Value at risk under dependence and heavy-tailedness: models with common shocks

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Outline

Empirical literature

- Many time series in economics and finance: **heavy-tailed**

  $P(X > x) \sim x^{-\alpha}, \ \ \alpha > 0$

  - Mandelbrot (1963): $\alpha \approx 1.7$ for historical daily changes of cotton prices

- Returns on various stocks and stock indices:
  - $1 < \alpha < 2$ (Fama, 1963, 1965)
  - $3 < \alpha < 5$ (Jansen and de Vries, 1991)
  - $2 < \alpha < 4$ (Loretan and Phillips, 1994)
  - $1.5 < \alpha < 2$ (McCulloch, 1996, 1997)
  - $0.9 < \alpha < 2$ (Rachev and Mittnik, 2000)
  - $\alpha \approx 3$ (Gopikrishnan et. al., 1999, Gabaix et. al., 2003)

Empirical literature, ctd.

- Trading volume: $\alpha \approx 1.5$
- Number of trades: $\alpha \approx 3.4$

  (Gopikrishnan et. al., 2000, Plerou et. al., 2000, Gabaix, 2003)
- Firm sizes, city sizes, sizes of largest mutual funds: $\alpha \approx 1$

  (Gabaix, 1999, Axtell, 2001, Gabaix et. al., 2003)
- Embrechts, Klupperberg & Mikosch (1997), Beirlant, Goegebeur, Teugels, Segers (2004), Rachev, Menn & Fabozzi (2005)

Empirical literature, ctd.

- $\alpha < 2$, slightly or subtantially < 1 for a number of time series (Rachev and Mittnik, 2000)
- $\alpha < 1$: Some exchange rates (Rachev and Mittnik, 2000)
- α **considerably less** than one for
  - Profits, sizes of **technological innovations** (Scherer, Harhoff and Kukies, 2004, Silverberg and Verspagen, 2004)
  - Loss distributions of **operational risks** (Nešlehova, Embrechts & Chavez-Demoulin)
- Economic losses from natural disasters (e.g., earthquakes): $\alpha \in [0.6, 1.5]$ (Ibragimov, Jaffee & Walden, 2006)
- Profits from **motion pictures**: $1 < \alpha < 2$ (De Vany and Walls, 2004)

Main contributions

- **Diversification** and **portfolio VaR** for **heavy-tailed** and **dependent** risks
  - Models with multiple **common shocks**
- **Portfolio VaR** in **balanced** models:

  All risks are available for portfolio formation
  - **Diversification** is **optimal** for **moderately** heavy-tailed dependent risks

    $\alpha > 1$ for **both** the **common shock** and **idiosyncratic** parts of risks
  - **Inferior** for **extremely** heavy-tailed dependent risks

    $\alpha < 1$ for **both** the **common shock** and **idiosyncratic** parts of risks

Main contributions

- **Diversification** in **unbalanced** models with **common shocks**:

  Optimal even though there is **extreme heavy-tailedness**

  Portfolios of indices of **heavy-tailed dependent** risks:

  - $\alpha < 1$ in **common shocks**, $\alpha > 1$ in **idiosyncratic** parts: **diversification** on the **level** of **individual** risks is optimal
  - $\alpha > 1$ in **common shocks**, $\alpha < 1$ in **idiosyncratic** parts: **diversification** on the **level** of **indices** is optimal

- **Contrast to variance** comparisons: **heavy-tailedness helps diversification!**
- Similar conclusions: **efficiency comparisons** of **linear estimators** in **random effects** location models

