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BOUNDS ON MOMENTS OF SYMMETRIC STATISTICS

R. IBRAGIMOV and SH. SHARAKHMETOV

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Abstract

In this paper we prove analogues of Khintchine, Marcinkiewicz–Zygmund and Rosenthal moment inequalities for symmetric statistics of arbitrary order in not identically distributed random variables. We also construct an example that shows the significance of each term in the obtained Rosenthal-type inequalities for symmetric statistics and obtain results concerning the rate of growth of the best constants in the inequalities.

In what follows, $A(\cdot)$, $B(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$ denote constants depending on parameters in the parentheses only, not necessarily the same from one place to another, and C and C_1 denote absolute constants, not necessarily the same from one place to another. The following inequalities are well known for sums of independent random variables (r.v's).

Rosenthal inequalities ([21]). *If $\xi_1, \dots, \xi_n$ are independent nonnegative r.v's with $\mathbf{E}\xi_i^t < \infty$, $i = 1, \dots, n$, $1 \leqq t < \infty$, then*

$$\max\left(\sum_{i=1}^n \mathbf{E}\xi_i^t, \left(\sum_{i=1}^n \mathbf{E}\xi_i\right)^t\right) \leqq \mathbf{E}\left(\sum_{i=1}^n \xi_i\right)^t \leqq B(t) \max\left(\sum_{i=1}^n \mathbf{E}\xi_i^t, \left(\sum_{i=1}^n \mathbf{E}\xi_i\right)^t\right). \tag{1}$$

If $\xi_1, \dots, \xi_n$ are independent r.v's with $\mathbf{E}\xi_i = 0$, $\mathbf{E}|\xi_i|^t < \infty$, $i = 1, \dots, n$, $2 \leqq t < \infty$, then

$$A(t) \max\left(\sum_{i=1}^n \mathbf{E}|\xi_i|^t, \left(\sum_{i=1}^n \mathbf{E}\xi_i^2\right)^{t/2}\right) \leqq \mathbf{E}\left|\sum_{i=1}^n \xi_i\right|^t \leqq B(t) \max\left(\sum_{i=1}^n \mathbf{E}|\xi_i|^t, \left(\sum_{i=1}^n \mathbf{E}\xi_i^2\right)^{t/2}\right). \tag{2}$$

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KHINTCHINE INEQUALITIES ([13]). *If* $\epsilon_1, \ldots, \epsilon_n$ *are independent symmetric Bernoulli r.v's, that is* $\mathbf{P}(\epsilon_i = 1) = \mathbf{P}(\epsilon_i = -1) = 1/2$, $i = 1, \ldots, n$, *then*

$$A(t)\biggl(\sum_{i=1}^n a_i^2\biggr)^{t/2} \leqq \mathbf{E}\biggl|\sum_{i=1}^n a_i\epsilon_i\biggr|^t \leqq B(t)\biggl(\sum_{i=1}^n a_i^2\biggr)^{t/2}, \tag{3}$$

$t > 0$, $a_i \in R$, $i = 1, \ldots, n$.

MARCINKIEWICZ–ZYGMUND INEQUALITIES ([19]). *If* $\xi_1, \ldots, \xi_n$ *are independent r.v's with* $\mathbf{E}\xi_i = 0$, $\mathbf{E}|\xi_i|^t < \infty$, $i = 1, \ldots, n$, $1 \leqq t < \infty$, *then*

$$A(t)\mathbf{E}\biggl(\sum_{i=1}^n \xi_i^2\biggr)^{t/2} \leqq \mathbf{E}\biggl|\sum_{i=1}^n \xi_i\biggr|^t \leqq B(t)\mathbf{E}\biggl(\sum_{i=1}^n \xi_i^2\biggr)^{t/2}. \tag{4}$$

A number of papers have focused on extensions of inequalities (1)–(4) in the case of symmetric statistics and related problems (e.g., Serfling [22], Krakowiak and Szulga [18], McConnell and Taqqu [20], De la Peña [3], De la Peña and Klass [5], Koroljuk and Borovskikh [17], Sharakhmetov [23], Borovskikh and Korolyuk [1], Ibragimov [8], Klass and Nowicki [14]–[16], Ibragimov and Sharakhmetov [9], [10] and Ibragimov et al. [11]). McConnell and Taqqu [20] and Krakowiak and Szulga [18] obtained generalizations of inequalities (3) and (4) in the case of multilinear forms. De la Peña [3] proved extensions of inequalities (3) and (4) for U-statistics of second order. Sharakhmetov [23] proved analogues of Rosenthal inequalities (1) and (2) for symmetric statistics of second order in identically distributed r.v's and independently obtained generalizations of inequalities (3) and (4) for that class of symmetric statistics. Klass and Nowicki [14], [15] obtained Rosenthal-type inequalities for generalized moments of symmetric statistics of second order with nonnegative kernels in not necessarily identically distributed r.v's. Ibragimov and Sharakhmetov [10] obtained extensions of inequalities (1) and (2) in the case of symmetric statistics of second order in not necessarily identically distributed r.v's and proved analogues of inequalities (3) and (4) using those results. One should note that the method used in [10] allows one to obtain, using the results in [12] and [26], analogues of upper Rosenthal inequalities (1) and (2) for the t-th moment of symmetric statistics of second order with the constants $(Ct/\ln t)^{2t}$ having the best possible rate of growth (see Remark 3 after Corollary 4 and Remark 5 at the end of the paper). Klass and Nowicki [16] obtained results concerning estimates for moments of symmetric statistics of arbitrary order.

The main goal of this paper is to extend the results obtained in Sharakhmetov [23] and Ibragimov and Sharakhmetov [10] in the case of symmetric statistics of arbitrary order in not identically distributed r.v's. More precisely, we obtain analogues of Khintchine, Marcinkiewicz–Zygmund and Rosenthal moment inequalities (1)–(4) for those objects (Theorems 1–4). We also construct an example that shows the significance of each term in the obtained Rosenthal-type inequalities for symmetric statistics (Remark 2) and obtain results concerning the rate of growth of the best constants in the inequalities (Remarks 3 and 5). The results obtained in the paper were announced in Ibragimov and Sharakhmetov [9].

Let $1 \leqq m \leqq n$, $t > 0$, and let $X_1, \ldots, X_n$ be independent, not necessarily identically distributed r.v's on a probability space $(\Omega, \mathfrak{I}, \mathbf{P})$.

Denote by M_m the group of all permutations of the set $\{1, 2, \ldots, m\}$. Let $F(t, m)$ be the class of functions $Y_{i_1,\ldots,i_m} : \mathbf{R}^m \to \mathbf{R}$, $1 \leqq i_k \leqq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \ldots, m$, that satisfy the conditions

$$\mathbf{E}|Y_{i_1,\ldots,i_m}(X_{i_1}, \ldots, X_{i_m})|^t < \infty,$$
$$Y_{i_1,\ldots,i_m}(X_{i_1}, \ldots, X_{i_m}) = Y_{i_{\pi(1)},\ldots,i_{\pi(m)}}(X_{i_{\pi(1)}}, \ldots, X_{i_{\pi(m)}}) \quad \text{a.s.}$$

for all $1 \leqq i_1 < i_2 < \cdots < i_m \leqq n$ and all $\pi \in M_m$.

In what follows, we assume that if $Z_1, Z_2, \ldots$ are r.v's on $(\Omega, \mathfrak{I}, \mathbf{P})$, then $\sigma(Z_1, Z_2, \ldots, Z_k) = (\emptyset, \Omega)$ for $k = 0$.

Denote by $G(t, m)$ the subset of $F(t, m)$ consisting of functions Y such that

$$\mathbf{E}(Y_{i_1,\ldots,i_m}(X_{i_1}, \ldots, X_{i_m}) | X_{i_{\pi(1)}}, \ldots, X_{i_{\pi(m-1)}}) = 0 \quad \text{a.s.}$$

for all $1 \leqq i_1 < i_2 < \cdots < i_m \leqq n$ and all $\pi \in M_m$.

Definition. Let $\mathfrak{I}'$ be a σ-algebra on a probability space $(\Omega, \mathfrak{I}, \mathbf{P})$. An $\mathfrak{I}'$-measurable r.v. X is called *conditionally symmetric* (on $\mathfrak{I}'$) if $\mathbf{P}(X > a \,|\, \mathfrak{I}') = \mathbf{P}(X < a \,|\, \mathfrak{I}')$ for all $a \geqq 0$.

Let $G'(t, m)$ be a subset of $G(t, m)$ consisting of functions Y for which the r.v. $Y_{i_1,\ldots,i_m}(X_{i_1}, \ldots, X_{i_m})$ is conditionally symmetric on the σ-algebra $\sigma(X_{i_{\pi(1)}}, \ldots, X_{i_{\pi(m-1)}})$ for all $1 \leqq i_1 < i_2 < \cdots < i_m \leqq n$, $\pi \in M_m$.

It is easy to see that if $\mathbf{E}X_i = 0$, $\mathbf{E}|X_i|^t < \infty$, $i = 1, \ldots, n$, then the functions $Y_{i_1,\ldots,i_m} : \mathbf{R}^m \to \mathbf{R}$; $Y_{i_1,\ldots,i_m}(x_1, x_2, \ldots, x_m) = x_1 x_2 \ldots x_m$, $x_k \in \mathbf{R}$, $1 \leqq i_k \leqq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \ldots, m$, belong to the class $G(t, m)$. If the r.v's $X_1, \ldots, X_n$ are symmetric and $\mathbf{E}|X_i|^t < \infty$, $i = 1, \ldots, n$, then the above-defined functions Y belong to the class $G'(t, m)$.

For $Y \in F(t, m)$, denote by

$$T_{n,m} = \sum_{1 \leqq i_1 < \cdots < i_m \leqq n} Y_{i_1,\ldots,i_m}(X_{i_1}, X_{i_2}, \ldots, X_{i_m})$$

a symmetric statistic (U-statistic) in r.v's $X_1, \ldots, X_n$.

