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Analogues of Khintchine, Marcinkiewicz–Zygmund and Rosenthal Inequalities for Symmetric Statistics

R. IBRAGIMOV

Central Michigan University

Sh. SHARAKHMETOV

Tashkent State Economic University

ABSTRACT. In this paper we prove the analogues of Khintchine, Marcinkiewicz–Zygmund and Rosenthal moment inequalities for symmetric statistics of second order in not identically distributed random variables.

Key words: Khintchine, Marcinkiewicz–Zygmund, Rosenthal inequalities, martingale-difference sequence, moment, symmetric statistics

In many fields of mathematics such as probability theory, mathematical statistics, multiple stochastic integration, harmonic analysis, operator theory, quantum mechanics etc. the necessity of investigation of symmetric statistics, in particular, of their moments behaviour, appears (see Bonami, 1970; Serfling, 1980; Rosinski & Szulga, 1982; Sjorgen, 1982; Rosinski & Woyczynski, 1984, 1986; Cambanis *et al.*, 1985; Krakowiak & Szulga, 1986; Korolyuk & Borovskikh, 1989; de la Pena & Klass, 1994; for complete information).

In the case of linear statistics the following inequalities are well-known (in what follows $A(t)$, $B(t)$ are constants depending on t only, not necessarily the same in the different places).

Theorem 1 (Rosenthal, 1970)
If $\xi_1, \ldots, \xi_n$ are independent non-negative random variables (r.v.s) with finite tth moment, $1 \leqslant t < \infty$, then

$$\max\left(\sum_{i=1}^n E\xi_i^t, \left(\sum_{i=1}^n E\xi_i\right)^t\right) \leqslant E\left(\sum_{i=1}^n \xi_i\right)^t \leqslant B(t)\max\left(\sum_{i=1}^n E\xi_i^t, \left(\sum_{i=1}^n E\xi_i\right)^t\right). \quad (1)$$

Theorem 2 (Rosenthal, 1970)
If $\xi_1, \ldots, \xi_n$ are independent r.v.s with $E\xi_i = 0$, $E|\xi_i|^t < \infty$, $i = 1, \ldots, n$, $2 \leqslant t < \infty$, then

$$A(t)\max\left(\sum_{i=1}^n E|\xi_i|^t, \left(\sum_{i=1}^n E\xi_i^2\right)^{t/2}\right) \leqslant E\left|\sum_{i=1}^n \xi_i\right|^t$$
$$\leqslant B(t)\max\left(\sum_{i=1}^n \mathrm{E}|\xi_i|^t, \left(\sum_{i=1}^n E\xi_i^2\right)^{t/2}\right). \quad (2)$$

Theorem 3 (Khintchine (1923), Marcinkiewicz & Zygmund (1937))
If $\xi_1, \ldots, \xi_n$ are independent r.v.s with $E\xi_i = 0$, $E|\xi_i|^t < \infty$, $i = 1, \ldots, n$, $1 \leqslant t < \infty$, then

$$A(t)E\left(\sum_{i=1}^{n}\xi_i^2\right)^{t/2} \leqslant \mathrm{E}\left|\sum_{i=1}^{n}\xi_\mathrm{i}\right|^t \leqslant B(t)E\left(\sum_{i=1}^{n}\xi_i^2\right)^{t/2}. \tag{3}$$

The best constants in inequalities (1) and (2) (in the latter one for the case $t = 2m$) were found in Ibragimov & Sharakhmetov (1998, 1999a, b). The exact constant in inequality (2) for symmetric r.v.s was found in Figiel *et al*. (1997) and Ibragimov & Sharakhmetov (1995, 1997). The authors' results on the extremal problems and the best constants in Rosenthal's inequalities (1) and (2) and their proofs were presented in Ibragimov (1997).

In Sharakhmetov (1995a, b) the following analogues of inequalities (1)–(3) for non-linear statistics were proved.

Theorem 4 (Sharakhmetov, 1995, 1999)
If $t \geqslant 2$, $X_1, \ldots, X_n$ *are independent identically distributed r.v.s taking values in some measurable space* $(\mathscr{E}A)$, $Y\colon \mathscr{E}^2 \to \mathbb{R}$ *is a symmetric function satisfying the conditions* $E|Y(X_1, X_2)|^t < \infty$, $E(Y(X_1, X_2)/X_1) = 0$ *(a.s.), then*

$$A(t)\max(n^2E|Y(X_1, X_2)|^t, n^{t/2+1}E(E(Y^2(X_1, X_2)/X_1))^{t/2}, n^t(EY^2(X_1, X_2))^{t/2})$$
$$\leqslant E\left|\sum_{1\leqslant i,< i_2\leqslant n} Y(Xi_1, Xi_2)\right|^t \leqslant B(t)\max(n^2E|Y(X_1, X_2)|^t,$$
$$n^{t/2+1}E(E(Y^2(X_1, X_2)/X_1))^{t/2}, n^t(EY^2(X_1, X_2))^{t/2}),$$
$$A(t)E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Y^2(Xi_1, Xi_2)\right)^{t/2} \leqslant E\left|\sum_{1\leqslant i_1<i_2\leqslant n} Y(Xi_1, Xi_2)\right|^t$$
$$\leqslant B(t)E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Y^2(Xi_1, Xi_2)\right)^{t/2}.$$

The main goal of this paper is to extend the results obtained in Sharakhmetov (1995, 1999) to the case of symmetric statistics in not identically distributed r.v.s.