Majorization & Diversification

- w **majorized** by v ($\mathbf{w} \prec \mathbf{v}$):

$$\sum_{i=1}^{k} w_{[i]} \leq \sum_{i=1}^{k} v_{[i]}, \quad k = 1, ..., N-1,$$

$$\sum_{i=1}^{N} w_{[i]} = \sum_{i=1}^{N} v_{[i]},$$

$$w_{[1]} \geq \ldots \geq w_{[N]}, \quad v_{[1]} \geq \ldots \geq v_{[N]}$$

- $w \prec v \iff$ Components of w **less diverse**

$$\underbrace{(\frac{1}{N}, ..., \frac{1}{N})}_{N} \prec (w_1, ..., w_N) \prec \underbrace{(1, 0, ..., 0)}_{N}$$

- $\forall w \in \mathcal{I}_N = \{w = (w_1, ..., w_N) \in \mathbf{R}_+^N : \sum_{i=1}^{N} w_i = 1\}$.

Majorization & Diversification

- Precise formalization of **diversification**:

  $w \prec v$: portfolio with weights w is **more diversified**

- **Equal** weights $\left(\frac{1}{N}, ..., \frac{1}{N}\right)$: **Most diversified**

- **One** risk $\left(1, 0, ..., 0\right)$: **Least diversified**

- **Schur-convex** ϕ: $a \prec b \Longrightarrow \phi(a) \leq \phi(b)$
- **Schur-concave** ϕ : $a \prec b \Longrightarrow \phi(a) \geq \phi(b)$
  - $\phi(a) = \sum_{i=1}^{n} f(a_i)$ **Schur-convex** iff f convex
  - • $\phi(a) = \sum_{i=1}^{n} f(a_i)$ **Schur-concave** iff f concave

Stable distributions

- $S_\alpha(\sigma)$: the symmetric **stable** distribution

  CF: $E(e^{ixX}) = exp\{-\sigma^\alpha |x|^\alpha\}$

  - $\alpha = 2$: **Gaussian**
  - $\alpha = 1$ **Cauchy**
  - **Lévy** $(\alpha = 1/2)$: one sided

  α : **tail heaviness;** $E|X|^p < \infty$ iff $p < \alpha$

Heavy-tailed classes

- $\overline{\mathcal{CS}}$ (**moderately** heavy-tailed): **convolutions** of $S_\alpha(\sigma)$, $\alpha > 1$

$$X = Y_1 + ... + Y_k, \quad Y_i \sim S_{\alpha_i}(\sigma_i), \quad \alpha_i > 1$$

- $\underline{\mathcal{CS}}$ (Extremely heavy-tailed): **convolutions** of $S_\alpha(\sigma)$, $\alpha < 1$

$$X = Y_1 + ... + Y_k, \quad Y_i \sim S_{\alpha_i}(\sigma_i), \quad \alpha_i < 1$$

- Sum of independent stable; **same** α = stable α.

  Does not hold for **different** α
- $\overline{\mathcal{CS}}$: **wider** than **all stable** with $\alpha > 1$
- $\underline{\mathcal{CS}}$: **wider** than **all stable** with $\alpha < 1$
- Cauchy: **boundary** between $\underline{\mathcal{CS}}$ and $\overline{\mathcal{CS}}$

Models with common shocks

- $Y_{ij} = R_i + C_j + U_{ij}, \quad i = 1, ..., r, j = 1, ..., c$
  - R_i : "**row** effects" **common shocks**, C_j : "**column** effects" **common shocks**, U_{ij} : **idiosyncratic** parts of Y_{ij}
  - R_i : affect all risks in ith row (ith country)
  - C_j : affect all risks in jth column (jth industry)
  - Independent of each other and i.i.d. among themselves (extensions to dependence)
  - Many problems in finance, risk management & insurance
    - Total loss $Z = \sum_{i=1}^{r} Z_i$, r **risk types**, **business lines** or **classes** of risks
    - Many real-world frameworks: **dependent** severity variables Y_j
    - Models for **operational risks**
    - **Multiline** vs. **monoline** insurance

Models with common shocks

- $Y_{ij} = R_i + U_{ij}, \;\; i = 1, ..., r, j = 1, ..., c$

  Balanced models: **all** *rc* **risks** Y_{ij} **available** for portfolio formation