The following Theorems 1 and 2 give extensions of Rosenthal inequalities (1) and (2) in the case of general symmetric statistics $T_{n,m}$. The upper inequalities given by Theorems 1 and 2 (as well as by Corollaries 1–4) hold with the constants $B(t, m) = (Ct/\ln t)^{mt}$ which have the best possible rate of growth (see Remarks 3 and 5 below).

Theorem 1. *If* $t \geqq 1$, $Y_{i_1,\dots,i_m} : \mathbf{R}^m \to \mathbf{R}$, $1 \leqq i_k \leqq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \dots, m$, *are nonnegative functions from the class* $F(t, m)$, *then*

$$\max_{k=0,m} \max_{1\leqq j_1<\dots<j_k\leqq m} \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m, s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} \mathbf{E}(Y_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m}) \,|\, X_{i_{j_1}},\dots,X_{i_{j_k}})\Bigg)^t$$

$$\leqq \mathbf{E}T_{n,m}^t \tag{5}$$

$$\leqq B(t,m) \max_{k=0,m} \max_{1\leqq j_1<\dots<j_k\leqq m} \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m, s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} \mathbf{E}(Y_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m}) \,|\, X_{i_{j_1}},\dots,X_{i_{j_k}})\Bigg)^t.$$

Theorem 2. *If* $t \geqq 2$, $Y_{i_1,\dots,i_m} : \mathbf{R}^m \to \mathbf{R}$, $1 \leqq i_k \leqq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \dots, m$, *are functions from the class* $G(t, m)$, *then*

$$A(t,m) \max_{k=0,m} \max_{1\leqq j_1<\dots<j_k\leqq m} \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} \mathbf{E}(Y^2_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m}) \,|\, X_{i_{j_1}},\dots,X_{i_{j_k}})\Bigg)^{t/2}$$

$$\leqq \mathbf{E}|T_{n,m}|^t \tag{6}$$

$$\leqq B(t,m) \max_{k=0,m} \max_{1\leqq j_1<\dots<j_k\leqq m} \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} \mathbf{E}(Y^2_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m}) \,|\, X_{i_{j_1}},\dots,X_{i_{j_k}})\Bigg)^{t/2}.$$

The following Theorem 3 gives generalizations of Khintchine and Marcinkiewicz–Zygmund inequalities (3) and (4).

Theorem 3. *If* $t \geqq 1$, $Y_{i_1,\dots,i_m} : \mathbf{R}^m \to \mathbf{R}$, $1 \leqq i_k \leqq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \dots, m$, *are functions from the class* $G(t, m)$, *then*

$$A(t,m)\mathbf{E}\Bigg(\sum_{1\leqq i_1<\dots<i_m\leqq n} Y^2_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m})\Bigg)^{t/2} \leqq \mathbf{E}|T_{n,m}|^t \tag{7}$$

for $t \geqq 2$,

$$\mathbf{E}|T_{n,m}|^t \leqq B(t,m)\mathbf{E}\Bigg(\sum_{1\leqq i_1<\cdots<i_m\leqq n} Y^2_{i_1,\ldots,i_m}(X_{i_1},\ldots,X_{i_m})\Bigg)^{t/2} \tag{8}$$

for $t \geqq 1$. *If* $t>0$, $Y_{i_1,\ldots,i_m}:\mathbf{R}^m\to\mathbf{R}$, $1\leqq i_k\leqq n$; $i_r\neq i_s$, $r\neq s$; $k,r,s=1,\ldots,m$, *are functions from the class* $G'(t,m)$, *then*

$$\mathbf{E}\Bigg(\sum_{1\leqq i_1<\cdots<i_m\leqq n} Y^2_{i_1,\ldots,i_m}(X_{i_1},\ldots,X_{i_m})\Bigg)^{t/2} \leqq \mathbf{E}|T_{n,m}|^t \tag{9}$$

for $t \geqq 2$,

$$\mathbf{E}|T_{n,m}|^t \leqq B(t,m)\mathbf{E}\Bigg(\sum_{1\leqq i_1<\cdots<i_m\leqq n} Y^2_{i_1,\ldots,i_m}(X_{i_1},\ldots,X_{i_m})\Bigg)^{t/2} \tag{10}$$

for $t>0$, *where* $B(t,m)=1$ *for* $0<t\leqq 2$.

Remark 1. From Theorem 1 it follows that, for $t \geqq 2$, the analogues of Rosenthal inequalities (6) and Marcinkiewicz–Zygmund inequalities (7), (8) are equivalent to within a constant, that is if inequalities (6) hold for a symmetric statistics $T_{n,m}$, then inequalities (7) and (8) hold as well (in general, with other constants) and vice versa.

In the case of identically distributed r.v's $X_1,\ldots,X_n$ and kernels Y that do not depend on the summation indices, inequalities (5) and (6) have a simpler form.

Corollary 1. *If* $t \geqq 1$, $X_1,\ldots,X_n$ *are independent identically distributed r.v's,* $Y:\mathbf{R}^m\to\mathbf{R}$ *is a nonnegative function satisfying the conditions*

$$\mathbf{E}Y^t(X_1,X_2,\ldots,X_m)<\infty,$$
$$Y(X_1,X_2,\ldots,X_m)=Y(X_{\pi(1)},X_{\pi(2)},\ldots,X_{\pi(m)}) \quad a.s.$$

for all $\pi\in M_m$, *then*

$$A(t,m)\max_{k=0,m} n^{t(m-k)+k}\mathbf{E}(\mathbf{E}(Y(X_1,X_2,\ldots,X_m)|X_1,\ldots,X_k))^t \leqq \mathbf{E}T^t_{n,m}$$
$$\leqq B(t,m)\max_{k=0,m} n^{t(m-k)+k}\mathbf{E}(\mathbf{E}(Y(X_1,X_2,\ldots,X_m)|X_1,\ldots,X_k))^t.$$

Corollary 2. *If* $t \geqq 2$, $X_1,\ldots,X_n$ *are independent identically distributed r.v's,* $Y:\mathbf{R}^m\to\mathbf{R}$ *is a function satisfying the conditions*

$$\mathbf{E}|Y(X_1,X_2,\ldots,X_m)|^t<\infty,$$

$$\mathbf{E}(Y(X_1, X_2, \dots, X_m)|X_1, \dots, X_{m-1}) = 0 \quad a.s.,$$
$$Y(X_1, X_2, \dots, X_m) = Y(X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(m)}) \quad a.s.$$

for all $\pi \in M_m$, *then*

$$A(t,m) \max_{k=0,m} n^{t/2(m-k)+k} \mathbf{E}(\mathbf{E}(Y^2(X_1, X_2, ..., X_m)|X_1, \dots, X_k))^{t/2} \leqq \mathbf{E}|T_{n,m}|^t$$
$$\leqq B(t,m) \max_{k=0,m} n^{t/2(m-k)+k} \mathbf{E}(\mathbf{E}(Y^2(X_1, X_2, \dots, X_m)|X_1, \dots, X_k))^{t/2}.$$

Remark 2. Let us bring an example that shows that all terms in the expression

$$\varphi_n = \max_{k=0,m} n^{t/2(m-k)+k} \mathbf{E}(\mathbf{E}(Y^2(X_1, X_2, \dots, X_m)|X_1, \dots, X_k))^{t/2}, \quad t > 2$$

(Corollary 2), and, therefore, all terms in the expression

$$\psi_n = \max_{k=0,m} n^{t(m-k)+k} \mathbf{E}(\mathbf{E}(Y(X_1, X_2, \dots, X_m)|X_1, \dots, X_k))^{t}, \quad t > 1$$

(Corollary 1) are significant, that is no one of them can be omitted. Let $0 \leqq k \leqq m$, $t > 2$, $n \geqq 2$, $a = 1/t - n^{-t/2}$, and let the r.v's $X_1, \dots, X_n$ have an exponential distribution with parameter 1: $\mathbf{P}(X_i \leqq x) = 0$, $x \leqq 0$; $\mathbf{P}(X_i \leqq x) = 1 - e^{-x}$, $x > 0$, $i = 1, \dots, n$ (the property of the above distribution which is important to us is $\mathbf{E}\exp(aX_i) = (1-a)^{-1}$, $0 < a < 1$, $i = 1, \dots, n$). Set

$$Y(x_1, ..., x_m) = \sum_{\pi \in M_m} \Bigg[\exp\Bigg(a \sum_{i=1}^{k} x_{\pi(i)} - a \sum_{i=k+1}^{m} x_{\pi(i)} \Bigg) + \sum_{l=1}^{m} (-1)^l \sum_{1 \leqq j_1 < \dots < j_l \leqq m}$$
$$\mathbf{E}\Bigg(\exp\Bigg(a \sum_{i=1}^{k} X_{\pi(i)} - a \sum_{i=k+1}^{m} X_{\pi(i)} \Bigg) \Bigg| X_s = x_s, s \in \{1, ..., m\} \setminus \{j_1, ..., j_l\} \Bigg) \Bigg]$$
$$= \sum_{\pi \in M_m} \Bigg[\exp\Bigg(a \sum_{i=1}^{k} x_{\pi(i)} - a \sum_{i=k+1}^{m} x_{\pi(i)} \Bigg)$$
$$+ \sum_{l=1}^{m} (-1)^l \sum_{1 \leqq j_1 < \dots < j_l \leqq m} \prod_{s=1}^{l} (1/(1 - a \operatorname{sign}(k + 0.5 - j_s))) \times$$
$$\times \exp\Bigg(a \sum_{\substack{i=1 \\ i \neq j_1, \dots, j_l}}^{k} x_{\pi(i)} - a \sum_{\substack{i=k+1 \\ i \neq j_1, \dots, j_l}}^{m} x_{\pi(i)} \Bigg) \Bigg],$$

where $\operatorname{sign}(k + 0.5 - j_s) = (k + 0.5 - j_s)/|k + 0.5 - j_s|$. It is not difficult to see that $Y \in G(t,m)$. The expressions

$$n^{t/2(m-l)+l} \mathbf{E}(\mathbf{E}(Y^2(X_1, X_2, \dots, X_m)|X_1, \dots, X_l))^{t/2}, \quad l = 0, \dots, k,$$

have the rate of growth $n^{t/2(m-l)+l}(\mathbf{E}\exp(taX_i))^l = n^{tm/2+l}$ as $n\to\infty$, and the expressions

$$n^{t/2(m-l)+l}\mathbf{E}(\mathbf{E}(Y^2(X_1,X_2,\dots,X_m)|X_1,\dots,X_l))^{t/2}, \quad l=k+1,\dots,n,$$

have the rate of growth $n^{t/2(m-l)+l}(\mathbf{E}\exp(taX_i))^k = n^{t(m+k)/2-(t/2-1)l}$. Since $tm/2+l<tm/2+k$, $l=0,\dots,k-1$, and $t(m+k)/2-(t/2-1)l<tm/2+k$, $l=k+1,\dots,n$, the rate of growth of ϕ_n is the same as the rate of growth of the term $n^{t/2(m-k)+k}\mathbf{E}(\mathbf{E}(Y^2(X_1,X_2,\dots,X_m)|X_1,\dots,X_k))^{t/2}$, that proves its significance.