Let $X_1, \ldots, X_n$ be independent, not necessarily identically distributed r.v.s taking values in some measurable space $(\mathscr{E}A)$, $t \geqslant 1$.

Let $F(t)$ be a class of functions $Yi_1i_2\colon \mathscr{E}^2 \to \mathbb{R}$, $1 \leqslant i_1 \leqslant n$, $1 \leqslant i_2 \leqslant n$, $i_1 \neq i_2$, satisfying the conditions

$$Yi_1i_2(Xi_1, Xi_2) = Yi_2i_1(Xi_2, Xi_1) \text{ (a.s.)},$$
$$E|Yi_1i_2(Xi_1, Xi_2)|^t < \infty$$

for all $1 \leqslant i_1 < i_2 \leqslant n$. Denote by $G(t)$ the subset of $F(t)$ consisting of functions Y such that

$$E(Yi_1i_2(Xi_1, Xi_2)/Xi_1) = 0 \text{ (a.s.)},$$
$$E(Yi_1i_2(Xi_1, Xi_2)/Xi_2) = 0 \text{ (a.s.)}$$

for all $1 \leqslant i_1 < i_2 \leqslant n$. For $Y \in F(t)$ set

$$T_n = \sum_{1\leqslant i_1<i_2\leqslant n} Yi_1i_2(Xi_1, Xi_2).$$

In the following we shall write for brevity

$$Yi_1 i_2 = Yi_1 i_2(Xi_1, Xi_2),\ 1 \leqslant i_1 < i_2 \leqslant n.$$

In addition to that, denote

$$Vi_2 = \sum_{i_1=1}^{i_2-1} Yi_1 i_2, \quad i_2 = 2, \ldots, n,$$

$$Wi_1 = \sum_{i_2=i_1+1}^{n} Yi_1 i_2, \quad i_1 = 1, \ldots, n-1.$$

Theorem 5
If $t \geqslant 1$, $Yi_1 i_2 \colon \mathscr{E}^2 \to \mathbb{R}$, $1 \leqslant i_1 \leqslant n$, $1 \leqslant i_2 \leqslant n$, $i_1 \neq i_2$, *are non-negative functions belonging to the class* $F(t)$, *then*

$$\max\left[\sum_{1\leqslant i_1<i_2\leqslant n} EY^t_{i_1 i_2}, \sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1 i_2/Xi_1)\right)^t, \sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1 i_2/Xi_2)\right)^t, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1 i_2\right)^t\right] \leqslant ET^t_n \leqslant B(t) \max$$
$$\times\left[\sum_{1\leqslant i_1<i_2\leqslant n} EY^t_{i_1 i_2}, \sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1 i_2/Xi_1)\right)^t, \sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1 i_2/Xi_2)\right)^t, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1 i_2\right)^t\right]. \quad (4)$$

Remark 1. In their recently published paper, Klass & Nowicki (1997) independently obtained the analogues of inequalities (4) for generalized moments of symmetric statistics with non-negative kernels with the help of methods different from those used in the present paper.

Theorem 6
If $t \geqslant 2$, $Yi_1 i_2 \colon \mathscr{E}^2 \to \mathbb{R}$, $1 \leqslant i_1 \leqslant n$, $1 \leqslant i_2 \leqslant n$, $i_1 \neq i_2$, *are functions belonging to the class* $G(t)$, *then*

$$A(t)\max\left[\sum_{1\leqslant i_1<i_2\leqslant n} E|Yi_1 i_2|^t, \sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/2}, \sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1^2 i_2/Xi_2)\right)^{t/2}, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2}\right] \leqslant E|T_n|^t$$
$$\leqslant B(t)\max\left[\sum_{1\leqslant i_1<i_2\leqslant n} E|Yi_1 i_2|^t, \sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/2}, \sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1^2 i_2/Xi_2)\right)^{t/2}, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2}\right]. \quad (5)$$

Theorem 7
If $t \geqslant 2$, $Yi_1 i_2 \colon \mathscr{E}^2 \to \mathbb{R}$, $1 \leqslant i_1 \leqslant n$, $1 \leqslant i_2 \leqslant n$, $i_1 \neq i_2$, *are functions belonging to the class* $G(t)$, *then*

$$A(t)E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Yi_1^2 i_2\right)^{t/2} \leqslant E|T_n|^t \leqslant B(t)E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Yi_1^2 i_2\right)^{t/2}. \tag{6}$$

