

Exact Estimates for Moments of Random Bilinear Forms

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Received March 4, 1999; revised September 17, 1999

The present paper concentrates on the analogues of Rosenthal's inequalities for ordinary and decoupled bilinear forms in symmetric random variables. More specifically, we prove the exact moment inequalities for these objects in terms of moments of their individual components. As a corollary of these results we obtain the explicit expressions for the best constant in the analogues of Rosenthal's inequality for ordinary and decoupled bilinear forms in identically distributed symmetric random variables in the case of the fixed number of random variables.

KEY WORDS: Random bilinear forms; moment inequalities; decoupling; symmetric statistics.

1. INTRODUCTION

In recent years, several studies have focused on moment and probability inequalities for multilinear forms and symmetric statistics (see, in particular, Serfling (1980), Krakowiak and Szulga (1986), McConell and Taqqu (1986), de la Peña (1992), de la Peña and Klass (1994), Koroljuk and Borovskikh (1994), de la Peña and Montgomery-Smith (1995),

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Sharakhmetov (1995, 1997), Ibragimov and Sharakhmetov (1996a, 1998a, 1999), Borovskikh and Korolyuk (1997), Ibragimov (1997), Klass and Nowicki (1997a, b) and Giné *et al.* (2000)). Interest in such inequalities is motivated by their applications in limit theorems, multiple stochastic integration, harmonic analysis, operator theory, quantum mechanics, theory of income inequality and species' diversity measurement, etc. (see, in addition to the abovementioned papers, Bonami (1970), Rosiński and Szulga (1982), Sjörgen (1982), Rosiński and Woyczyński (1984, 1986), Cambanis *et al.* (1985) and Kwapien and Woyczyński (1992)). Furthermore, the bounds on moments for symmetric statistics can also be applied in investment theory and in testing for chaos in time series data based on the notion of correlation integral, which has the form of symmetric statistics (see Cecen and Erkal (1996a, b)).

In the case of linear statistics (sums of independent random variables (r.v.'s)), the exact moment estimates are given by the well-known Khintchine, Marcinkiewicz–Zygmund and Rosenthal inequalities (see Khintchine (1923), Marcinkiewicz and Zygmund (1937), Rosenthal (1970)). Let us remind the latter ones ($A_i(\cdot)$, $B_i(\cdot)$ denote constants depending on parameters in parentheses only).

Theorem 1. If ξ_1, \dots, ξ_n are independent mean zero r.v.'s with finite t th moment, $2 < t < \infty$, then

$$\begin{aligned} A_1(t) \max \left(\sum_{i=1}^n E |\xi_i|^t, \left(\sum_{i=1}^n E \xi_i^2 \right)^{t/2} \right) \\ \leq E \left| \sum_{i=1}^n \xi_i \right|^t \\ \leq B_1(t) \max \left(\sum_{i=1}^n E |\xi_i|^t, \left(\sum_{i=1}^n E \xi_i^2 \right)^{t/2} \right) \end{aligned} \quad (1)$$

The exact upper constants in inequality (1) (case $t = 2m$) and in its analogue for nonnegative r.v.'s were found in Ibragimov and Sharakhmetov (1996b, 1998a, b). The best constant in inequality (1) for symmetric r.v.'s was independently found by Figiel *et al.* (1997) and Ibragimov and Sharakhmetov (1995, 1997). The results obtained by Ibragimov and Sharakhmetov (1996b, 1997, 1998a, b) and their proofs were presented in Ibragimov (1997). Concerning refinements and extensions of Rosenthal's inequalities and related problems see also Prokhorov (1962), Nagaev and

Pinelis (1977), Pinelis (1980, 1994), Pinelis and Utev (1984), Johnson *et al.* (1985), Utev (1985), Talagrand (1989), Hitczenko (1990, 1994), Nagaev (1990, 1998), Kwapien and Szulga (1991) and Peshkir and Shiryaev (1995).