The following Corollaries 3–5, which are immediate consequences of Theorems 1–3, give analogues of Rosenthal, Khintchine and Marcinkiewicz–Zygmund inequalities for multilinear forms.

Corollary 3. *If $t\geqq 1$, $X_1,\dots,X_n$ are independent nonnegative random variables with $\mathbf{E}X_k^t<\infty$, $k=1,\dots,n$, then*

$$\max_{k=0,m}\max_{1\leqq j_1<\dots<j_k\leqq m}\sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n}\prod_{l=1}^{k}\mathbf{E}X_{i_{j_l}}^t\times$$
$$\times\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}}\ \prod_{\substack{l\neq j_1,\dots,j_k\\ l=1,\dots,m}}\mathbf{E}X_{i_l}\Bigg)^t\leqq\mathbf{E}\Bigg(\sum_{1\leqq i_1<\dots<i_m\leqq n}X_{i_1}\dots X_{i_m}\Bigg)^t$$
$$\leqq B(t,m)\max_{k=0,m}\max_{1\leqq j_1<\dots<j_k\leqq m}\sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n}\prod_{l=1}^{k}\mathbf{E}X_{i_{j_l}}^t\times$$
$$\times\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}}\ \prod_{\substack{l\neq j_1,\dots,j_k\\ l=1,\dots,m}}\mathbf{E}X_{i_l}\Bigg)^t.$$

Corollary 4. *If $t\geqq 2$, $X_1,\dots,X_n$ are independent r.v's with $\mathbf{E}X_k=0$, $\mathbf{E}|X_k|^t<\infty$, $k=1,\dots,n$, then*

$$A(t,m)\max_{k=0,m}\max_{1\leqq j_1<\dots<j_k\leqq m}\sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n}\prod_{l=1}^{k}\mathbf{E}|X_{i_{j_l}}|^t\times$$
$$\times\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}}\ \prod_{\substack{l\neq j_1,\dots,j_k\\ l=1,\dots,m}}\mathbf{E}X_{i_l}^2\Bigg)^{t/2}\leqq\mathbf{E}\Bigg|\sum_{1\leqq i_1<\dots<i_m\leqq n}X_{i_1}\dots X_{i_m}\Bigg|^t$$
$$\leqq B(t,m)\max_{k=0,m}\max_{1\leqq j_1<\dots<j_k\leqq m}\sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n}\prod_{l=1}^{k}\mathbf{E}|X_{i_{j_l}}|^t\times$$

$$\times\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}}\ \prod_{\substack{l\neq j_1,\dots,j_k\\ l=1,\dots,m}}\mathbf{E}X_{i_l}^2\Bigg)^{t/2}.$$

Remark 3. The best constants in the inequalities given by Corollaries 3 and 4 in the case of bilinear forms for identically distributed nonnegative or symmetric r.v's were found in Ibragimov et al. [11]. The following examples show that the best constants $B_1(t,m)$ and $B_2(t,m)$ in the analogues of upper inequalities given by Corollaries 3 and 4 for L_t-norms, and, therefore, the best constants in the analogues of right-hand inequalities (5) and (6) for L_t-norms grow not slower than $(t/\ln t)^m$ as $t\to\infty$. From these examples it follows that the rate of growth of the constants $(Ct/\ln t)^{mt}$ in the inequalities given by Theorems 1 and 2 and Corollaries 1–4 is the best possible. Let $t>1$ be a sufficiently large number and let $n\geqq m\geqq 1$. Let, as in Proposition 2.9 in [12], $X_1,\dots,X_n$ be independent r.v's with the distribution $\mathbf{P}(X_i=1)=\ln t/t$, $\mathbf{P}(X_i=0)=1-\ln t/t$, $i=1,\dots,n$. Then (here and in what follows, $C_n^m=n!/(m!(n-m)!)$ and $\|U\|_t=(\mathbf{E}|U|^t)^{1/t}$ denotes the L_t-norm of a random variable U)

$$\Bigg\|\sum_{1\leqq i_1<\dots<i_m\leqq n}X_{i_1}\dots X_{i_m}\Bigg\|_t\geqq C_n^m\Bigg(\mathbf{P}\Bigg(\sum_{1\leqq i_1<\dots<i_m\leqq n}X_{i_1}\dots X_{i_m}=C_n^m\Bigg)\Bigg)^{1/t}$$
$$=C_n^m(\mathbf{P}(X_i=1))^{n/t}\geqq A(m)n^m(\ln t/t)^{n/t}.$$

By Corollaries 1 and 3,

$$A(m)n^m(\ln t/t)^{n/t}\leqq B_1(t,m)\max_{k=0,m}n^{(m-k)+k/t}(\mathbf{E}X_1^t)^{k/t}(\mathbf{E}X_1)^{m-k}.$$

Therefore,

$$B_1(t,m)\geqq A(m)\min_{k=0,m}n^{k-k/t}(\ln t/t)^{n/t-(m-k)-k/t}.$$

Choosing n such that $n-1\leqq t/\ln t\leqq n$, we get

$$B_1(t,m)\geqq A(m)(t/\ln t)^m(\ln t/t)^{1/t+1/\ln t}$$
$$\geqq A(m)(t/\ln t)^m(\ln t)^{1/\ln t}/(t^{1/\ln t}t^{1/t})=A(m)(t/\ln t)^m(\ln t)^{1/\ln t}/(et^{1/t}).$$

Since the function $y(x)=x^{1/x}$ is decreasing in $e\leqq x<\infty$, for all $t\geqq e^e$ we have $(\ln t)^{1/\ln t}\geqq t^{1/t}$, that is $B_1(t,m)\geqq A(m)(t/\ln t)^m/e$. Therefore, the constant $B_1(t,m)$ grows not slower than $(t/\ln t)^m$ as $t\to\infty$. Now let $t>2$ be a sufficiently large number, $n\geqq m\geqq 1$, and let, as in Proposition 4.3 in [12], $X_1,\dots,X_n$ be independent r.v's with the distribution $\mathbf{P}(X_i=1)=$

$\mathbf{P}(X_i=-1)=\ln t/t$, $\mathbf{P}(X_i=0)=1-2\ln t/t$, $i=1,\dots,n$. Then, similarly to the above,

$$\Big\|\sum_{1\leqq i_1<\dots<i_m\leqq n} X_{i_1}\dots X_{i_m}\Big\|_t \geqq A(m)n^m(\ln t/t)^{n/t}.$$

Using Corollaries 2 and 4 and choosing n such that $n-1\leqq t/\ln t\leqq n$, we easily obtain that $B_2(t,m)\geqq A(m)(t/\ln t)^m(\ln t/t)^{1/t+1/\ln t}$ and, similarly to the above, we get $B_2(t,m)\geqq A(m)(t/\ln t)^m$, that is the constant $B_2(t,m)$ grows not slower than $(t/\ln t)^m$ as $t\to\infty$.

The following corollary complements the results obtained by Krakowiak and Szulga [18] and McConnell and Taqqu [20].

Corollary 5. *Let $a_{i_1,\dots,i_m}\in R$, $1\leqq i_1<i_2<\dots<i_m\leqq n$. If $t\geqq 1$, $X_1,\dots,X_n$ are independent r.v's with $\mathbf{E}X_k=0$, $\mathbf{E}|X_k|^t<\infty$, $k=1,\dots,n$, then*

$$A(t,m)\mathbf{E}\Big(\sum_{1\leqq i_1<\dots<i_m\leqq n} a^2_{i_1,\dots,i_m}X^2_{i_1}\dots X^2_{i_m}\Big)^{t/2} \leqq \mathbf{E}\Big|\sum_{1\leqq i_1<\dots<i_m\leqq n} a_{i_1,\dots,i_m}X_{i_1}\dots X_{i_m}\Big|^t \tag{11}$$

for $t\geqq 2$,

$$\mathbf{E}\Big|\sum_{1\leqq i_1<\dots<i_m\leqq n} a_{i_1,\dots,i_m}X_{i_1}\dots X_{i_m}\Big|^t \leqq B(t,m)\mathbf{E}\Big(\sum_{1\leqq i_1<\dots<i_m\leqq n} a^2_{i_1,\dots,i_m}X^2_{i_1}\dots X^2_{i_m}\Big)^{t/2} \tag{12}$$

for $t\geqq 1$. If $t>0$, $X_1,\dots,X_n$ are independent symmetric r.v's with $\mathbf{E}|X_k|^t<\infty$, $k=1,\dots,n$, then

$$\mathbf{E}\Big(\sum_{1\leqq i_1<\dots<i_m\leqq n} a^2_{i_1,\dots,i_m}X^2_{i_1}\dots X^2_{i_m}\Big)^{t/2} \leqq \mathbf{E}\Big|\sum_{1\leqq i_1<\dots<i_m\leqq n} a_{i_1,\dots,i_m}X_{i_1}\dots X_{i_m}\Big|^t \tag{13}$$

for $t \geqq 2$,

$$\mathbf{E}\Big|\sum_{1\leqq i_1<\cdots<i_m\leqq n} a_{i_1,\dots,i_m} X_{i_1}\dots X_{i_m}\Big|^t \leqq B(t,m)\mathbf{E}\Big(\sum_{1\leqq i_1<\cdots<i_m\leqq n} a^2_{i_1,\dots,i_m} X^2_{i_1}\dots X^2_{i_m}\Big)^{t/2} \tag{14}$$

for $t>0$, *where* $B(t,m)=1$ *for* $0<t\leqq 2$.