Corollary 1
If $t \geqslant 1$, $X_1, \ldots, X_n$ *are independent non-negative r.v.s with* $EX_k^t < \infty$, $k = 1, \ldots, n$, *then*

$$\max\left[\sum_{1\leqslant i_1<i_2\leqslant n} EXi_1^t EXi_2^t, \sum_{i_1=1}^{n-1} EXi_1^t\left(\sum_{i_2=i+1}^{n} EXi_2\right)\right]^t, \sum_{i_2=2}^{n} EXi_2^t\left(\sum_{i_1=1}^{i_2-1} EXi_1\right)^t,$$
$$\left(\sum_{1\leqslant i_1<i_2\leqslant n} EXi_1 EXi_2\right)^t\Bigg] \leqslant E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Xi_1 X_2\right)^t \leqslant B(t)\max$$
$$\Bigg[\sum_{1\leqslant i_1<i_2\leqslant n} EXi_1^t EXi_2^t, \sum_{i_1=1}^{n-1} EXi_1^t\left(\sum_{i_2=i_1+1}^{n} EXi_2\right)^t,$$
$$\sum_{i_2=2}^{n} EXi_2^t\left(\sum_{i_1=1}^{i_2-1} EXi_1\right)^t, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EXi_1 EXi_2\right)^t\Bigg].$$

Corollary 2
If $t \geqslant 2$, $X_1, \ldots, X_n$ *are independent r.v.s with* $EX_k = 0$, $E|X_k|^t < \infty$, $k = 1, \ldots, n$, *then*

$$A(t)\max\Bigg[\sum_{1\leqslant i_1<i_2\leqslant n} E|Xi_1|^t E|Xi_2|^t, \sum_{i_1=1}^{n-1} E|Xi_1|^t\left(\sum_{i_2=i_1+1}^{n} EXi_2^2\right)^{t/2},$$
$$\sum_{i_2=2}^{n} E|Xi_2|^t\left(\sum_{i_1=1}^{i_2-1} EXi_1^2\right)^{t/2}, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EXi_1^2 EXi_2^2\right)^{t/2}\Bigg] \leqslant E\left|\sum_{1\leqslant i_1<i_2\leqslant n} Xi_1 Xi_2\right|^t$$
$$\leqslant B(t)\max\Bigg[\sum_{1\leqslant i_1<i_2\leqslant n} E|Xi_1|^t E|Xi_2|^t, \sum_{i_1=1}^{n-1} E|Xi_1|^t\left(\sum_{i_2=i_1+1}^{n} EXi_2^2\right)^{t/2},$$
$$\sum_{i_2=2}^{n} E|Xi_2|^t\left(\sum_{i_1=1}^{i_2-1} EXi_1\right)^{t/2}, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EXi_1^2 EXi_2^2\right)^{t/2}\Bigg].$$

Corollary 3
If $t \geqslant 2$, $ai_1 i_2 \in \mathbb{R}$, $1 \leqslant i_1 < i_2 \leqslant n$, $X_1, \ldots, X_n$ *are independent r.v.s with* $EX_k = 0$, $E|X_k|^t < \infty$, $k = 1, \ldots, n$, *then*

$$A(t)E\left(\sum_{1\leqslant i_1<i_2\leqslant n} ai_1^2 i_2 Xi_1^2 Xi_2^2\right)^{t/2} \leqslant E\left|\sum_{1\leqslant i_1<i_2\leqslant n} ai_1 i_2 Xi_1 Xi_2\right|^t$$
$$\leqslant B(t)E\left(\sum_{1\leqslant i_1<i_2\leqslant n} ai_1^2 i_2 Xi_1^2 Xi_2^2\right)^{t/2}.$$

Remark 2. In Krakowiak & Szulga (1986) and McConell & Taqqu (1986) it was shown that in the case of symmetric r.v.s $X_1, \ldots, X_n$ the inequality given by corollary 3 is true for all $t > 0$. Using this result it is not difficult to obtain that corollary 3 is true under the restriction $t \geqslant 1$ (see also de la Pena & Klass, 1994, th. 2.3).

Let us formulate a number of preliminary facts needed for the proof of theorems.

Given a sequence of σ-algebras $(\mathscr{F}_n)$ on some probability space $(\Omega, \mathscr{F}, P)$, we denote by $E_k(\cdot) = E(\cdot/\mathscr{F}_k)$ the conditional expectation operator (with the convention that $E_0 = E$, the expectation operator). A sequence (Y_n) of integrable r.v.s is called a forward martingale-difference sequence (relative to $(\mathscr{F}_n)$) if

(a) $\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots$,
(b) Y_n is $\mathscr{F}_n$-measurable,
(c) $E_{n-1} Y_n = 0$ (a.s.), $n \geqslant 1$,
and a reverse martingale-difference sequence if

(a′) $\mathscr{F}_1 \supseteq \mathscr{F}_2 \supseteq \ldots$,
(b′) Y_n is $\mathscr{F}_n$-measurable,
(c′) $E_{n+1} Y_n = 0$ (a.s.), $n \geqslant 1$.