Recently, Sharakhmetov (1995, 1997), Ibragimov and Sharakhmetov (1996a, 1998a, 1999, 2000) (see also Ibragimov (1997)), Klass and Nowicki (1997a, b) and Giné *et al.* (2000) obtained analogues of Rosenthal's inequality (1) and its analogue for nonnegative r.v.'s in the case of symmetric statistics. Ibragimov and Sharakhmetov (2000) also showed the significance of each term in the analogues of Rosenthal's bounds for symmetric statistics of arbitrary order. Ibragimov (1997) showed that the best constants in the analogues of Rosenthal's inequalities for L_t -norms of symmetric statistics grow not slower than $(t/\ln t)^m$, as $t \rightarrow \infty$, where m is the order of a symmetric statistic. Giné *et al.* (2000) showed that the actual rate of growth of the above constants is $(t/\ln t)^m$.

The qualitative difference of the results on Rosenthal's inequalities for nonlinear statistics from the linear case is the exact constants in them are unknown yet. The main goal of the present paper is to fill partially this gap in the case of bilinear forms. More specifically, we obtain the explicit expressions for the best constant in the analogues of Rosenthal's inequalities for ordinary and decoupled bilinear forms in identically distributed symmetric r.v.'s in the case of fixed number of r.v.'s. The proof of the expressions for the best constants in the non-linear analogues of Rosenthal inequalities is based on a theorem, which extends the extremal results obtained in Utev (1985) and Ibragimov and Sharakhmetov (1996b, 1997) in the case of bilinear forms and gives the exact estimates for moments of random bilinear forms in terms of moment characteristics of their particular components. To our knowledge, this theorem and its proof are the first attempt to apply methods which were used to investigate the extremal problems in moment inequalities for sums of independent r.v.'s for non-linear statistics. The results obtained in the present paper can be extended to the case of nonnegative random variables, multilinear forms of arbitrary order and generalized moments; these extensions will be presented elsewhere.

2. MAIN RESULTS

Let $t > 2$, $X_1, Y_1, X_2, Y_2, \dots, X_n, Y_n$ be independent symmetric r.v.'s with finite t th moment. Let $a_i \geq 0$, $b_i \geq 0$, $c_i \geq 0$, $d_i \geq 0$, $a_i^t \leq b_i$, $c_i^t \leq d_i$, $i = 1, \dots, n$. Set

$$(X, n) = (X_1, \dots, X_n), \quad (Y, n) = (Y_1, \dots, Y_n)$$

$$M_1(n, a, b) = \{(X, n) : EX_i^2 = a_i^2, E|X_i|^t = b_i, i = 1, \dots, n\}$$

$$M_1(n, c, d) = \{(Y, n) : EY_i^2 = c_i^2, E|Y_i|^t = d_i, i = 1, \dots, n\}$$

$$M_2(n, a, b) = \{(X, n) : EX_i^2 \leq a_i^2, E|X_i|^t \leq b_i, i = 1, \dots, n\}$$

$$M_2(n, c, d) = \{(Y, n) : EY_i^2 \leq c_i^2, E|Y_i|^t \leq d_i, i = 1, \dots, n\}$$

Let $U_i(a_i, b_i, t)$, $V_i(c_i, d_i, t)$, $i = 1, \dots, n$, be independent r.v.'s such that

$$P(U_i(a_i, b_i, t) = 0) = 1 - (a_i^t/b_i)^{2/(t-2)}$$

$$P(U_i(a_i, b_i, t) = \pm (b_i/a_i^2)^{1/(t-2)}) = (1/2)(a_i^t/b_i)^{2/(t-2)}$$

$$P(V_i(c_i, d_i, t) = 0) = 1 - (c_i^t/d_i)^{2/(t-2)}$$

$$P(V_i(c_i, d_i, t) = \pm (d_i/c_i^2)^{1/(t-2)}) = (1/2)(c_i^t/d_i)^{2/(t-2)}$$

and let U_i , V_i , $i = 1, \dots, n$, be independent r.v.'s with distribution

$$P(U_i = \pm 1) = P(V_i = \pm 1) = 1/2, \quad i = 1, \dots, n$$

The following theorem extends the results obtained in Utev (1985) and Ibragimov and Sharakhmetov (1997) on the nonlinear case and gives the explicit bounds for moments of random bilinear forms in terms of moment characteristics of their particular components.