Remark 4. As a referee has pointed out, one can also obtain inequalities (11) and (12) by using a decoupling and symmetrization argument (e.g., De la Peña and Giné [4]) and the multilinear version of Khintchine inequality.

Let us formulate a number of preliminary facts needed for the proof of theorems.

Given a sequence of σ-algebras $(\mathfrak{J}_n)$ on some probability space $(\Omega,\mathfrak{J},\mathbf{P})$, we denote by $\mathbf{E}_{k-1}(\cdot)=\mathbf{E}(\cdot|\mathfrak{J}_{k-1})$ the conditional expectation operator (with the convention that $\mathbf{E}_0=\mathbf{E}$, the expectation operator). A sequence (Y_n) of integrable r.v's on $(\Omega,\mathfrak{J},\mathbf{P})$ is called a martingale-difference sequence (with respect to $(\mathfrak{J}_n)$) if

(a) $\mathfrak{J}_0=(\emptyset,\Omega)\subseteqq\mathfrak{J}_1\subseteqq\mathfrak{J}_2\subseteqq\cdots\subseteqq\mathfrak{J}$,
(b) Y_n is $\mathfrak{J}_n$-measurable,
(c) $\mathbf{E}_{n-1}Y_n=0$ a.s., $n\geqq 1$.

Lemma 1 (Burkholder [2]). *Let* $\mathfrak{J}_0=(\emptyset,\Omega)\subseteqq\mathfrak{J}_1\subseteqq\mathfrak{J}_2\subseteqq\cdots\subseteqq\mathfrak{J}_n\subseteqq\mathfrak{J}$ *be a nondecreasing sequence of* σ*-algebras on a probability space* $(\Omega,\mathfrak{J},\mathbf{P})$, *and let* X_k, $k=1,\dots,n$, *be a sequence of nonnegative* $\mathfrak{J}_k$*-measurable r.v's such that* $\mathbf{E}X_k^t<\infty$, $k=1,\dots,n$, $1\leqq t<\infty$. *Then*

$$\mathbf{E}\Big(\sum_{k=1}^n X_k\Big)^t \leqq B(t)\max\bigg(\sum_{k=1}^n \mathbf{E}X_k^t, \mathbf{E}\Big(\sum_{k=1}^n \mathbf{E}_{k-1}X_k\Big)^t\bigg).$$

Lemma 2 (Burkholder [2]). *Let* X_k, $k=1,\dots,n$, *be a martingale-difference sequence with respect to* $\mathfrak{J}_0=(\emptyset,\Omega)\subseteqq\mathfrak{J}_1\subseteqq\mathfrak{J}_2\subseteqq\cdots\subseteqq\mathfrak{J}_n\subseteqq\mathfrak{J}$ *such that* $\mathbf{E}|X_k|^t<\infty$, $k=1,\dots,n$, $2\leqq t<\infty$. *Then*

$$A(t)\max\bigg(\sum_{k=1}^n \mathbf{E}|X_k|^t, \mathbf{E}\Big(\sum_{k=1}^n \mathbf{E}_{k-1}X_k^2\Big)^{t/2}\bigg) \leqq \mathbf{E}\Big|\sum_{k=1}^n X_k\Big|^t \leqq B(t)\max\bigg(\sum_{k=1}^n \mathbf{E}|X_k|^t, \mathbf{E}\Big(\sum_{k=1}^n \mathbf{E}_{k-1}X_k^2\Big)^{t/2}\bigg).$$

Using the fact that if $\{X_n\}$ are independent symmetric r.v's and $\{\varepsilon_n\}$ are independent symmetric Bernoulli r.v's independent of $\{X_n\}$ then the joint distribution of $\{X_n\}$, $\{X_n\varepsilon_n\}$ and $\{|X_n|\,\varepsilon_n\}$ is the same, we obtain the following

LEMMA 3 (Egorov [6]). *Let $\{X_n\}$ be a sequence of independent symmetric r.v's. Then these r.v's are conditionally independent of the σ-algebra spanned by the sequence $\{|X_n|\}$.*

LEMMA 4. *Let r.v's $X_1,\dots,X_n$ satisfy the conditions*

$$\mathbf{E}(X_kX_l \mid |X_k|,|X_l|)=0$$

for $k,l=1,\dots,n$, $k\neq l$. If $t\geqq 2$, $\mathbf{E}|X_k|^t<\infty$, $k=1,\dots,n$, then

$$\mathbf{E}\Big(\sum_{i=1}^n X_i^2\Big)^{t/2}\leqq \mathbf{E}\Big|\sum_{i=1}^n X_i\Big|^t.$$

If $0<t\leqq 2$, $\mathbf{E}|X_k|^t<\infty$, $k=1,\dots,n$, then

$$\mathbf{E}\Big|\sum_{i=1}^n X_i\Big|^t\leqq \mathbf{E}\Big(\sum_{i=1}^n X_i^2\Big)^{t/2}.$$

PROOF. Let $t\geqq 2$. By Jensen's inequality

$$\begin{aligned}\mathbf{E}\Big|\sum_{i=1}^n X_i\Big|^t &= \mathbf{E}\bigg(\mathbf{E}\bigg(\Big|\sum_{i=1}^n X_i\Big|^t \,\Big|\, |X_1|,\dots,|X_n|\bigg)\bigg)\\ &\geqq \mathbf{E}\bigg(\mathbf{E}\bigg(\Big(\sum_{i=1}^n X_i\Big)^2 \,\Big|\, |X_1|,\dots,|X_n|\bigg)\bigg)^{t/2} \qquad (15)\\ = \mathbf{E}\bigg(\sum_{i=1}^n X_i^2 &+ \sum_{1\leqq k<l\leqq n}\mathbf{E}(X_kX_l \mid |X_k|,|X_l|)\bigg)^{t/2} = \mathbf{E}\Big(\sum_{i=1}^n X_i^2\Big)^{t/2}.\end{aligned}$$

In the case $0<t<2$ one just needs to change the sign of inequality in (15) to the inverse one.

PROOF OF THEOREM 1. Let $m\geqq 2$. For $1\leqq i_1<\dots<i_m\leqq n$, denote $Y(i_1,\dots,i_m)=Y_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m})$. For $1\leqq j_1<\dots<j_k\leqq m$, $1\leqq i_{j_1}<\dots<i_{j_k}\leqq n$, $k=1,\dots,m-1$, denote

$$Y(i_{j_1},\dots,i_{j_k})=\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} Y(i_1,\dots,i_m).$$

Write $E_{i_1,\dots,i_s}(\cdot)$ for $\mathbf{E}(\cdot\mid X_{i_1},\dots,X_{i_s})$, $1\leqq i_1<\dots<i_s\leqq n$, $s=0,\dots,m$. By nonnegativity of the functions Y and Jensen's inequality we have

$$\mathbf{E}T_{n,m}^t\geqq \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n}\mathbf{E}Y^t(i_{j_1},\dots,i_{j_k})$$

$$\geqq \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}(E_{i_{j_1},\dots,i_{j_k}} Y(i_{j_1,\dots,i_{j_k}}))^t$$

for $1\leqq j_1<\dots<j_k\leqq m$, $k=0,\dots,m$. Consequently,

$$\mathbf{E}T^t_{n,m} \geqq \max_{k=0,m} \max_{1\leqq j_1<\dots<j_k\leqq m} \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq m} \mathbf{E}(E_{i_{j_1},\dots,i_{j_k}} Y(i_{j_1},\dots,i_{j_k}))^t.$$

Therefore, left-hand inequality (5) holds.

The r.v's

$$Y(i_m)=\sum_{1\leqq i_1<\dots<i_{m-1}\leqq i_m-1} Y_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_{m-1}},X_{i_m}), \quad i_m=m,\dots,n,$$

are functions of the r.v's $X_1,X_2,\dots,X_{i_m}$; and for $k+1\leqq i_{k+1}<\dots<i_m\leqq n$, $k=1,\dots,m-1$, the r.v's

$$Y(i_k,\dots,i_m)=\sum_{1\leqq i_1<\dots<i_{k-1}\leqq i_k-1} Y_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_k},X_{i_{k+1}},\dots,X_{i_m}),$$
$$i_k=k,\dots,i_{k+1}-1,$$

are functions of the r.v's $X_1,X_2,\dots,X_{i_k},X_{i_{k+1}},\dots,X_{i_m}$. Therefore, the r.v's $Y(i_m)$, $i_m=m,\dots,n$, are measurable with respect to the σ-algebras $\sigma(X_1,X_2,\dots,X_{i_m})$; and for $k+1\leqq i_{k+1}<\dots<i_m\leqq n$, $k=1,\dots,m-1$, the r.v's $Y(i_k,\dots,i_m)$, $i_k=k,\dots,i_{k+1}-1$, are measurable with respect to the σ-algebras $\sigma(X_1,X_2,\dots,X_{i_k},X_{i_{k+1}},\dots,X_{i_m})$.

Using Lemma 1, we obtain

$$\begin{aligned}\mathbf{E}T^t_{n,m} &= \mathbf{E}\Bigg(\sum_{i_m=m}^{n} Y(i_m)\Bigg)^t \\ &\leqq B(t)\max\Bigg(\sum_{i_m=m}^{n}\mathbf{E}Y^t(i_m), \mathbf{E}\Bigg(\sum_{i_m=m}^{n} E_{1,\dots,i_m-1}Y(i_m)\Bigg)^t\Bigg),\end{aligned}\tag{16}$$

$$\begin{aligned}\mathbf{E}Y^t(i_k,\dots,i_m) &= \mathbf{E}\Bigg(\sum_{i_{k-1}=k-1}^{i_k-1} Y(i_{k-1},\dots,i_m)\Bigg)^t \\ &\leqq B(t)\max\Bigg(\sum_{i_{k-1}=k-1}^{i_k-1}\mathbf{E}Y^t(i_{k-1},\dots,i_m), \\ &\quad \mathbf{E}\Bigg(\sum_{i_{k-1}=k-1}^{i_k-1} E_{1,\dots,i_{k-1}-1,i_k,\dots,i_m} Y(i_{k-1},\dots,i_m)\Bigg)^t\Bigg)\end{aligned}\tag{17}$$

for all $k \leqq i_k < \cdots < i_m \leqq n$, $k = 2, \ldots, m$.