Lemma 1 (see Burkholder, 1973; Hitczenko, 1990)
Let $\mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \cdots \subseteq \mathscr{F}_n$ be an increasing sequence of σ-algebras on some probability space $(\Omega, \mathscr{F}, P)$, X_k, $k = 1, \ldots, n$, be a sequence of non-negative $\mathscr{F}_k$-measurable r.v.s such that $EX_k^t < \infty$, $1 \leqslant t < \infty$. Then

$$E\left(\sum_{k=1}^n X_k\right)^t \leqslant B(t) \max\left(\sum_{k=1}^n EX_k^t,\; E\left(\sum_{k=1}^n E_{k-1} X_k\right)^t\right).$$

Lemma 2 (see Burkholder, 1973; Korolyuk & Borovskikh, 1988)
If (X_k) is a forward martingale-difference sequence relative to $(\mathscr{F}_k)$ such that *$E|X_k|^t < \infty$, $k = 1, \ldots, n$, $2 \leqslant t < \infty$, then*

$$A(t)\max\left(\sum_{k=1}^n E|X_k|^t,\; E\left(\sum_{k=1}^n E_{k-1} X_k^2\right)^{t/2}\right) \leqslant E\left|\sum_{k=1}^n X_k\right|^t$$

$$\leqslant B(t)\max\left(\sum_{k=1}^n E|X_k|^t,\; E\left(\sum_{k=1}^n E_{k-1} X_k^2\right)^{t/2}\right).$$

Lemma 3
Let $\{X_n\}_{n=1}^\infty$ be a sequence of non-negative r.v.s,

$$A_t = \sum_{n=1}^\infty EX_n^t < \infty,$$

$$B_y = \left(\sum_{n=1}^\infty EX_n^y\right)^{1/y} < \infty.$$

Then

$$A_s \leqslant (A_t^{s-y} B_y^{y(t-s)})^{1/(t-y)} \quad \forall\, 1 \leqslant y < s < t \tag{7}$$

$$A_t B_y^s \leqslant \max(A_{t+s},\, B_y^{t+s}) \quad \forall\ 1 \leqslant y < t,\ s \geqslant 0, \tag{8}$$

$$\max(A_s,\, B_y^s) \leqslant (\max(A_t,\, B_y^t))^{s/t} \quad \forall\ 1 \leqslant y < s < t. \tag{9}$$

Proof. Inequality (7) follows from the convexity of function $f(t) = \ln A_t$. Using (7), we have that

$$A_t \leqslant (A_{t+s}^{t-y} B_y^{ys})^{1/(t+s-y)}, \quad 1 \leqslant y < t,\ s > 0.$$

Consequently,

$$A_t B_y^s \leqslant A_{t+s}^{(t-y)/(t+s-y)} (B_y^{t+s})^{s/(t+s-y)} \leqslant \max(A_{t+s},\, B_y^{t+s}), \quad 1 \leqslant y < t,\ s \geqslant 0.$$

Besides, on the strength of (7),

$$A_s \leqslant A_t^{(s-y)/(t-y)} B_y^{y(t-s)/(t-y)} \leqslant (\max(A_t,\, B_y^t))^{(s-y)/(t-y)} (\max(A_t,\, B_y^t))^{y(t-s)/(t(t-y))}$$

$$\leqslant (\max(A_t,\, B_y^t))^{s/t}, \quad 1 \leqslant y < s < t.$$

Thus, inequalities (8), (9) are true. The proof is complete.

Proof of theorem 5. By non-negativity of functions Y and Jensen's inequality we have

$$E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Yi_1 i_2\right)^t \geqslant \sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^t i_2,$$

$$E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Yi_1 i_2\right)^t \geqslant \sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} Yi_1 i_2\right)^t \geqslant \sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1 i_2/Xi_1)\right)^t,$$

$$E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Yi_1 i_2\right)^t \geqslant \sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} Yi_1 i_2\right)^t \geqslant \sum_{i_2=1}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1 i_2/Xi_2)\right)^t,$$

$$E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Yi_1 i_2\right)^t \geqslant \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1 i_2\right)^t.$$

The left-hand inequality (4) is true.