Theorem 4. If $2 < t < 4$, then

$$\begin{aligned} & \sup_{(X, n) \in M_k(n, a, b)} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \\ &= \sum_{1 \leq i < j \leq n} (b_i - a_i^t)(b_j - a_j^t) \\ &+ \sum_{i=1}^n (b_i - a_i^t) E \left| \sum_{j \neq i}^n a_j U_j \right|^t + E \left| \sum_{1 \leq i < j \leq n} a_i a_j U_i U_j \right|^t \end{aligned} \quad (2)$$

$$\begin{aligned} & \sup_{(X, n) \in M_k(n, a, b), (Y, n) \in M_l(n, c, d)} E \left| \sum_{1 \leq i, j \leq n, i \neq j} X_i Y_j \right|^t \\ &= \sum_{1 \leq i, j \leq n, i \neq j} (b_i - a_i^t)(d_j - c_j^t) \\ &+ \sum_{j=1}^n (d_j - c_j^t) E \left| \sum_{i=1, i \neq j}^n a_i U_i \right|^t + \sum_{i=1}^n (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^n c_j V_j \right|^t \\ &+ E \left| \sum_{1 \leq i, j \leq n, i \neq j} a_i c_j U_i V_j \right|^t, \quad k, l = 1, 2 \end{aligned} \quad (3)$$

If $3 \leq t < 4$, then

$$\begin{aligned} & \inf_{(X, n) \in \mathcal{M}_1(n, a, b)} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \\ &= E \left| \sum_{1 \leq i < j \leq n} U_i(a_i, b_i, t) U_j(a_j, b_j, t) \right|^t \end{aligned} \quad (4)$$

$$\begin{aligned} & \inf_{(X, n) \in \mathcal{M}_1(n, a, b), (Y, n) \in \mathcal{M}_1(n, c, d)} E \left| \sum_{1 \leq i, j \leq n, i \neq j} X_i Y_j \right|^t \\ &= E \left| \sum_{1 \leq i, j \leq n, i \neq j} U_i(a_i, b_i, t) V_j(c_j, d_j, t) \right|^t \end{aligned} \quad (5)$$

If $t \geq 4$, then

$$\begin{aligned} & \sup_{(X, n) \in \mathcal{M}_k(n, a, b)} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \\ &= E \left| \sum_{1 \leq i < j \leq n} U_i(a_i, b_i, t) U_j(a_j, b_j, t) \right|^t \end{aligned} \quad (6)$$

$$\begin{aligned} & \sup_{(X, n) \in \mathcal{M}_k(n, a, b), (Y, n) \in \mathcal{M}_l(n, c, d)} E \left| \sum_{1 \leq i, j \leq n, i \neq j} X_i Y_j \right|^t \\ &= E \left| \sum_{1 \leq i, j \leq n, i \neq j} U_i(a_i, b_i, t) V_j(c_j, d_j, t) \right|^t, \quad k, l = 1, 2 \end{aligned} \quad (7)$$

$$\begin{aligned} & \inf_{(X, n) \in \mathcal{M}_1(n, a, b)} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \\ &= \sum_{1 \leq i < j \leq n} (b_i - a_i^t)(b_j - a_j^t) \\ &+ \sum_{i=1}^n (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^n a_j U_j \right|^t + E \left| \sum_{1 \leq i < j \leq n} a_i a_j U_i U_j \right|^t \end{aligned} \quad (8)$$

