqq 0,\ p \in [3,4).$$

Then

$$\mathbf{E}\,|\xi + \eta|^p \geqq \mathbf{E}\,\bigl|\varepsilon(a,\, b,\, p) + \eta\bigr|^p.$$

Proof. The proof of Lemma 7 follows the pattern of Lemma 7.1 in [8]. According to Lemma 3 it is necessary only to reverse the inequality signs in (7.2) and (7.3).

Proof of the proposition. Inequalities (2) and (4) easily follow from Lemmas 6 and 7 by induction.

We prove (3). Let

$$D_1^n(A,\, B) = \Bigl\{(a,n),\,(b,n)\colon\ a_i \geqq 0,\ b_i \geqq 0,\ a_i^p \leqq b_i,\ i = 1,\dots,n,$$
$$\sum_{i=1}^n a_i^2 = B^2,\ \sum_{i=1}^n b_i = A\Bigr\},$$
$$D_2^n(A,\, B) = \Bigl\{(a,n),\,(b,n)\colon\ a_i \geqq 0,\ b_i \geqq 0,\ a_i^p \leqq b_i,\ i = 1,\dots,n,$$
$$\sum_{i=1}^n a_i^2 \leqq B^2,\ \sum_{i=1}^n b_i \leqq A\Bigr\},$$
$$D_1(A,\, B) = \bigcup_{n=1}^\infty D_1^n(A,\, B), \quad D_2(A,\, B) = \bigcup_{n=1}^\infty D_2^n(A,\, B).$$

Since (see [10])

$$\mathbf{E}\,\Bigl|\sum_{i=1}^n a_i\varepsilon_i\Bigr|^p \leqq \frac{2^{p/2}}{\pi^{1/2}}\,\Gamma\Bigl(\frac{p+1}{2}\Bigr)\,B^p$$

for $\sum_{i=1}^n a_i^2 = B^2$, it follows that

$$\begin{aligned}
\sup_{(\xi,n)\in U_k(A,B)} \mathbf{E}\,\Bigl|\sum_{i=1}^n \xi_i\Bigr|^p &= \sup_{(a,n),(b,n)\in D_k(A,B)}\ \sup_{(\xi,n)\in M_1(n,a,b)} \mathbf{E}\,\Bigl|\sum_{i=1}^n \xi_i\Bigr|^p \\
&= \sup_{(a,n),(b,n)\in D_k(A,B)} \sum_{i=1}^n (b_i - a_i^p) + \mathbf{E}\,\Bigl|\sum_{i=1}^n a_i\varepsilon_i\Bigr|^p \\
&\leqq A + \frac{2^{p/2}}{\pi^{1/2}}\,\Gamma\Bigl(\frac{p+1}{2}\Bigr)\,B^p.
\end{aligned}$$

We know that

$$A + \sup_n B^p\biggl(\mathbf{E}\,\Bigl|\sum_{i=1}^n \frac{\varepsilon_i}{n^{1/2}}\Bigr|^p - n^{1-p/2}\biggr) = A + \frac{2^{p/2}}{\pi^{1/2}}\,\Gamma\Bigl(\frac{p+1}{2}\Bigr)\,B^p;$$

hence,

$$\sup_{(\xi,n)\in U_k(A,B)} \mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p = A + \frac{2^{p/2}}{\pi^{1/2}}\,\Gamma\Big(\frac{p+1}{2}\Big)\,B^p.$$

Proof of Theorem 1.

$$\sup_{(\xi,n)\in U_1(D,D^{1/p})} \mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p \leqq \sup_{(\xi,n)\in U(D)} \mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p \leqq \sup_{(\xi,n)\in U_2(D,D^{1/p})} \mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p$$

and relations (3) and (6) we obtain

$$\sup_{(\xi,n)\in U(D)} \mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p = \left(1 + \frac{2^{p/2}}{\pi^{1/2}}\,\Gamma\Big(\frac{p+1}{2}\Big)\right) D, \quad 2 < p < 4,$$

$$\sup_{(\xi,n)\in U(D)} \mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p = \mathbf{E}\,|\theta_1 - \theta_2|^p D, \quad p \geqq 4.$$

This fact and the obvious relation

$$C_p^* = \sup_{D>0}\left(\sup_{(\xi,n)\in U(D)} \frac{1}{D}\,\mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p\right)^{1/p}$$

yield the statement of the theorem.

Before proving Theorem 2 we formulate the following lemma.

LEMMA 8 ([7]). *For $p = 2s$, $s \in \mathbf{N}$, the following relation is valid*:

$$\sup_{(\xi,n)\in U_1(A,B)} \mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p = \sum_{j=1}^{s}\Gamma_{j,s}\big(A^{s-j}B^{2s(j-1)}\big)^{1/(s-1)},$$

where

$$\Gamma_{j,s} = (2s)!\sum_{r=1}^{j}\sum\prod_{k=1}^{r}\frac{((2m_k)!)^{-j_k}}{j_k!},$$

and the internal sum is ranged over all positive integers $m_1 > \cdots > m_r$ and $j_1,\dots,j_r$, meeting the conditions $m_1j_1 + \cdots + m_rj_r = s$, $j_1 + \cdots + j_r = j$.