Let us prove right-hand inequality (4). It is obvious that r.v.s $Vi_2,\ i_2 = 2, \ldots, n$ are measurable with respect to σ-algebras $\sigma(X_1, X_2, \ldots, Xi_2)$, r.v.s $Yi_1 i_2,\ i_1 = 1, \ldots, i_2 - 1$ are measurable with respect to σ-algebras $\sigma(X_1, X_2, \ldots, Xi_1, Xi_2)$, $i_2 = 2, \ldots, n$. On the strength of lemma 1 the following inequalities are true:

$$ET_n^t \leqslant B(t) \max\left(\sum_{i_2=2}^{n} EYi_2^t,\ E\left(\sum_{i_2=2}^{n} E(Yi_2/X_1, \ldots, Xi_2 - 1)\right)^t\right), \tag{10}$$

$$EVi_2^t \leqslant B(t) \max\left(\sum_{i_1=1}^{i_2-1} EYi_1^t i_2,\ E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1 i_2/Xi_2)\right)^t\right). \tag{11}$$

(10) and (11) imply that ($B(t)$ changes from line to line)

$$ET_n^t \leqslant B(t) \max\left(\sum_{1 \leqslant i_1 < i_2 \leqslant n} EYi_1 i_2, \sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1 i_2 / Xi_2) \right)^t, \right.$$

$$\left. E\left(\sum_{1 \leqslant i_1 < i_2 \leqslant n} E(Yi_1 i_2 / Xi_1) \right)^t \right). \quad (12)$$

By Rosenthal's inequality for independent non-negative r.v.s (inequality (1))

$$E\left(\sum_{1 \leqslant i_1 < i_2 \leqslant n} E(Yi_1 i_2 / Xi_1) \right)^t = E\left(\sum_{i_1=1}^{n} \sum_{i_2=i_1+1}^{n} E(Yi_1 i_2 / Xi_1) \right)^t$$

$$\leqslant B(t) \max\left(\sum_{i_1=1}^{n} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1 i_2 / Xi_1) \right)^t, \left(\sum_{1 \leqslant i_1 < i_2 \leqslant n} EYi_1 i_2 \right)^t \right). \quad (13)$$

Right-hand inequality (4) follows from (12) and (13). The proof is complete.

Proof of theorem 6. It is easy to see that under the assumptions of theorem 6 $\{Vi_2,\ i_2 = 2, \ldots, n\}$ is a forward martingale-difference sequence with respect to σ-algebras $\sigma(X_1, X_2, \ldots, Xi_2)$, $\{Yi_1 i_2,\ i_1 = 1, \ldots, i_2 - 1\}$ is a forward martingale-difference sequence with respect to σ-algebras $\sigma(X_1, X_2, \ldots, Xi_1, Xi_2)$, $i_2 = 2, \ldots, n$, $\{Wi_1,\ i_1 = 1, \ldots, n-1\}$ is a reverse martingale-difference sequence with respect to σ-algebras $\sigma(Xi_1, \ldots, X_n)$, $\{Yi_1 i_2,\ i_2 = i_1 + 1, \ldots, n\}$ is a reverse martingale-difference sequence with respect to σ-algebras $\sigma(Xi_1, Xi_2, \ldots, X_n)$, $i_1 = 1, \ldots, n-1$.

On the strength of lemma 2

$$E|T_n|^t \geqslant A(t) \max\left(\sum_{i_2=2}^{n} E|Vi_2|^t, E\left(\sum_{i_2=2}^{n} E(Vi_2^2 / X_1, \ldots, Xi_2 - 1) \right)^{t/2} \right), \quad (14)$$

$$E|T_n|^t \geqslant A(t) \max\left(\sum_{i_1=1}^{n-1} E|Wi_1|^t, E\left(\sum_{i_1=1}^{n-1} E(Wi_1^2 / Xi_1 + 1, \ldots, X_n) \right)^{t/2} \right), \quad (15)$$

$$E|Vi_2|^t \geqslant A(t) \max\left(\sum_{i_1=1}^{i_2-1} E|Yi_1 i_2|^t, E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1{}_1^2 i_2 / Xi_2) \right)^{t/2} \right), \quad (16)$$

$$E|Wi_1|^t \geqslant A(t) \max\left(\sum_{i_2=i_1+1}^{n} E|Yi_1 i_2|^t, E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2 / Xi_1) \right)^{t/2} \right). \quad (17)$$