From (16) and (17) it follows that

$$\begin{aligned} \mathbf{E}T_{n,m}^t \leqq B(t,m) \max_{k=1,m-1} \Bigg(& \sum_{1 \leqq i_1 < \cdots < i_m \leqq n} \mathbf{E}Y^t(i_1, i_2, \ldots, i_m), \\ & \mathbf{E}\Bigg(\sum_{1 \leqq i_1 < \cdots < i_m \leqq n} E_{i_1, i_2, \ldots, i_{m-1}} Y(i_1, i_2, \ldots, i_m) \Bigg)^t, \\ & \sum_{k+1 \leqq i_{k+1} < \cdots < i_m \leqq m} \mathbf{E}\Bigg(\sum_{1 \leqq i_1 < \cdots < i_k \leqq i_{k+1} - 1} E_{i_1, i_2, \ldots, i_{k-1}, i_{k+1}, \ldots, i_m} \\ & Y(i_1, i_2, \ldots, i_m) \Bigg)^t \Bigg). \end{aligned} \tag{18}$$

Suppose that for all statistics

$$T_{n,l} = \sum_{1 \leqq i_1 < \cdots < i_l \leqq n} Y_{i_1, \ldots, i_l}(X_{i_1}, X_{i_2}, \ldots, X_{i_l}), \qquad 1 \leqq l \leqq m - 1,$$

where $Y_{i_1, \ldots, i_l} : \mathbf{R}^l \to \mathbf{R}$, $1 \leqq i_k \leqq n$; $i_r \neq i_s$, $r \neq s$; $k, r, s = 1, \ldots, l$, are nonnegative functions satisfying the conditions

$$\mathbf{E}Y_{i_1, \ldots, i_l}^t(X_{i_1}, \ldots, X_{i_l}) < \infty,$$
$$Y_{i_1, \ldots, i_l}(X_{i_1}, \ldots, X_{i_l}) = Y_{i_{\pi(1)}, \ldots, i_{\pi(l)}}(X_{i_{\pi(1)}}, \ldots, X_{i_{\pi(l)}}) \quad \text{a.s.},$$

$1 \leqq i_1 < i_2 < \cdots < i_l \leqq n$, $\pi \in M_l$, the estimate

$$\mathbf{E}T_{n,l}^t \leqq B(t,l) \max_{s=0,l} \max_{1 \leqq j_1 < \cdots < j_s \leqq l} \sum_{1 \leqq i_{j_1} < \cdots < i_{j_s} \leqq n} \mathbf{E}(E_{i_{j_1}, \ldots, i_{j_s}} Y(i_{j_1}, \ldots, i_{j_s}))^t$$

has already been proved. Then we have

$$\begin{aligned} & \mathbf{E}\Bigg(\sum_{1 \leqq i_1 < \cdots < i_m \leqq n} E_{i_1, i_2, \ldots, i_{m-1}} Y(i_1, i_2, \ldots, i_m) \Bigg)^t \\ & = \mathbf{E}\Bigg(\sum_{1 \leqq i_1 < \cdots < i_{m-1} \leqq n-1} \sum_{i_m = i_{m-1}+1}^{n} E_{i_1, i_2, \ldots, i_{m-1}} Y(i_1, i_2, \ldots, i_m) \Bigg)^t \\ & \leqq B(t, m-1) \max_{s=0,m-1} \max_{1 \leqq j_1 < \cdots < j_s \leqq m-1} \sum_{1 \leqq i_{j_1} < \cdots < i_{j_s} \leqq n-1} \\ & \qquad \mathbf{E}(E_{i_{j_1}, \ldots, i_{j_s}} Y(i_{j_1}, \ldots, i_{j_s}))^t. \end{aligned} \tag{19}$$

Furthermore,

$$\mathbf{E}\Bigl(\sum_{1\leqq i_1<\cdots<i_k\leqq i_{k+1}-1} E_{i_1,i_2,\dots,i_{k-1},i_{k+1},\dots,i_m} Y(i_1,i_2,\dots,i_m)\Bigr)^t \tag{20}$$

$$=\mathbf{E}\Bigl(\sum_{1\leqq i_1<\cdots<i_{k-1}\leqq i_{k+1}-2}\ \sum_{i_k=i_{k-1}+1}^{i_{k+1}-1} E_{i_1,i_2,\dots,i_{k-1},i_{k+1},\dots,i_m} Y(i_1,i_2,\dots,i_m)\Bigr)^t$$

$$\leqq B(t,k-1)\max_{s=0,k-1}\ \max_{1\leqq j_1<\cdots<j_s\leqq k-1}$$

$$\sum_{1\leqq i_{j_1}<\cdots<i_{j_s}\leqq i_{k+1}-2} \mathbf{E}(E_{i_{j_1},\dots,i_{j_s},i_{k+1},\dots,i_m} Y(i_{j_1},\dots,i_{j_s}))^t$$

for all $k+1\leqq i_{k+1}<\cdots<i_m\leqq n,\ k=2,\dots,m-1$. Substituting (19) and (20) into (18), we obtain

$$\mathbf{E}T_{n,m}^t\leqq B(t,m)\max_{k=0,m}\ \max_{1\leqq j_1<\cdots<j_k\leqq m}\ \sum_{1\leqq i_{j_1}<\cdots<i_{j_k}\leqq n} \mathbf{E}(E_{i_{j_1},\dots,i_{j_k}} Y(i_{j_1},\dots,i_{j_k}))^t.$$

Therefore, right-hand inequality (5) holds for all symmetric statistics $T_{n,m}$ of order m. Since for $m=1$ estimate (5) holds as inequality (1), by induction principle the proof is complete.

Proof of Theorems 2 and 3. Let us first prove relations (7) and (9) for $t\geqq 2$, (8) for $1\leqq t<2$ and (10) for $0<t\leqq 2$. Let $Y_{i_1,\dots,i_m}:\mathbf{R}^m\to\mathbf{R}$, $1\leqq i_k\leqq n;\ i_r\neq i_s,\ r\neq s;\ k,r,s=1,\dots,m$, be functions from the class $G'(t,m)$, and let $t>0$. Let $1\leqq i_1<i_2<\cdots<i_m\leqq n$, $1\leqq j_1<j_2<\cdots<j_m\leqq n$, $(i_1,i_2,\dots,i_m)\neq(j_1,j_2,\dots,j_m)$ and let $\{i_{\alpha_1},\dots,i_{\alpha_k}\}=\{i_1,\dots,i_m\}\cap\{j_1,\dots,j_m\}$. Since the functions Y belong to $G'(t,m)$ and the r.v's X_{i_s},X_{j_s}, $i_s,j_s\notin\{i_{\alpha_1},\dots,i_{\alpha_k}\}$ are independent, then the r.v's $Y(i_1,\dots,i_m)$ and $Y(j_1,\dots,j_m)$ are conditionally independent and conditionally symmetric on the σ-algebra $\sigma(X_{i_{\alpha_1}},\dots,X_{i_{\alpha_k}})$, and from Lemma 3 we get

$$\begin{aligned}\mathbf{E}(Y(i_1,\dots,i_m)Y(j_1,\dots,j_m)\,|\,X_{i_{\alpha_1}},\dots,X_{i_{\alpha_k}},|Y(i_1,\dots,i_m)|,|Y(j_1,\dots,j_m)|)\\ =\mathbf{E}(Y(i_1,\dots,i_m)\,|\,X_{i_{\alpha_1}},\dots,X_{i_{\alpha_k}},|Y(i_1,\dots,i_m)|)\times\\ \times\mathbf{E}(Y(j_1,\dots,j_m)\,|\,X_{i_{\alpha_1}},\dots,X_{i_{\alpha_k}},|Y(j_1,\dots,j_m)|)=0.\end{aligned}$$

Therefore, $\mathbf{E}(Y(i_1,\dots,i_m)Y(j_1,\dots,j_m)\ |\ |Y(i_1,\dots,i_m)|,|Y(j_1,\dots,j_m)|)=0$ for all $1\leqq i_1<i_2<\cdots<i_m\leqq n$, $1\leqq j_1<j_2<\cdots<j_m\leqq n$, $(i_1,i_2,\dots,i_m)\neq(j_1,j_2,\dots,j_m)$, and from Lemma 4 we obtain

$$\mathbf{E}\Bigl(\sum_{1\leqq i_1<\cdots<i_m\leqq n} Y^2(i_1,\dots,i_m)\Bigr)^{t/2}\leqq\mathbf{E}|T_{n,m}|^t$$

for $t \geqq 2$ and

$$\mathbf{E}|T_{n,m}|^t \leqq \mathbf{E}\bigg(\sum_{1\leqq i_1<\dots<i_m\leqq n} Y^2(i_1,\dots,i_m)\bigg)^{t/2}$$

for $0<t\leqq 2$. Therefore, the relations (9) for $t\geqq 2$ and (10) for $0<t\leqq 2$ hold.