- **Unbalanced** model: $Y_{ij} = R_i + U_{ij}, \;\; j = 1, ..., n_i, i = 1, ..., r$

- $Y(w) = \sum_{i,j} w_{ij} Y_{ij}$: return on **portfolio** of Y_{ij}
  - **Most** diversified, **equal** weights: $\underline{w}_{rc} = \underbrace{\Big(1/(rc), 1/(rc), ..., 1/(rc)\Big)}_{rc}$
  - **Least** diversified, **one risk**: $\overline{w}_{rc} = \underbrace{\Big(1, 0, ..., 0)\Big)}_{rc}$

Portfolio VaR: Moderately heavy tails

- $VaR_\alpha(Z)$: **value at risk** of Z at level α

  $(1-\alpha)-$quantile: $P(Z > x) = \alpha$

- $R_i, C_j, U_{ij} \sim \overline{\mathcal{CS}}$ (**convolutions of stables**, $\alpha > 1$)

  moderately heavy-tailed, finite first moment:

  Diversification $\Longrightarrow$

  Decrease in riskiness of $Y(w) = \sum_{i,j} w_{ij} X_{ij}$

$$\underbrace{VaR_q[Y(\underline{w}_{rc})]}_{\textit{Equal weights: Most diversified}} \leq VaR_q[Y(w)] \leq \underbrace{VaR_q[Y(\overline{w}_{rc})]}_{\textit{One risk: Least diversified}}$$

Portfolio VaR: Extremely heavy tails

- $R_i, C_j, U_{ij} \sim \underline{\mathcal{CS}}$ (**convolutions of stables**, $\alpha < 1$)

  extremely-tailed, infinite first moment:

  Diversification $\Longrightarrow$

  Increase in riskiness of $Y(w) = \sum_{i,j} w_{ij} X_{ij}$

$$\underbrace{VaR_q[Y(\underline{w}_{rc})]}_{\textit{Equal weights: Most diversified}} \geq VaR_q[Y(w)] \geq \underbrace{VaR_q[Y(\overline{w}_{rc})]}_{\textit{One risk: Least diversified}}$$

- Similar to independent $X_i,\ Y(w) = \sum_{i=1}^{n} w_i X_i$
  - Diversification: optimal for VaR if $\alpha > 1$
  - Suboptimal for portfolio VaR if $\alpha < 1$

When heavy-tailedness helps: VaR for financial indices

- $n_1 \geq ... \geq n_r \geq 1$, $\sum_{i=1}^{n_r} n_i = n$
- r **equally weighted indices** $i = 1, ..., r$ comprised of

  $Y_{ij} = R_i + U_{ij}, \;\; j = 1, ..., n_i$
- **Return on index** i : $Z_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij} = R_i + \frac{1}{n_i} \sum_{j=1}^{n_i} U_{ij}$
- $Z(w)$: **portfolio** of indices $i = 1, ..., r$ with weights w:

$$Z(w) = \sum_{i=1}^{r} w_i Z_i = \sum_{i=1}^{r} \frac{(Y_{i1} + ... + Y_{in_i})}{n_i} w_i =$$

$$\sum_{i=1}^{r} w_i R_i + \sum_{i=1}^{r} \frac{(U_{i1} + ... + U_{in_i})}{n_i} w_i.$$

VaR for financial indices

Index return decomposition

- $Z(w) = \underbrace{R(w)}_{\textit{common shock portfolio}} + \underbrace{U(\tilde{w})}_{\textit{idiosyncratic risk portfolio}}$

  $R(w) = \sum_{i=1}^{n} w_i R_i$

  $U(\tilde{w}) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \tilde{w}_{ij} U_{ij}$