Proof of Theorem 2.

$$\sup_{(\xi,n)\in U_k(D,D^{1/p})} \mathbf{E}\left|\sum_{i=1}^{n}\xi_i\right|^p = \left(\sum_{j=1}^{s}\Gamma_{j,s}\right) D = T_{2s}D, \quad k = 1, 2,$$

where T_{2s} is the number of partitions of a $(2s)$-set into parts each of which contains an even number of elements (see [11, p. 280–281]). Hence we deduce, following the pattern of demonstrating Theorem 1, that $C_{2s}^* = T_{2s}^{1/(2s)}$. Theorem 2 is proved.

Proof of the corollary. We establish a stronger result:

$$C_p^* = \frac{p}{e\log p}\left(1 + o\Big(\frac{\log^2\log p}{\log p}\Big)\right).$$

Theorem 1 implies

$$(C_p^*)^p = \frac{2}{e}\sum_{m=1}^{\infty}\sum_{k=0}^{\infty}\frac{m^p}{k!(m+k)!\,2^{m+2k}}.$$

First we observe that

$$\frac{2}{e}\sum_{m=1}^{\infty}\frac{m^p}{2^m m!} \leqq (C_p^*)^p \leqq \frac{2}{e}\sum_{m=1}^{\infty}\frac{m^p}{2^m m!}\sum_{k=0}^{\infty}\frac{1}{2^{2k}k!} \leqq \frac{32}{15e}\sum_{m=1}^{\infty}\frac{m^p}{2^m m!}. \tag{9}$$

Set

$$S=\sum_{m=1}^{\infty}\frac{m^p}{2^m m!}.$$

We find the asymptotics of S in a manner similar to the procedure for calculating the asymptotics of Bell's numbers [11, Chapter V, section 8].

We write S as a sum of three summands

$$S=S_1+S_2+S_3=\sum_{m=1}^{\mu-1}+\sum_{m=\mu}^{\nu}+\sum_{m=\nu+1}^{\infty},$$

where $\mu,\nu=[e^r/2+Ap^{1/2}]$, r is a unique solution of the equation $re^r=2p$ and A is a positive constant.

By Stirling's formula we find (here all the quantities o(1) are of exponential order)

$$\begin{aligned}
S_2&=\frac{1}{(2\pi)^{1/2}}\sum_{-Ap^{1/2}}^{Ap^{1/2}}\Big(m+\Big[\frac{e^r}{2}\Big]\Big)^{p-m-[e^r/2]-1/2}\Big(\frac{e}{2}\Big)^{[e^r/2]+m}\big(1+o(1)\big)\\
&=\frac{1}{(2\pi(r+1))^{1/2}}\Big(\frac{e^r}{2}\Big)^{p-e^r/2}\Big(\frac{e}{2}\Big)^{e^r/2}\\
&\quad\times\sum_{-Ap^{1/2}}^{Ap^{1/2}}\exp\Big(-\frac{m^2r(r+1)}{2p}\Big)\Big(\frac{r\,(r+1)}{m}\Big)^{1/2}\big(1+o(1)\big)\\
&=\frac{1}{(2\pi(r+1))^{1/2}2^p}\exp\Big(p\Big(r+\frac{1}{r}-1\Big)\Big)\int_{-A(p(p+1))^{1/2}}^{A(p(p+1))^{1/2}}\exp\Big(-\frac{y^2}{2}\Big)\,dy\,\big(1+o(1)\big)\\
&=\frac{1}{(r+1)^{1/2}2^p}\exp\Big(p\Big(r+\frac{1}{r}-1\Big)\Big)\big(1+o(1)\big).
\end{aligned}$$

By an analogy with [11] one can show that for some A

$$\frac{S_1}{S_2}=O\Big(\frac{1}{r}\Big(\frac{r}{p}\Big)^{(r+1)/2}\Big),\qquad \frac{S_3}{S_2}=O\Big(\frac{1}{r}\Big(\frac{r}{p}\Big)^{(r+1)/2}\Big).$$

Therefore,

$$S=\frac{1}{(r+1)^{1/2}2^p}\exp\Big(p\Big(r+\frac{1}{r}-1\Big)\Big)\big(1+o(1)\big),$$

where r is a unique solution of the equation $re^r=2p$.

Taking into account the relation

$$r=\log 2p-\log\log 2p+O\Big(\frac{\log\log p}{\log p}\Big),$$

which is valid for large p, we deduce from (9) that

$$C_p^*=S^{1/p}\Big(1+o\Big(\frac{1}{p}\Big)\Big)=0.5\,\exp\Big(r+\frac{1}{r}-1\Big)\Big(1+o\Big(\frac{\log p}{p}\Big)\Big)$$

$$= \frac{p}{e \log p} \left(1 + o\Big(\frac{\log^2 \log p}{\log p}\Big)\right).$$

The corollary is proved.

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