It is easy to check that if functions $Yi_1 i_2 \colon \mathscr{E}^2 \to \mathbb{R}$, $1 \leqslant i_1 \leqslant n$, $1 \leqslant i_2 \leqslant n$, $i_1 \neq i_2$, satisfy the conditions

$$E(Yi_1 i_2(Xi_1, Xi_2)/Xi_1) = 0 \text{ (a.s.)},$$

$$E(Yi_1 i_2(Xi_1, Xi_2)/Xi_2) = 0 \text{ (a.s.)}$$

for all $1 \leqslant i_1 < i_2 \leqslant n$, then r.v.s $Yi_1 i_2(Xi_1, Xi_2)$ are orthogonal, that is

$$EYi_1 i_2(Xi_1, Xi_2) Yj_1 j_2(Xj_1, Xj_2) = 0 \quad (18)$$

for $(i_1, i_2) \neq (j_1, j_2)$. By (18) and Jensen's inequality we have

$$E\left(\sum_{i_2=2}^{n} E(V_{i_2}^2/X_1, \ldots, X_{i_2-1})\right)^{t/2} \geqslant \left(\sum_{i_2=2}^{n} EV_{i_2}^2\right)^{t/2} = \left(\sum_{1\leqslant i_1<i_2\leqslant n} EY_{i_1 i_2}^2\right)^{t/2}. \tag{19}$$

Left-hand inequality (5) follows from (14)–(17) and (19).

Let us prove the remaining part of theorem 6. By lemma 2

$$E|T_n|^t \leqslant B(t) \max\left(\sum_{i_2=2}^{n} E|V_{i_2}|^t,\ E\left(\sum_{i_2=2}^{n} E(V_{i_2}^2/X_1, \ldots, X_{i_2-1})\right)^{t/2}\right),$$

$$E|V_{i_2}|^t \leqslant B(t) \max\left(\sum_{i_1=1}^{i_2-1} E|Y_{i_1 i_2}|^t,\ E\left(\sum_{i_1=1}^{i_2-1} E(Y_{i_1 i_2}^2/X_{i_2})\right)^{t/2}\right).$$

Therefore,

$$E|T_n|^t \leqslant B(t) \max\left(\sum_{1\leqslant i_1<i_2\leqslant n} E|Y_{i_1 i_2}|^t,\ \sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Y_{i_1 i_2}^2/X_{i_2})\right)^{t/2},\ E\left(\sum_{i_2=1}^{n} E(V_{i_2}^2/X_1, \ldots, X_{i_2-1})\right)^{t/2}\right). \tag{20}$$

We have

$$\begin{aligned} E\left(\sum_{i_2=2}^{n} E(V_{i_2}^2/X_1, \ldots, X_{i_2-1})\right)^{t/2} &= E\left(\sum_{1\leqslant i_1<i_2\leqslant n} E(Y_{i_1 i_2}^2/X_{i_1}) + 2\sum_{i_2=3}^{n}\sum_{1\leqslant k<l\leqslant i_2-1} E(Y_{k i_2} Y_{l i_2}/X_k, X_l)\right)^{t/2} \\ &\leqslant 2^{t/2-1} E\left(\sum_{1\leqslant i_1<i_2\leqslant n} E(Y_{i_1 i_2}^2/X_{i_1})\right)^{t/2} \\ &\quad + 2^{t/2} E\left|\sum_{1\leqslant k<l\leqslant n-1}\ \sum_{i_2=l+1}^{n} E(Y_{k i_2} Y_{l i_2}/X_k, X_l)\right|^{t/2}. \end{aligned} \tag{21}$$

It follows from inequality (1) that for all $t \geqslant 2$

$$\begin{aligned} E&\left(\sum_{1\leqslant i_1<i_2\leqslant n} E(Y_{i_1 i_2}^2/X_{i_1})\right)^{t/2} \\ &\leqslant B(t) \max\left(\sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Y_{i_1 i_2}^2/X_{i_1})\right)^{t/2},\ \left(\sum_{1\leqslant i_1<i_2\leqslant n} EY_{i_1 i_2}^2\right)^{t/2}\right). \end{aligned} \tag{22}$$

It is not difficult to check that the functions $Z_{kl}: \mathscr{E}^2 \to \mathbb{R}$, $1 \leqslant k \leqslant n$, $1 \leqslant l \leqslant n$, $k \neq l$, defined by the conditions

$$Z_{kl}(x, y) = \sum_{i_2=l+1}^{n} E(Y_{k i_2} Y_{l i_2}/X_k = x, X_l = y),$$

belong to the class $G(t/2)$.

On the strength of the orthogonality of the functions Z (relation (18)) and Jensen's inequality we have for $2 \leqslant t < 4$

$$E\left|\sum_{1\leqslant k<l\leqslant n-1}\sum_{i_2=l+1}^{n} E(Yki_2 Yli_2/X_k, X_l)\right|^{t/2}$$

$$\leqslant \left(\sum_{1\leqslant k<l\leqslant n-1} E\left(\sum_{i_2=l+1}^{n} E(Yki_2 Yli_2/X_k, X_l)\right)^2\right)^{t/4}. \tag{23}$$

Let us show that

$$\left(\sum_{1\leqslant k<l\leqslant n-1} E\left(\sum_{i_2=l+1}^{n} E(Yki_2 Yli_2/X_k, X_l)\right)^2\right)^{t/4} \leqslant \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2} \tag{24}$$

for all $2 \leqslant t < 4$. Indeed, by Schwarz inequality

$$|E(Yki_2 Yli_2/X_k, X_l| \leqslant (E(Yki_2^2/X_k))^{1/2}(E(Yli_2^2/X_l))^{1/2}. \tag{25}$$