  - $\tilde{w} = (\tilde{w}_{11}, ..., \tilde{w}_{1n_1}, ..., \tilde{w}_{r1}, ..., \tilde{w}_{rn_r}) = (\frac{w_1}{n_1} e_1, ..., \frac{w_r}{n_r} e_r)$
  - $\tilde{w}_{ij} = w_i / n_i$, $j = 1, ..., n_i$
  - $e = \underbrace{(1, ..., 1)}_{N} \in \mathbf{R}^N$, $N \geq 1$: vectors of ones

Diversification weights

- $w^{(1)} = \underline{w}_r = \big(1/r, ..., 1/r\big)$

  Diversification on the level of **indices** $i = 1, ..., r$ with returns $Z_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}$

$$Z(w^{(1)}) = \underbrace{\frac{1}{r}\sum_{i=1}^{r} Z_i}_{\textit{Equal weights: Most diversified}} = \frac{1}{r}\sum_{i=1}^{r} \frac{(Y_{i1} + ... + Y_{in_i})}{n_i} =$$

$$\frac{1}{r}\sum_{i=1}^{r} R_i + \frac{1}{r}\sum_{i=1}^{r} \frac{(U_{i1} + ... + U_{in_i})}{n_i}$$

Diversification weights

- $w^{(2)} = \left(\frac{n_1}{n}, \frac{n_2}{n}, \dots, \frac{n_r}{n}\right)$:

$$Z(w^{(2)}) = \sum_{i=1}^{r} \frac{n_i}{n} Z_i = \underbrace{\frac{1}{n} \sum_{i=1}^{r} \sum_{j=1}^{n_i} Y_{ij}}_{\textit{Equal weights: Most diversified}}$$

(Full) **Diversification** on the level of **underlying risks** Y_{ij}

Variance: Suboptimal diversification

- **Variance** comparisons: **Suboptimal diversification**

$$Var[Z(w)] = \underbrace{Var[R(w)]}_{common\ shock} + \underbrace{Var[U(\tilde{w})]}_{idiosyncratic} = V_R(w) + V_U(w)$$

- $\underbrace{V_R(w^{(1)})}_{Diversified\ indices} \leq \underbrace{V_R(w^{(2)})}_{Diversified\ risks}$ **but**

  $\underbrace{V_U(w^{(1)})}_{Diversified\ indices} \geq \underbrace{V_U(w^{(2)})}_{Diversified\ risks}$

  Opposite for common shock and idiosyncratic parts

- Statistics literature going back to W. G. Cochran (1954), Koch (1967), Low (1970), Marshall and Olkin (1979), Birkes *et al.* (1981), EL-Bassiouni (2000)

Finite variance: Efficient weights

- Statistics literature, **efficient** estimation of **location**:

  $w_i(c) = \frac{n_i[(n_i-1)c+1]^{-1}}{\sum_{i=1}^{r} n_i[(n_i-1)c+1]^{-1}}, \quad 0 \le c \le 1$

  - **Minimal complete class** in variance minimization
  - $w(1) = w^{(1)} = \underline{w}_r$ (**Diversification** of **indices**, Z_i)
  - $w(0) = w^{(2)}$ (**Diversification** of risks Y_{ij})
  - Known **intraclass correlation** $\gamma = \sigma_R^2/(\sigma_R^2 + \sigma_U^2)$: $w(\gamma) = argmin_w var[Z(w)]$

- Efficiency: $\text{eff}(c,\gamma) = var[Z(w(c))] / \underbrace{var[Z(w(\gamma))]}_{minimum\ variance}$

- $Z(w(c^*))$: **Maximizes** (over $c \in [0,1]$) **minimum** possible **efficiency** $\min_{\gamma \in [0,1]} \text{eff}(c,\gamma)$

Finite variance: Efficient weights

- $nV_U(w(c^*)) = rV_R(w(c^*)) \iff$
  $n\sum_{i=1}^r w_i^2(c^*)/n_i = r\sum_{i=1}^r w_i^2(c^*)$