In the case of the class $G(t,m)$ we use a symmetrization argument. Let $Y_{i_1,\dots,i_m}:\mathbf{R}^m\to\mathbf{R}$, $1\leqq i_k\leqq n$; $i_r\neq i_s$, $r\neq s$; $k,r,s=1,\dots,m$, be functions from the class $G(t,m)$, $t\geqq 1$. In what follows $X_1',\dots,X_n'$ denote independent r.v's independent of $X_1,\dots,X_n$ and such that for $k=1,\dots,n$ the r.v's X_k' and X_k have the same distribution. Denote $Z(i_1,\dots,i_m)=Y_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m})-Y_{i_1,\dots,i_m}(X_{i_1}',...,X_{i_m}')$. For $1\leqq i_1<i_2<\dots<i_m\leqq n$, $1\leqq j_1<j_2<\dots<j_m\leqq n$, $(i_1,i_2,\dots,i_m)\neq(j_1,j_2,\dots,j_m)$, the r.v's $Z(i_1,...,i_m)$ and $Z(j_1,\dots,j_m)$ are conditionally independent and conditionally symmetric on the σ-algebra $\sigma(X_{i_{\alpha_1}},\dots,X_{i_{\alpha_k}},X_{i_{\alpha_1}}',\dots,X_{i_{\alpha_k}}')$, where $\{i_{\alpha_1},\dots,i_{\alpha_k}\}=\{i_1,\dots,i_m\}\cap\{j_1,\dots,j_m\}$. Consequently, by Lemma 3,

$$\begin{aligned}\mathbf{E}(Z(i_1,\dots,i_m)Z(j_1,\dots,j_m)\,|\,&X_{i_{\alpha_1}},\dots,X_{i_{\alpha_k}},X_{i_{\alpha_1}}',\dots,X_{i_{\alpha_k}}',\\ &|Z(i_1,\dots,i_m)|,|Z(j_1,\dots,j_m)|)\\ =\mathbf{E}(Z(i_1,\dots,i_m)\,|\,&X_{i_{\alpha_1}},\dots,X_{i_{\alpha_k}},X_{i_{\alpha_1}}',\dots,X_{i_{\alpha_k}}',|Z(i_1,\dots,i_m)|)\times\\ \times\mathbf{E}(Z(j_1,\dots,j_m)\,|\,&X_{i_{\alpha_1}},\dots,X_{i_{\alpha_k}},X_{i_{\alpha_1}}',\dots,X_{i_{\alpha_k}}',|Z(j_1,\dots,j_m)|)=0.\end{aligned}$$

Therefore, $\mathbf{E}(Z(i_1,\dots,i_m)Z(j_1,\dots,j_m)\,|\,|Z(i_1,\dots,i_m)|,|Z(j_1,\dots,j_m)|)=0$ for all $1\leqq i_1<i_2<\dots<i_m\leqq n$, $1\leqq j_1<j_2<\dots<j_m\leqq n$, $(i_1,i_2,\dots,i_m)\neq(j_1,j_2,\dots,j_m)$, and Lemma 4 implies that

$$\text{(21)}\qquad \mathbf{E}\bigg(\sum_{1\leqq i_1<\dots<i_m\leqq n} Z^2(i_1,\dots,i_m)\bigg)^{t/2}\leqq \mathbf{E}\bigg|\sum_{1\leqq i_1<\dots<i_m\leqq n} Z(i_1,\dots,i_m)\bigg|^t$$

for $t\geqq 2$ and

$$\text{(22)}\qquad \mathbf{E}\bigg|\sum_{1\leqq i_1<\dots<i_m\leqq n} Z(i_1,\dots,i_m)\bigg|^t\leqq \mathbf{E}\bigg(\sum_{1\leqq i_1<\dots<i_m\leqq n} Z^2(i_1,\dots,i_m)\bigg)^{t/2}$$

for $0<t\leqq 2$. From Jensen's inequality, the convexity of the function $|x|^t$, $t\geqq 1$, and the estimate

$$\text{(23)}\qquad |x+y|^t\leqq 2^{t-1}(|x|^t+|y|^t),$$

$x,y\in\mathbf{R}$, $t\geqq 1$, it follows that

$$\text{(24)}\qquad \mathbf{E}|T_{n,m}|^t\leqq \mathbf{E}\bigg|\sum_{1\leqq i_1<\dots<i_m\leqq n} Z(i_1,\dots,i_m)\bigg|^t\leqq 2^t\mathbf{E}|T_{n,m}|^t$$

for $t \geqq 1$. From (23) for $t=2$ and the inequality $|x+y|^t \leqq |x|^t + |y|^t$, $x, y \in \mathbf{R}$, $0 < t \leqq 1$, we get

$$\mathbf{E}\Big(\sum_{1\leqq i_1<\dots<i_m\leqq n} Z^2(i_1,\dots,i_m)\Big)^{t/2} \tag{25}$$

$$\leqq 2^{t/2}\mathbf{E}\Big(\sum_{1\leqq i_1<\dots<i_m\leqq n} (Y^2_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m}) + Y^2_{i_1,\dots,i_m}(X'_{i_1},\dots,X'_{i_m})\Big)^{t/2}$$

$$\leqq 2^{t/2+1}\mathbf{E}\Big(\sum_{1\leqq i_1<\dots<i_m\leqq n} Y^2_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m})\Big)^{t/2}$$

for $1 \leqq t < 2$. Let $t \geqq 2$. From Jensen's inequality it follows that

$$\mathbf{E}\Big(\sum_{1\leqq i_1<\dots<i_m\leqq n} Z^2(i_1,\dots,i_m)\Big)^{t/2} \tag{26}$$

$$\geqq \max_{k=0,m} \max_{1\leqq j_1<\dots<j_k\leqq m} \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}\Big(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} Z^2(i_1,\dots,i_m)\Big)^{t/2}$$

$$\geqq \max_{k=0,m} \max_{1\leqq j_1<\dots<j_k\leqq m} \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}\Big(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}}$$

$$\mathbf{E}(Z^2(i_1,\dots,i_m) \,|\, X_{i_{j_1}},\dots,X_{i_{j_k}}, X'_{i_{j_1}},\dots,X'_{i_{j_k}})\Big)^{t/2}.$$

Since $Y_{i_1,\dots,i_m} \in G(t,m)$, for $1 \leqq j_1 < \dots < j_k \leqq m$, $k = 0,\dots,m$, $t \geqq 2$ we have

(27)

$$\sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}\Big(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} \mathbf{E}(Z^2(i_1,\dots,i_m) \,|\, X_{i_{j_1}},\dots,X_{i_{j_k}}, X'_{i_{j_1}},\dots,X'_{i_{j_k}})\Big)^{t/2}$$

$$= \sum_{1\leqq i_{j_1}<\dots<i_{j_k}\leqq n} \mathbf{E}\Big(\sum_{\substack{i_s:1\leqq s\leqq m,\, s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} \mathbf{E}(Y^2_{i_1,\dots,i_m}(X_{i_1},\dots,X_{i_m}) \,|\, X_{i_{j_1}},\dots,X_{i_{j_k}})$$

$$+\mathbf{E}(Y^2_{i_1,\dots,i_m}(X'_{i_1},\dots,X'_{i_m}) \,|\, X'_{i_{j_1}},\dots,X'_{i_{j_k}})\Big)^{t/2}$$

$$\geqq \sum_{1\leqq i_{j_1}<\cdots<i_{j_k}\leqq n} 2\mathbf{E}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m,\ s\neq j_1,\ldots,j_k,\\ 1\leqq i_1<\cdots<i_m\leqq n}} \mathbf{E}(Y^2_{i_1,\ldots,i_m}(X_{i_1},\ldots,X_{i_m})\,|\,X_{i_{j_1}},\ldots,X_{i_{j_k}})\Bigg)^{t/2}.$$

Using Theorem 1, from (26) and (27) we obtain

$$\begin{aligned}&\mathbf{E}\Bigg(\sum_{1\leqq i_1<\cdots<i_m\leqq n} Z^2(i_1,\ldots,i_m)\Bigg)^{t/2}\\ &\geqq A(t,m)\mathbf{E}\Bigg(\sum_{1\leqq i_1<\cdots<i_m\leqq n} Y^2(i_1,\ldots,i_m)\Bigg)^{t/2}\end{aligned}\tag{28}$$

for $t\geqq 2$. (21), (22), (24), (25) and (28) imply relations (7) for $t\geqq 2$ and (8) for $1\leqq t<2$.

Let us prove right-hand inequality (6). Let $Y_{i_1,\ldots,i_m}:\mathbf{R}^m\to\mathbf{R}$, $1\leqq i_k\leqq n$; $i_r\neq i_s$, $r\neq s$; $k,r,s=1,\ldots,m$, be functions from the class $G(t,m)$, $t\geqq 2$. From now on, write $E_{i_1,\ldots,i_s}$ for $\mathbf{E}(\cdot\,|\,X_{i_1},\ldots,X_{i_s},X'_{i_1},\ldots,X'_{i_s})$, $1\leqq i_1<\cdots< i_s\leqq n$, $s=0,1,\ldots,m+1$. It is easy to see that the r.v's

$$Y(i_m)=\sum_{1\leqq i_1<\cdots<i_{m-1}\leqq i_m-1} Y_{i_1,\ldots,i_m}(X_{i_1},\ldots,X_{i_{m-1}},X_{i_m}),\quad i_m=m,\ldots,n,$$

form a martingale-difference sequence with respect to the σ-algebras $\sigma(X_1,X_2,\ldots,X_{i_m})$.