$$\begin{aligned} & \inf_{(X, n) \in \mathcal{M}_1(n, a, b), (Y, n) \in \mathcal{M}_1(n, c, d)} E \left| \sum_{1 \leq i, j \leq n, i \neq j} X_i Y_j \right|^t \\ &= \sum_{1 \leq i, j \leq n, i \neq j} (b_i - a_i^t)(d_j - c_j^t) \\ &+ \sum_{j=1}^n (d_j - c_j^t) E \left| \sum_{i=1, i \neq j}^n a_i U_i \right|^t + \sum_{i=1}^n (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^n c_j V_j \right|^t \\ &+ E \left| \sum_{1 \leq i, j \leq n, i \neq j} a_i c_j U_i V_j \right|^t \end{aligned} \quad (9)$$

Remark. The expressions in relations (2)–(9) are of a simple structure and their values can be easily calculated for given sequences $a_i, b_i, c_i, d_i, i = 1, \dots, n$.

Let us fix $t > 2$ and $n \geq 1$. From the results obtained in Ibragimov and Sharakhmetov (1999) and decoupling theorems for symmetric statistics (see McConell and Taquq (1986) and de la Peña and Montgomery–Smith (1995)) it follows that for all independent identically distributed symmetric r.v.'s $X_1, \dots, X_n, \bar{X}_1, \dots, \bar{X}_n$ with finite t th moment the following Rosenthal-type inequalities are true ($C_n^2 = n(n-1)/2$):

$$E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \leq B_4(t, n) \max(C_n^2 (E |X_1|^t)^2, (C_n^2)^{t/2} (EX_1^2)^t) \quad (10)$$

$$E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \leq B_5(t, n) \max(n^2 (E |X_1|^t)^2, n^t (EX_1^2)^t) \quad (11)$$

$$E \left| \sum_{1 \leq i \neq j \leq n} X_i \bar{X}_j \right|^t \leq B_6(t, n) \max(C_n^2 (E |X_1|^t)^2, (C_n^2)^{t/2} (EX_1^2)^t) \quad (12)$$

$$E \left| \sum_{1 \leq i \neq j \leq n} X_i \bar{X}_j \right|^t \leq B_7(t, n) \max(n^2 (E |X_1|^t)^2, n^t (EX_1^2)^t) \quad (13)$$

The following theorems give the explicit expressions for the best constants in inequalities (10) and (11).

Theorem 5. The exact constant in inequality (10) is given by

$$\begin{aligned} B_4^*(t, n) &= C_n^2 (1/(C_n^2)^{1/2} - 1/(C_n^2)^{t/2})^2 \\ &\quad + (1/(C_n^2)^{1/2} - 1/(C_n^2)^{t/2}) n / (C_n^2)^{t/4} E \left| \sum_{i=2}^n U_i \right|^t \\ &\quad + E \left| \sum_{1 \leq i < j \leq n} U_i U_j / (C_n^2)^{1/2} \right|^t, \quad 2 < t < 4 \end{aligned} \quad (14)$$

$$\begin{aligned} B_4^*(t, n) &= E \left| \sum_{1 \leq i < j \leq n} U_i (1/(C_n^2)^{1/4}, 1/(C_n^2)^{1/2}, t) \right. \\ &\quad \left. \times U_j (1/(C_n^2)^{1/4}, 1/(C_n^2)^{1/2}, t) \right|^t, \quad t \geq 4 \end{aligned} \quad (15)$$

Theorem 6. The exact constant in inequality (11) is given by

$$B_5^*(t, n) = C_n^2(1/n - 1/n^t)^2 + (1/n^{t/2} - 1/n^{3t/2-1}) E \left| \sum_{i=2}^n U_i \right|^t \\ + E \left| \sum_{1 \leq i < j \leq n} U_i U_j / n \right|^t, \quad 2 < t < 4 \quad (16)$$

$$B_5^*(t, n) = E \left| \sum_{1 \leq i < j \leq n} U_i(1/n^{1/2}, 1/n, t) U_j(1/n^{1/2}, 1/n, t) \right|^t, \quad t \geq 4 \quad (17)$$

Theorems 7 and 8 below give the explicit expressions for the exact constants in inequalities (12) and (13).