Consequently,

$$\left(\sum_{1\leqslant k<l\leqslant n-1} E\left(\sum_{i_2=l+1}^{n} E(Yki_2 Yli_2/X_k, X_l)\right)^2\right)^{t/4}$$

$$\leqslant \left(\sum_{1\leqslant k<l\leqslant n-1} E\left(\sum_{i_2=l+1}^{n} (E(Yki_2^2/X_k))^{1/2}(E(Yli_2^2/X_l))^{1/2}\right)^2\right)^{t/4}. \tag{26}$$

From (26) and Cauchy's inequality

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right), \tag{27}$$

where $a_i, b_i \geqslant 0$, it follows that

$$\left(\sum_{1\leqslant k<l\leqslant n-1} E\left(\sum_{i_2=l+1}^{n} E(Yki_2 Yli_2/X_k, X_l)\right)^2\right)^{t/4}$$

$$\leqslant \left(\sum_{1\leqslant k<l\leqslant n-1}\left(\sum_{i_2=l+1}^{n} EYki_2^2\right)\left(\sum_{i_2=l+1}^{n} EYli_2^2\right)\right)^{t/4} \leqslant \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2}.$$

(23), (24) imply

$$E\left|\sum_{1\leqslant k<l\leqslant n-1}\sum_{i_2=l+1}^{n} E(Yki_2 Yli_2/X_k, X_l)\right|^{t/2} \leqslant \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2} \tag{28}$$

for all $2 \leqslant t < 4$. Using (20)–(22), (28) we obtain that right-hand inequality (5) is true for $2 \leqslant t < 4$.

For $2 \leqslant t < \infty$ denote $a(t) = [t/2]$. We have proved that right-hand inequality (5) is true for all t such that $a(t) = 1$. Take t such that $a(t) = m \geqslant 2$ and suppose that right-hand inequality (5) is true for all t with $a(t) = m - 1$.

On the strength of inductive hypothesis

$$E\left|\sum_{1\leqslant k<l\leqslant n-1}\sum_{i_2=l+1}^{n}E(Yki_2Yli_2/X_k,X_l)\right|^{t/2}$$
$$\leqslant B(t/2)\max\left[\sum_{1\leqslant k<l\leqslant n-1}E\left|\sum_{i_2=l+1}^{n}E(Yki_2Yli_2/X_k,X_l)\right|^{t/2},\right.$$
$$\sum_{t=2}^{n-1}E\left(\sum_{k=1}^{l-1}E\left(\left(\sum_{i_2=l+1}^{n}E(Yki_2Yli_2/X_k,X_l)\right)^2\Big/X_l\right)\right)^{t/4},$$
$$\sum_{k=2}^{n-1}E\left(\sum_{l=k+1}^{n-1}E\left(\sum_{i_2=l+1}^{n}E(Yki_2Yli_2/X_k,X_l)\right)^2\Big/X_k\right)^{t/4},$$
$$\left.\left[\sum_{1\leqslant k<l\leqslant n-1}E\left(\sum_{i_2=l+1}^{n}E(Yki_2Yli_2/X_k,X_l)\right)^2\right]^{t/2}\right]. \tag{29}$$

From (25), (27) it follows that

$$\sum_{1\leqslant k<l\leqslant n-1}E\left|\sum_{i_2=l+1}^{n}E(Yki_2Yli_2/X_k,X_l)\right|^{t/2}$$
$$\leqslant\sum_{1\leqslant k<l\leqslant n-1}E\left(\sum_{i_2=l+1}^{n}(E(Yki_2^2/X_k))^{1/2}(E(Yli_2^2/X_l))^{1/2}\right)^{t/2}$$
$$\leqslant\sum_{1\leqslant k<l\leqslant n-1}E\left(\sum_{i_2=l+1}^{n}E(Yki_2^2/X_k)\right)^{t/4}E\left(\sum_{i_2=l+1}^{n}E(Yli_2^2/X_l)\right)^{t/4}$$
$$\leqslant\left[\sum_{i_1=1}^{n-1}E\left(\sum_{i_2=i_1=1}^{n}E(Yi_1^2i_2/Xi_1)\right)^{t/4}\right]^2. \tag{30}$$