- "**Middle variance**" weights:

$$w_i^{(3)} = \frac{n_i(n-n_i)}{n^2 - \sum_{s=1}^m n_s^2}, \quad i = 1, ..., r$$

$$\underbrace{V_R[Z(w^{(1)})]}_{\textit{Diversified indices}} \leq V_R[Z_{w^{(3)}}] \leq \underbrace{V_R[Z_{w^{(2)}}]}_{\textit{Diversified risks}}$$

$$\underbrace{V_U[Z(w^{(1)})]}_{\textit{Diversified indices}} \geq V_U[Z_{w^{(3)}}] \geq \underbrace{V_U[Z_{w^{(2)}}]}_{\textit{Diversified risks}}$$

When heavy-tailedness helps: VaR for financial indices

- $n_1 \geq ... \geq n_r \geq 1$, $\sum_{i=1}^{n_r} n_i = n$
- r **equally weighted indices** of $Y_{ij} = R_i + U_{ij}, \;\; j = 1, ..., n_i$
- $Z_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}$, $Z(w) = \sum_{i=1}^{r} w_i Z_i$
- $R_i \sim \underline{\mathcal{CS}} : \alpha < 1$, **extremely heavy-tailed**

  $U_{ij} \sim \overline{\mathcal{CS}} : \alpha > 1$, **moderately heavy-tailed**
- $w_i(c) = \frac{n_i[(n_i-1)c+1]^{-1}}{\sum_{i=1}^{r} n_i[(n_i-1)c+1]^{-1}}, \;\; 0 \leq c \leq 1$

  Minimal complete class, $w_i(c^*)$: **maximin efficiency**
- $VaR_q[Z(w(c))]$ is **increasing** in $c \in [0, 1]$

When heavy-tailedness helps

- $w(1) = w^{(1)} = \underline{w}_r$ (**Diversification** of **indices**, Z_i)
- $w(0) = w^{(2)}$ (**Diversification** of risks Y_{ij})
- $\underbrace{VaR_q[Z(w^{(1)})]}_{\textit{Diversified } \textbf{indices}} \geq VaR_q[Z(w(c))] \geq \underbrace{VaR_q[Z(w^{(2)})]}_{\textit{Diversified } \textbf{risks}}$
  - $\forall c \in [0,1]$; Holds for $c = c^*$, "optimal" **maximin efficiency**
- $\underbrace{VaR_q[Z(w^{(1)})]}_{\textit{Diversified } \textbf{indices}} \geq VaR_q[Z(w^{(3)})] \geq \underbrace{VaR_q[Z(w^{(2)})]}_{\textit{Diversified } \textbf{risks}}$
- **Diversification** on the level of **individual risks** Y_{ij} is **optimal**

When heavy-tailedness helps

- $R_i \sim \overline{\mathcal{CS}} : \alpha > 1$, **moderately heavy-tailed**

  $U_{ij} \sim \underline{\mathcal{CS}} : \alpha < 1$, **extremely heavy-tailed**

- $VaR_q[Z(w(c))]$ is **decreasing** in $c \in [0, 1]$

- $\underbrace{VaR_q[Z(w^{(1)})]}_{\textit{Diversified } \textbf{indices}} \leq VaR_q[Z(w(c))] \leq \underbrace{VaR_q[Z(w^{(2)})]}_{\textit{Diversified } \textbf{risks}}$

  - $w(1) = w^{(1)} = \underline{w}_r$, $w(0) = w^{(2)}$
  - $\forall c \in [0, 1]$; Holds for $c = c^*$, "optimal" **maximin efficiency**

- $\underbrace{VaR_q[Z(w^{(1)})]}_{\textit{Diversified } \textbf{indices}} \leq VaR_q[Z(w^{(3)})] \leq \underbrace{VaR_q[Z(w^{(2)})]}_{\textit{Diversified } \textbf{risks}}$

- **Diversification** of **indices** $Z_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}$ is **optimal**