Using Lemma 2, we have

$$\begin{aligned}&\mathbf{E}|T_{n,m}|^t=\mathbf{E}\Bigg|\sum_{i_m=m}^{n} Y(i_m)\Bigg|^t\\ &\leqq B(t)\max\Bigg(\sum_{i_m=m}^{n}\mathbf{E}|Y(i_m)|^t,\mathbf{E}\Bigg(\sum_{i_m=m}^{n}E_{1,\ldots,i_m-1}Y^2(i_m)\Bigg)^{t/2}\Bigg).\end{aligned}\tag{29}$$

Suppose that right-hand inequality (6) is already proven for symmetric statistics of order $m-1$. Then we have

$$\begin{aligned}&\mathbf{E}|Y(i_m)|^t\leqq B(t,m-1)\max_{k=0,m-1}\ \max_{1\leqq j_1<\cdots<j_k\leqq m-1}\ \sum_{1\leqq i_{j_1}<\cdots<i_{j_k}\leqq i_m-1}\\ &\mathbf{E}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m-1,s\neq j_1,\ldots,j_k,\\ 1\leqq i_1<\cdots<i_{m-1}\leqq i_m-1}} E_{i_{j_1},\ldots,i_{j_k},i_m}Y^2(i_1,\ldots,i_m)\Bigg)^{t/2}.\end{aligned}\tag{30}$$

For $1 \leqq i_1 < \cdots < i_m \leqq n$, denote $\overline{Y}(i_1, ..., i_m) = Y_{i_1,...,i_m}(X_{i_1}, ..., X_{i_{m-1}}, X'_{i_m})$. Also, denote $V_{n,m} = \sum_{1 \leqq i_1 < \cdots < i_m \leqq n} Y_{i_1,...,i_m}(X_{i_1}, ..., X_{i_{m-1}}, X'_{i_m})$. From Jensen inequality and the induction hypothesis we have

$$\mathbf{E}\Big(\sum_{i_m=m}^{n} E_{1,...,i_m-1} Y^2(i_m)\Big)^{t/2}$$

$$\leqq \mathbf{E}|V_{n,m}|^t \leqq B(t, m-1) \max_{k=0,m-1} \max_{1 \leqq j_1 < \cdots < j_k \leqq m-1} \sum_{1 \leqq i_{j_1} < \cdots < i_{j_k} \leqq n-1}$$

$$\mathbf{E}\Big(\sum_{\substack{i_s: 1 \leqq s \leqq m-1, s \neq j_1,...,j_k, \\ 1 \leqq i_1 < \cdots < i_{m-1} \leqq i_m - 1}} \mathbf{E}\Big(\Big(\sum_{i_m = i_{m-1}+1}^{n} \tag{31}$$

$$\overline{Y}(i_1, ..., i_m)\Big)^2 \Big| X_{i_{j_1}}, ..., X_{i_{j_k}}, X'_1, ..., X'_n\Big)\Big)^{t/2}.$$

Similarly to the proof of Theorem 6 in [10], for $0 \leqq k \leqq m-1$, $1 \leqq j_1 < \cdots < j_k \leqq m-1$, $1 \leqq i_{j_1} < \cdots < i_{j_k} \leqq n-1$, we have

$$\mathbf{E}\Big(\sum_{\substack{i_s: 1 \leqq s \leqq m-1, s \neq j_1,...,j_k, \\ 1 \leqq i_1 < \cdots < i_{m-1} \leqq n-1}} \mathbf{E}\Big(\Big(\sum_{i_m=i_{m-1}+1}^{n} \overline{Y}(i_1, ..., i_m)\Big)^2 \Big| X_{i_{j_1}}, ..., X_{i_{j_k}}, X'_1, ..., X'_n\Big)\Big)^{t/2}$$

$$= \mathbf{E}\Big(\sum_{\substack{i_s: 1 \leqq s \leqq m-1, s \neq j_1,...,j_k, \\ 1 \leqq i_1 < \cdots < i_{m-1} \leqq n-1}} \sum_{i_m=i_{m-1}+1}^{n} E_{i_{j_1},...,i_{j_k},i_m} \overline{Y}^2(i_1, ..., i_m) \tag{32}$$

$$+2 \sum_{\substack{i_s: 1 \leqq s \leqq m-1, s \neq j_1,...,j_k, \\ 1 \leqq i_1 < \cdots < i_{m-1} \leqq n-2}} \sum_{i_{m-1}+1 \leqq i_m < i_{m+1} \leqq n}$$

$$E_{i_{j_1},...,i_{j_k},i_m,i_{m+1}}\Big(\overline{Y}(i_1, ..., i_{m-1}, i_m)\overline{Y}(i_1, ..., i_{m-1}, i_{m+1})\Big)\Big)^{t/2}$$

$$\leqq 2^{t/2-1}\mathbf{E}\Big(\sum_{i_m=m}^{n} \sum_{\substack{i_s: 1 \leqq s \leqq m-1, s \neq j_1,...,j_k, \\ 1 \leqq i_1 < \cdots < i_{m-1} \leqq i_m - 1}} E_{i_{j_1},...,i_{j_k},i_m} \overline{Y}^2(i_1, ..., i_m)\Big)^{t/2}$$

$$+2^{t-1}\mathbf{E}\Big| \sum_{m \leqq i_m < i_{m+1} \leqq n} \sum_{\substack{i_s: 1 \leqq s \leqq m-1, s \neq j_1,...,j_k, \\ 1 \leqq i_1 < \cdots < i_{m-1} \leqq i_m - 1}}$$

$$E_{i_{j_1},...,i_{j_k},i_m,i_{m+1}}\Big(\overline{Y}(i_1, ..., i_{m-1}, i_m)\overline{Y}(i_1, ..., i_{m-1}, i_{m+1})\Big)\Big|^{t/2}.$$

By Jensen inequality we have, similarly to the proof of relation (28) in [10], that the second term in (32) is not greater than

$$2^{t-1}\mathbf{E}\Bigg(\sum_{m\leqq i_m<i_{m+1}\leqq n} E_{i_{j_1},\dots,i_{j_k}}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m-1, s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}} E_{i_{j_1},\dots,i_{j_k},i_m,i_{m+1}}(\overline{Y}(i_1,\dots,i_{m-1},i_m)\overline{Y}(i_1,\dots,i_{m-1},i_{m+1}))\Bigg)^2\Bigg)^{t/4} \tag{33}$$

for $2\leqq t<4$. Using, in complete similarity to the proof of (28) in [10], Schwarz inequality

$$\begin{aligned}&|E_{i_{j_1},\dots,i_{j_k},i_m,i_{m+1}}\left(\overline{Y}(i_1,\dots,i_{m-1},i_m)\overline{Y}(i_1,\dots,i_{m-1},i_{m+1})\right)|\\ &\leqq\left(E_{i_{j_1},\dots,i_{j_k},i_m}\overline{Y}^2(i_1,\dots,i_{m-1},i_m)\right)^{1/2}\times\\ &\times\left(E_{i_{j_1},\dots,i_{j_k},i_{m+1}}\overline{Y}^2(i_1,\dots,i_{m-1},i_{m+1})\right)^{1/2}\end{aligned}$$

and Cauchy inequality $\sum_{i=1}^n a_ib_i\leqq(\sum_{i=1}^n a_i^2)^{1/2}(\sum_{i=1}^n b_i^2)^{1/2}$, $a_i,b_i\geqq 0$, we get that (33) is not greater than

$$\begin{aligned}&2^{t-1}\mathbf{E}\Bigg(\sum_{m\leqq i_m<i_{m+1}\leqq n} E_{i_{j_1},\dots,i_{j_k}}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m-1, s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}}\\ &\left(E_{i_{j_1},\dots,i_{j_k},i_m}\overline{Y}^2(i_1,\dots,i_{m-1},i_m)\right)^{1/2}\times\\ &\times\left(E_{i_{j_1},\dots,i_{j_k},i_{m+1}}\overline{Y}^2(i_1,\dots,i_{m-1},i_{m+1})\right)^{1/2}\Bigg)^2\Bigg)^{t/4}\end{aligned}$$

$$\begin{aligned}&\leqq 2^{t-1}\mathbf{E}\Bigg(\sum_{m\leqq i_m<i_{m+1}\leqq n} E_{i_{j_1},\dots,i_{j_k}}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m-1, s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}}\\ &E_{i_{j_1},\dots,i_{j_k},i_m}\overline{Y}^2(i_1,\dots,i_{m-1},i_m)\Bigg)\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m-1, s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}}\\ &E_{i_{j_1},\dots,i_{j_k},i_{m+1}}\overline{Y}^2(i_1,\dots,i_{m-1},i_{m+1})\Bigg)\Bigg)^{t/4}\end{aligned} \tag{34}$$

$$\leqq 2^{t-1}\mathbf{E}\Big(\sum_{\substack{i_s:1\leqq s\leqq m,s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} E_{i_{j_1},\dots,i_{j_k}}\overline{Y}^2(i_1,\dots,i_m)\Big)^{t/2}$$

$$= 2^{t-1}\mathbf{E}\Big(\sum_{\substack{i_s:1\leqq s\leqq m,s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} E_{i_{j_1},\dots,i_{j_k}} Y^2(i_1,\dots,i_m)\Big)^{t/2}$$

for $t \geqq 2$. For $t \geqq 4$, using Theorem 7 in [10] which gives Khintchine inequality for symmetric statistics of second order, Schwarz inequality and Cauchy inequality similarly to above, we get that the second term in (32) is not greater than

$$B(t/2)\mathbf{E}\Big(\sum_{m\leqq i_m<i_{m+1}\leqq n}\Big(\sum_{\substack{i_s:1\leqq s\leqq m-1,s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}}$$

$$E_{i_{j_1},\dots,i_{j_k},i_m,i_{m+1}}\left(\overline{Y}(i_1,\dots,i_{m-1},i_m)\overline{Y}(i_1,\dots,i_{m-1},i_{m+1})\right)\Big)^2\Big)^{t/4}$$

(35)
$$\leqq B(t/2)\mathbf{E}\Big(\sum_{m\leqq i_m<i_{m+1}\leqq n}\Big(\sum_{\substack{i_s:1\leqq s\leqq m-1,s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}}$$

$$\Big(E_{i_{j_1},\dots,i_{j_k},i_m}\overline{Y}^2(i_1,\dots,i_{m-1},i_m)\Big)^{1/2}\times$$

$$\times\left(E_{i_{j_1},\dots,i_{j_k},i_{m+1}}\overline{Y}^2(i_1,\dots,i_{m-1},i_{m+1})\right)^{1/2}\Big)^2\Big)^{t/4}$$

$$\leqq B(t/2)\mathbf{E}\Big(\sum_{i_m=m}^{n}\sum_{\substack{i_s:1\leqq s\leqq m-1,s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}} E_{i_{j_1},\dots,i_{j_k},i_m} Y^2(i_1,\dots,i_{m-1},i_m)\Big)^{t/2}.$$

Similarly to inequality (22) in [10], from upper inequality (1) we obtain

$$\mathbf{E}\Big(\sum_{i_m=m}^{n}\sum_{\substack{i_s:1\leqq s\leqq m-1,s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}} E_{i_{j_1},\dots,i_{j_k},i_m} Y^2(i_1,\dots,i_{m-1},i_m)\Big)^{t/2}$$