Besides,

$$\sum_{l=2}^{n-1}E\left[\sum_{k=1}^{l-1}E\left(\left(\sum_{i_2=l+1}^{n}E(Yki_2Yli_2/X_k,X_l)\right)^2\Big/X_l\right)\right]^{t/4}$$
$$\leqslant\sum_{l=2}^{n-1}E\left[\sum_{k=1}^{l-1}E\left(\left(\sum_{i_2=l+1}^{n}(E(Yki_2^2/X_k))^{1/2}(E(Yli_2^2/X_l))^{1/2}\right)^2\Big/X_l\right)\right]^{t/4}$$
$$\leqslant\sum_{l=2}^{n-1}E\left[\sum_{k=1}^{l-1}\left(\sum_{i_2=l+1}^{n}EYki_2^2\right)\left(\sum_{i_2=l+1}^{n}E(Yli_2^2/X_l)\right)\right]^{t/4}$$
$$=\sum_{l=2}^{n-1}\left(\sum_{k=1}^{l-1}\sum_{i_2=l+1}^{n}EYki_2^2\right)^{t/4}E\left(\sum_{i_2=l+1}^{n}E(Yli_2^2/X_l)\right)^{t/4}$$
$$\leqslant\sum_{i_1=1}^{n-1}\left(\sum_{1\leqslant i_1<i_2\leqslant n}EYi_1^2i_2\right)^{t/4}E\left(\sum_{i_2=i_1+1}^{n}E(Yi_1^2i_2/X_{i1})\right)^{t/4}. \tag{31}$$

Analogously,

$$\sum_{k=1}^{n-1} E\left[\sum_{l=k+1}^{n-1} E\left(\sum_{i_2=l+1}^{n} E(Yki_2\,Yli_2/X_k,\,X_l)\right)^2 \Big/ X_k\right]^{t/4}$$

$$\leqslant \sum_{k=1}^{n-1} E\left[\sum_{l=k+1}^{n-1} E\left(\left(\sum_{i_2=l+1}^{n} (E(Yki_2^2/X_k))^{1/2}(E(Yli_2^2/X_l))^{1/2}\right)^2 \Big/ X_k\right)\right]^{t/4}$$

$$\leqslant \sum_{k=1}^{n-1} E\left[\sum_{l=k+1}^{n-1} \left(\sum_{i_2=l+1}^{n} EYki_2^2\right)\left(\sum_{i_2=l+1}^{n} E(Yli_2^2/X_l)\right)\right]^{t/4}$$

$$= \sum_{k=1}^{n-1} \left(\sum_{l=k+1}^{n-1}\sum_{i_2=l+1}^{n} EYki_2^2\right)^{t/4} E\left(\sum_{i_2=l+1}^{n} E(Yli_2^2/X_l)\right)^{t/4}$$

$$\leqslant \sum_{i_1=1}^{n-1} E\left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/4} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/4}. \tag{32}$$

From lemma 3 it follows that

$$\left(\sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/4}\right)^2$$

$$\leqslant \max\left(\sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/2},\ \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2}\right), \tag{33}$$

$$\sum_{i_1=1}^{n-1} E\left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/4} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/4}$$

$$\leqslant \max\left(\sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/2},\ \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2}\right). \tag{34}$$

(30)–(34) imply that

$$E\left|\sum_{1\leqslant k<l\leqslant n-1}\sum_{i_2=l+1}^{n} E(Yki_2\,Yli_2/X_k,\,X_l)\right|^{t/2}$$

$$\leqslant B(t/2)\max\left(\sum_{i_1=1}^{n} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/2},\ \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2}\right). \tag{35}$$

From (20)–(22) and (35) it follows that right-hand inequality (5) is true for the chosen value of t with $a(t) = m$. By induction principle the proof is complete.

Proof of theorem 7. Theorem 5 implies that

$$\left[\sum_{1\leqslant i_1<i_2\leqslant n} E|Yi_1 i_2|^t,\ \sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/2},\right.$$

$$\sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1^2 i_2/Xi_2\right)^{t/2}, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2}\Bigg] \leqslant E\left(\sum_{1\leqslant i_1<i_2\leqslant n} Yi_1^2 i_2\right)^{t/2}$$

$$\leqslant B(t)\max\Bigg[\sum_{1\leqslant i_1<i_2\leqslant n} E|Yi_1 i_2|^t, \sum_{i_1=1}^{n-1} E\left(\sum_{i_2=i_1+1}^{n} E(Yi_1^2 i_2/Xi_1)\right)^{t/2},$$

$$\sum_{i_2=2}^{n} E\left(\sum_{i_1=1}^{i_2-1} E(Yi_1^2 i_2/Xi_2)\right)^{t/2}, \left(\sum_{1\leqslant i_1<i_2\leqslant n} EYi_1^2 i_2\right)^{t/2}\Bigg]. \quad (36)$$

Applying (5), (36) we obtain (6). The proof is complete.

The methods similar to those used in the present paper can be applied for the proof of exact moment inequalities in the case of generalized moments and symmetric statistics of arbitrary order (see Ibragimov, 1997; Ibragimov & Sharakhmetov, 1998).

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Rustam Ibragimov, Department of Economics, Central Michigan University, Mt. Pleasant, MI 48859, USA