Majorization & component VaR

Index return decomposition

- $Z(w) = \underbrace{R(w)}_{\text{common shock portfolio}} + \underbrace{U(\tilde{w})}_{\text{idiosyncratic risk portfolio}}$

  $R(w) = \sum_{i=1}^{n} w_i R_i$

  $U(\tilde{w}) = \sum_{i=1}^{r} \sum_{j=1}^{n_i} \tilde{w}_{ij} U_{ij}$

  - $\tilde{w} = (\tilde{w}_{11}, ..., \tilde{w}_{1n_1}, ..., \tilde{w}_{r1}, ..., \tilde{w}_{rn_r}) = (\frac{w_1}{n_1} e_1, ..., \frac{w_r}{n_r} e_r)$
  - $\tilde{w}_{ij} = w_i / n_i, \; j = 1, ..., n_i$
  - $e = \underbrace{(1, ..., 1)}_{N} \in \mathbf{R}^N, \; N \geq 1$: vectors of ones

Majorization & component VaR

- Weights at **common shocks** R_i :

$$\underbrace{w^{(1)}}_{\textit{Diversified indices}} \prec w(c') \prec w(c) \prec \underbrace{w^{(2)}}_{\textit{Diversified risks}}, \quad \forall 0 \leq c < c' \leq 1$$

$$\underbrace{w^{(1)}}_{\textit{Diversified indices}} \prec w^{(3)} \prec w(c) \prec \underbrace{w^{(2)}}_{\textit{Diversified risks}}$$

- Weights at **idiosyncratic risks** U_{ij} :

$$\underbrace{\tilde{w}^{(2)}}_{\textit{Diversified risks}} \prec \tilde{w}(c) \prec \tilde{w}(c') \prec \underbrace{\tilde{w}^{(1)}}_{\textit{Diversified indices}}, \quad \forall 0 \leq c < c' \leq 1$$

$$\underbrace{\tilde{w}^{(2)}}_{\textit{Diversified risks}} \prec \tilde{w}^{(3)} \prec \underbrace{\tilde{w}^{(1)}}_{\textit{Diversified indices}}$$

Majorization & component VaR

- $VaR_R(w)$: **Schur-convex** in w if $R_i \sim \overline{\mathcal{CS}}$, $\alpha > 1$

  $w \prec v \Longrightarrow VaR_R(w) \leq VaR_R(v)$

- $VaR_U(w)$: **Schur-convex** in $\tilde{w}$ if $U_{ij} \sim \underline{\mathcal{CS}}$, $\alpha < 1$

  $\tilde{w} \prec \tilde{v} \Longrightarrow VaR_U(\tilde{w}) \leq VaR_U(\tilde{v})$

- $VaR_R(w)$: **Schur-concave** in w if $R_i \sim \underline{\mathcal{CS}}$, $\alpha < 1$

  $w \prec v \Longrightarrow VaR_R(v) \leq VaR_R(w)$

- $VaR_U(w)$: **Schur-concave** in $\tilde{w}$ if $U_{ij} \sim \overline{\mathcal{CS}}$, $\alpha > 1$

  $\tilde{w} \prec \tilde{v} \Longrightarrow VaR_U(\tilde{v}) \leq VaR_U(\tilde{w})$

Majorization & component VaR

- $VaR_R(w)$: $R_i \sim \overline{\mathcal{CS}}$, $\alpha > 1$, $\forall 0 \leq c < c' \leq 1$

$$\underbrace{VaR_R(w^{(1)})}_{\text{Diversified indices}} \leq VaR(w(c')) \leq VaR(w(c)) \leq \underbrace{VaR(w^{(2)})}_{\text{Diversified risks}}$$

$$\underbrace{VaR(w^{(1)})}_{\text{Diversified indices}} \leq VaR(w^{(3)}) \leq VaR(w(c)) \leq \underbrace{VaR(w^{(2)})}_{\text{Diversified risks}}$$