(36)

$$\leqq B(t)\max\Bigg[\sum_{i_m=m}^{n}\mathbf{E}\Big(\sum_{\substack{i_s:1\leqq s\leqq m-1,s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_{m-1}\leqq i_m-1}} E_{i_{j_1},\dots,i_{j_k},i_m} Y^2(i_1,\dots,i_m)\Big)^{t/2},$$

$$\mathbf{E}\Bigg(\sum_{\substack{i_s:1\leqq s\leqq m, s\neq j_1,\dots,j_k,\\ 1\leqq i_1<\dots<i_m\leqq n}} E_{i_{j_1},\dots,i_{j_k}} Y^2(i_1,\dots,i_m)\Bigg)^{t/2}\Bigg]$$

for $t \geqq 2$. The latter estimates imply, together with (29)–(32), that upper inequality (6) holds for symmetric statistics of order m. Alternatively, one can show that the second summand in (32) is not greater than the right-hand side of (36) for $t \geqq 4$ using, in complete similarity to estimates (29)–(35) in [10], Rosenthal inequality for symmetric statistics of second order given by relation (5) in [10], the above inequality between (33) and (34) for $t \geqq 2$, Schwarz and Cauchy inequalities and Lemma 3 in [10] (note that the power at the last term in inequality (29) in [10] should read "$t/4$" instead of "$t/2$" and "$2 \leqq t < 4$" after relation (24) in [10] should read "$t \geqq 2$"; the relation and its analogue in the present setup hold for all $t \geqq 2$). This, together with estimates (29)–(32) and (36) and the inequality between (34) and the second summand in (32) for $2 \leqq t < 4$ implies that upper estimate (6) holds for symmetric statistics of order m. Since for $m = 1$ estimate (6) holds as inequality (2), by induction principle its proof is complete. Using Remark 1, we obtain (8) and (10) for $t \geqq 2$. (7) and Remark 1 imply left-hand inequality (6). The proof is complete.

Remark 5 (added in proof). It has come to our attention that in a recent paper citing our work, Giné et al. [7] proved results similar to those obtained in the present paper using a different approach. In particular, the authors showed that analogues of upper Rosenthal-type inequalities (5) and (6) for the t-th moment of decoupled symmetric statistics $W_{n,m} = \sum_{1\leqq i_1<\dots<i_m\leqq n} Y_{i_1,\dots,i_m}(X_{1,i_1},\dots,X_{m,i_m})$ of order m, where X_{pi}, $p = 1,\dots,m$, $i = 1,\dots,n$, are independent r.v's, hold with the constants $(Ct/\ln t)^{mt}$ that implies, by decoupling inequalities for symmetric statistics, similar results for regular symmetric statistics $T_{n,m}$. E.g., from the results in [7] and decoupling inequalities in [3] and [4, p. 99] it follows that the analogues of upper Rosenthal inequalities for regular symmetric statistics in (5) and (6) hold with the constants $A^t(m)(t/\ln t)^{mt}$. From Remark 3 in the present paper it follows that the rate of growth of the latter constants is the best possible. Below, we show that the approach applied in the present paper, namely, the use of Burkholder inequalities given by Lemmas 1 and 2 and moment inequalities for symmetric statistics of second order given by [10] in the proof of Theorems 1 and 2, allows one to obtain, using the results in [4, pp. 296–297], [12], [26] and [27], right-hand Rosenthal inequalities (5) and (6) for the t-th moment of regular symmetric statistics of order m with the constants $(Ct/\ln t)^{mt}$ of the best possible rate of growth, with C not depending on m (that is, one can take $A(m) = C^m$).

E.g., let, as in [12], $\mathrm{Log}\, t$ denote $\mathrm{Log}\, t = \max(1, \ln t)$. From [26] it follows that one can take $B(t) = (Ct/\mathrm{Log}\, t)^t$ in inequality (16). Assume (as in [7] and [25], we consider here first Rosenthal-type inequalities with sums of the terms instead of their maxima) that estimate (5) with the maxima replaced by sums and $B(t,l) = (Ct/\mathrm{Log}\, t)^{lt}$

(37)
$$\mathbf{E}T_{n,l}^t \leqq (Ct/\mathrm{Log}\, t)^{lt} \sum_{k=0}^{l} \sum_{1 \leqq j_1 < \cdots < j_k \leqq l} \sum_{1 \leqq i_{j_1} < \cdots < i_{j_l} \leqq n} \mathbf{E}\left(E_{i_{j_1},\dots,i_{j_l}} Y(i_{j_1},\dots,i_{j_l})\right)^t$$

holds for $l = m-1$ (the notations are the same as in the proof of Theorem 1). Then we have

$$\sum_{i_m=m}^{n} \mathbf{E}Y^t(i_m) \leqq (Ct/\mathrm{Log}\, t)^{(m-1)t} \sum_{i_m=m}^{n} \sum_{k=0}^{m-1} \sum_{1 \leqq j_1 < \cdots < j_k \leqq m-1} \sum_{1 \leqq i_{j_1} < \cdots < i_{j_k} \leqq i_m - 1} \mathbf{E}\left(E_{i_{j_1},\dots,i_{j_k},i_m} Y(i_{j_1},\dots,i_{j_k},i_m)\right)^t \tag{38}$$

and

$$\mathbf{E}\left(\sum_{i_m=m}^{n} E_{1,\dots,i_{m-1}} Y(i_m)\right)^t \leqq (Ct/\mathrm{Log}\, t)^{(m-1)t} \sum_{k=0}^{m-1} \sum_{1 \leqq j_1 < \cdots < j_k \leqq m-1} \sum_{1 \leqq i_{j_1} < \cdots < i_{j_k} \leqq i_m - 1} \mathbf{E}\left(E_{i_{j_1},\dots,i_{j_k}} Y(i_{j_1},\dots,i_{j_k},i_m)\right)^t. \tag{39}$$

Inequalities (16), (38) and (39) imply that inequality (37) holds for $l = m$. The fact that by [12] or [26] inequality (37) holds for $l = 1$ completes the proof of the inequality. Estimate (37) implies that upper inequality (5) holds with the constant $2^m (Ct/\mathrm{Log}\, t)^{mt} \leqq (2Ct/\mathrm{Log}\, t)^{mt}$.

In order to show that one can take $B(t,m) = (Ct/\ln t)^{mt}$ in upper inequality (6), first note that the proof of upper Rosenthal inequalities (5) and (6) for symmetric statistics of second order in [10] allows one to obtain, using the results in [12] and [26], the inequalities for the t-th moment of the statistics with the constants $(Ct/\ln t)^{2t}$ of the best possible, by Remark 3, rate of growth. E.g., from [26] it follows that one can take $B(t) = (C_1 t/\mathrm{Log}\, t)^t$ in inequalities (10), (11) and (13) in [10]; the latter three inequalities then evidently give the analogue of upper Rosenthal inequality (1) for symmetric statistics of second order with the constant $B(t) = 3(C_1 t/\mathrm{Log}\, t)^{2t}$. Similarly, one can take, again by [26], $B(t) = (C_1 t/\mathrm{Log}\, t)^t$ in the two estimates formulated between inequalities (19) and (20) in [10] and, by [12], $B(t) = (C_1 t/(2\mathrm{Log}\,(t/2)))^{t/2}$ in estimate (22) in [10]. Assuming in the induction hypothesis in the proof of Theorem 6 in [10] that the analogue of upper Rosenthal inequality (2) holds with the constant $B(t) = (Ct/(2\mathrm{Log}\,(t/2)))^t$ for all powers t of symmetric statistics of second order with $[t/2] = m-1$ gives inequality (35) in [10] with the same constant. The above mentioned estimates then together imply that the analogue of inequality (2) for symmetric statistics of second order holds with the constant $2(C_1t/\mathrm{Log}\,t)^{2t} + 2^{t/2-1}(C_1t/\mathrm{Log}\,t)^t(C_1t/(2\mathrm{Log}\,(t/2)))^{t/2} + 2^{t-1}(C_1t/\mathrm{Log}\,t)^t(Ct/(2\mathrm{Log}\,(t/2))^t$ for all powers t of symmetric statistics of second order with $[t/2] = m$ (note that the factor at the second summand in inequality (21) in [10] should read "2^{t-1}"). Choosing C such that the latter constant is not greater than $(Ct/\mathrm{Log}\,t)^{2t}$ completes the proof by induction.

Using the latter result, it is not difficult to get, applying inequality (2) with $B(t) = (C_1t/\ln t)^t$ (by [12] or [26]) and Rosenthal inequality for symmetric statistics of second order in [10] to estimate the right-hand side of (32) (see the end of the proof of Theorem 2), that both summands in the right-hand side of (32) are estimated by the right-hand side of (36) with $B(t) = (C_1t/\ln t)^t$. Assuming in the induction hypothesis in the proof of Theorem 2, as above, that right-hand inequality (6) with the maxima replaced by sums holds with the constant $B(t, m-1) = (Ct/\ln t)^{(m-1)t}$ for symmetric statistics of order $m-1$ and using the fact that, by [4, pp. 296–297] and [27], $\mathbf{E}|T_{n,m}|^t \leqq C_1^t\mathbf{E}|V_{n,m}|^t$, we then obtain that, for sufficiently large C, the same estimate holds with the constant $B(t,m) = (Ct/\ln t)^{mt}$ for $\mathbf{E}|T_{n,m}|^t$; this implies that upper inequality (6) holds with $B(t,m) = (Ct/\ln t)^{mt}$ for $T_{n,m}$.

Recently, De la Peña et al. [25] obtained sharp Rosenthal-type inequalities for infinite-degree U-statistics. Results related to those presented in this paper have been also obtained by Zhang [24], who proved moment inequalities for U-statistics using sub-Bernoulli functions.

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DEPARTMENT OF ECONOMICS
YALE UNIVERSITY
28 HILLHOUSE AVENUE
NEW HAVEN, CT 06511
U.S.A.

rustam.ibragimov@yale.edu

DEPARTMENT OF PROBABILITY THEORY
TASHKENT STATE ECONOMICS UNIVERSITY
UL. UZBEKISTANSKAYA 49
TASHKENT 700098
UZBEKISTAN

tim001@tseu.silk.org