Theorem 7. The exact constant in inequality (12) is given by

$$B_6^*(t, n) = 2C_n^2(1/(C_n^2)^{1/2} - 1/(C_n^2)^{t/2})^2 \\ + 2(1/(C_n^2)^{1/2} - 1/(C_n^2)^{t/2}) n/(C_n^2)^{t/4} E \left| \sum_{i=2}^n U_i \right|^t \\ + E \left| \sum_{1 \leq i, j \leq n, i \neq j} U_i V_j / (C_n^2)^{1/2} \right|^t, \quad 2 < t < 4 \quad (18)$$

$$B_6^*(t, n) = E \left| \sum_{1 \leq i \neq j \leq n} U_i(1/(C_n^2)^{1/4}, 1/(C_n^2)^{1/2}, t) \right. \\ \left. \times V_j(1/(C_n^2)^{1/4}, 1/(C_n^2)^{1/2}, t) \right|^t, \quad t \geq 4 \quad (19)$$

Theorem 8. The exact constant in inequality (13) is given by

$$B_7^*(t, n) = 2C_n^2(1/n - 1/n^t)^2 + 2(1/n^{t/2} - 1/n^{3t/2-1}) E \left| \sum_{i=2}^n U_i \right|^t \\ + E \left| \sum_{i \leq i, j \leq n, i \neq j} U_i V_j / n \right|^t, \quad 2 < t < 4 \quad (20)$$

$$B_7^*(t, n) = E \left| \sum_{1 \leq i, j \leq n, i \neq j} U_i(1/n^{1/2}, 1/n, t) V_j(1/n^{1/2}, 1/n, t) \right|^t, \\ t \geq 4 \quad (21)$$

3. PRELIMINARIES

Let us formulate some auxiliary steps needed for the proof of the theorems.

Lemma 1. If $2 < t < 4$, $z_1 \geq 0$, $z_2 \in \mathbf{R}$, $a \geq 0$, $b \geq 0$, $a^t \leq b$, X is a symmetric r.v. with $EX^2 \leq a^2$, $E|X|^t \leq b$, then

$$E|z_1 X + z_2|^t - bz_1^t \leq E|az_1 U + z_2|^t - a^t z_1^t \quad (22)$$

Proof. It suffices to consider the case $z_1 \neq 0$. From Lemma 5 in Ibragimov and Sharakhmetov (1997) it follows that

$$E|X + z_2/z_1|^t - b \leq E|aU + z_2/z_1|^t - a^t \quad (23)$$

Multiplying (23) by z_1^t we obtain (22). \square

Applying Lemma 7 in Ibragimov and Sharakhmetov (1997) and Lemmas 7.3 and 7.4 in Utev (1985) analogously to the proof of Lemma 1 above we easily obtain the following Lemmas 2–4.

Lemma 2. If $3 \leq t < 4$, $z_1, z_2 \in \mathbf{R}$, $a \geq 0$, $b \geq 0$, $a^t \leq b$, X is a symmetric r.v. with $EX^2 = a^2$, $E|X|^t = b$, then

$$E|z_1 X + z_2|^t \geq E|z_1 U(a, b, t) + z_2|^t$$

Lemma 3. If $t \geq 4$, $z_1, z_2 \in \mathbf{R}$, $a \geq 0$, $b \geq 0$, $a^t \leq b$, X is a symmetric r.v. with $EX^2 \leq a^2$, $E|X|^t \leq b$, then