- $VaR_U(\tilde{w})$: $U_{ij} \sim \underline{\mathcal{CS}}$, $\alpha < 1$, $\forall 0 \leq c < c' \leq 1$

$$\underbrace{VaR_R(\tilde{w}^{(1)})}_{\text{Diversified indices}} \leq VaR(\tilde{w}(c')) \leq VaR(\tilde{w}(c)) \leq \underbrace{VaR(\tilde{w}^{(2)})}_{\text{Diversified risks}}$$

$$\underbrace{VaR(\tilde{w}^{(1)})}_{\text{Diversified indices}} \leq VaR(\tilde{w}^{(3)}) \leq VaR(\tilde{w}(c)) \leq \underbrace{VaR(\tilde{w}^{(2)})}_{\text{Diversified risks}}$$

Finite variance: Suboptimal diversification

- **Variance** comparisons: **Suboptimal diversification**

$$Var[Z(w)] = \underbrace{Var[R(w)]}_{\text{common shock}} + \underbrace{Var[U(\tilde{w})]}_{\text{idiosyncratic}} = V_R(w) + V_U(w)$$

- $\underbrace{V_R(w^{(1)})}_{\textit{Diversified indices}} \leq \underbrace{V_R(w^{(3)})}_{\textit{Midlle variance}} \leq \underbrace{V_R(w^{(2)})}_{\textit{Diversified risks}}$ **but**

  $\underbrace{V_U(w^{(1)})}_{\textit{Diversified indices}} \geq \underbrace{V_R(w^{(3)})}_{\textit{Midlle variance}} \geq \underbrace{V_U(w^{(2)})}_{\textit{Diversified risks}}$

  Opposite for common shock and idiosyncratic parts

Extremely heavy tails in both common shock & idiosyncratic parts

- $Z(w) = \underbrace{R(w)}_{\text{common shock portfolio}} + \underbrace{U(\tilde{w})}_{\text{idiosyncratic risk portfolio}}$
- $R_i \sim \underline{\mathcal{CS}} : \alpha < 1$, **extremely heavy-tailed**

  $U_{ij} \sim \underline{\mathcal{CS}} : \alpha < 1$, **extremely heavy-tailed**
- $\underbrace{VaR_R(w^{(1)})}_{\text{Diversified indices}} \geq \underbrace{VaR_R(w^{(3)})}_{\text{Middle VaR}} \geq \underbrace{VaR_R(w^{(2)})}_{\text{Diversified risks}}$ **but**

  $\underbrace{VaR_U(w^{(1)})}_{\text{Diversified indices}} \leq \underbrace{VaR_R(w^{(3)})}_{\text{Middle VaR}} \leq \underbrace{V_U(w^{(2)})}_{\text{Diversified risks}}$

  Opposite for common shock and idiosyncratic parts
- **Suboptimal diversification**

Similar results

- **Indices**, **both row** & **column** common shock, **unbalanced**
- $Y_{ij} = R_i + C_j + U_{ij}, \;\; i = 1, ..., r, j = 1, ..., c$
- $n_{ij} \in \{0, 1\}$, $i = 1, ..., r$, $j = 1, ..., c$: indicators

  $n_{i0} = \sum_{j=1}^{c} n_{ij}$, $n_{0j} = \sum_{i=1}^{r} n_{ij}$, $n = \sum_{i=1}^{r} \sum_{j=1}^{c} n_{ij}$
- Equal weights $\underline{w}_{ij} = 1/(rc)$ (**Diversification** of risks)

  $\tilde{v}_{ij} = n_{i0}/(nc)$ (Portfolios of **column indices**)

  $\tilde{\tilde{v}}_{ij} = n_{0j}/(nr)$ (**Portfolios** of **row indices**)

  $\tilde{w}_{ij} = \frac{(n - n_{i0} - n_{0j} + n_{ij})n_{ij}}{n^2 - \sum_{i=1}^{r} n_{i0}^2 - \sum_{j=1}^{c} n_{0j}^2 + n}$ (Unbalanced models)
- **Variance** comparisons: **VaR** for **moderately heavy-tailed**
- **Reversals** for **extremely heavy-tailed**

Extensions: Multiple additive and multiplicative common shocks

- Same results:
- $Y_{ij} = R'_i + C'_j + U'_{ij}, \;\; i = 1, ..., r, \;\; j = 1, ..., c$

  $R'_i = \sum_{s=1}^{m_1} F_s R_{is}, \; C'_j = \sum_{s=1}^{m_2} G_s C_{js}, \; U'_{ij} = \sum_{s=1}^{m_3} H_s U_{ijs}$

  $F_s, G_s, H_s > 0$
- Y_{ij} : two **additive** common shocks R and C

  $m_1 + m_2 + m_3$ **multiplicative** common shocks F, G and H
- $R'_i, \; C'_j s, \; U'_{ij}$: **heavy-tailedness** and **dependence**
  - FR, GC, HU with $R, C, U \sim S_\alpha(\sigma, 0, 0)$, $\alpha < 1$: **extremely heavy-tailed**, infinite means
  - FR, GC, HU with $R, C, U \sim S_\alpha(\sigma, 0, 0)$, $\alpha > 1$: marginals with **power moments finite** up to a certain positive order

Multiple additive and multiplicative common shocks, efficiency

- $Y_{i_1,i_2,\dots,i_m} = \sum_{s=1}^{m} \sum_{1 \le j_1 < \dots < j_s \le m} U^{(j_1,\dots,j_s)}_{i_{j_1},\dots,i_{j_s}},\ 1 \le i_s \le N_s$
- **Efficiency** of **linear estimators** in **random effects** models
- $Y_{ij} = \mu + R_i + U_{ij},\ \ j = 1, \dots, n_i,\ \ i = 1, \dots, r$
- $Z(w) = \sum_{i=1}^{r} w_i Z_i,\ Z_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}$: Linear estimators
- $Z(w)$: **more efficient** than $Z(v)$ in the sense of **p**eakedness

  if $P\big(|Z(w) - \mu| > \epsilon\big) \le P\big(|Z(v) - \mu| > \epsilon\big)\ \forall \epsilon > 0$
- **Portfolio VaR** $\Longleftrightarrow$ **Efficiency** comparisons for $Z(w)$
- **Extreme heavy-tailedness**: $p-$efficient estimators

  Contrast with **variance** comparisons

Main contributions

- **Diversification** and **portfolio VaR** for **heavy-tailed** and **dependent** risks
  - Models with multiple **common shocks**
- **Portfolio VaR** in **balanced** models:

  All risks are available for portfolio formation
  - **Diversification** is **optimal** for **moderately** heavy-tailed dependent risks

    $\alpha > 1$ for **both** the **common shock** and **idiosyncratic** parts of risks
  - **Inferior** for **extremely** heavy-tailed dependent risks

    $\alpha < 1$ for **both** the **common shock** and **idiosyncratic** parts of risks

Main contributions

- **Heavy-tailedness** may help **diversification**
- **Diversification** in **unbalanced** models with **common shocks**:

  Optimal even though there is **extreme heavy-tailedness**

  Portfolios of indices of **heavy-tailed dependent** risks:

  - $\alpha < 1$ in **common shocks**, $\alpha > 1$ in **idiosyncratic** parts: **diversification** on the **level** of **individual** risks is optimal
  - $\alpha > 1$ in **common shocks**, $\alpha < 1$ in **idiosyncratic** parts: **diversification** on the **level** of **indices** is optimal

- **Contrast to variance** comparisons: **suboptimal**
- Similar analysis: **efficiency comparisons** of **linear estimators** in **random effects** location models