$$E|z_1 X + z_2|^t \leq E|z_1 U(a, b, t) + z_2|^t$$

Lemma 4. If $t \geq 4$, $z_1 \geq 0$, $z_2 \in \mathbf{R}$, $a \geq 0$, $b \geq 0$, $a^t \leq b$, X is a symmetric r.v. with $EX^2 = a^2$, $E|X|^t = b$, then

$$E|z_1 X + z_2|^t - bz_1^t \geq E|az_1 U + z_2|^t - a^t z_1^t$$

Lemma 5. Let $1 \leq k \leq n$, X_1, \dots, X_{k-1} , U_k , X_{k+1}, \dots, X_n be independent r.v.'s with $E|X_i|^t < \infty$, $i = 1, \dots, n$, $i \neq k$, $a_k, b_k \geq 0$, $a_k^t \leq b_k$, $c_i \in \mathbf{R}$, $i = 1, \dots, k-1$, and let F_1 be the set of symmetric r.v.'s X_k independent of X_1, \dots, X_{k-1} , X_{k+1}, \dots, X_n and satisfying the conditions $EX_k^2 \leq a_k^2$, $E|X_k|^t \leq b_k$, F_2 be the subset of F_1 consisting of r.v.'s X_k such that $EX_k^2 = a_k^2$, $E|X_k|^t = b_k$. If $2 < t < 4$, then

$$\begin{aligned}
& \sup_{X_k \in F_l} \left(\sum_{i=1}^{k-1} c_i E \left| \sum_{j=1, j \neq i}^n X_j \right|^t + E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \right) \\
&= \sum_{i=1}^{k-1} c_i E \left| a_k U_k + \sum_{j=1, j \neq i, k}^n X_j \right|^t + \sum_{i=1}^{k-1} c_i (b_k - a_k^t) \\
&\quad + (b_k - a_k^t) E \left| \sum_{j=1, j \neq k}^n X_j \right|^t \\
&\quad + E \left| a_k U_k \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t, \quad l = 1, 2
\end{aligned}$$

If $t \geq 4$, then

$$\begin{aligned}
& \inf_{X_k \in F_2} \left(\sum_{i=1}^{k-1} c_i E \left| \sum_{j=1, j \neq i}^n X_j \right|^t + E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \right) \\
&= \sum_{i=1}^{k-1} c_i E \left| a_k U_k + \sum_{j=1, j \neq i, k}^n X_j \right|^t + \sum_{i=1}^{k-1} c_i (b_k - a_k^t) \\
&\quad + (b_k - a_k^t) E \left| \sum_{j=1, j \neq k}^n X_j \right|^t \\
&\quad + E \left| a_k U_k \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t
\end{aligned}$$

Proof. From Lemmas 1 and 4 above and Lemma 5 in Ibragimov and Sharakhmetov (1997) it follows that it suffices to find a sequence of r.v.'s X_{mk} , $m = 1, 2, \dots$, independent of $X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n$ and satisfying the conditions $EX_{mk}^2 = a_k^2$, $E|X_{mk}|^t = b_k$,

$$\begin{aligned}
& \lim_{m \rightarrow \infty} E \left| X_{mk} + \sum_{j=1, j \neq i, k}^n X_j \right|^t \\
&= E \left| a_k U_k + \sum_{j=1, j \neq i, k}^n X_j \right|^t + b_k - a_k^t, \quad i = 1, \dots, k-1 \quad (24)
\end{aligned}$$

$$\begin{aligned}
& \lim_{m \rightarrow \infty} E \left| X_{mk} \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t \\
&= (b_k - a_k^t) E \left| \sum_{j=1, j \neq k}^n X_j \right|^t \\
&\quad + E \left| a_k U_k \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t \quad (25)
\end{aligned}$$

If $b_k = a_k^t$, then one can take $X_{mk} = a_k u_k$. Let $a_k^t < b_k$. Set $\delta_m = 1/m$,

$$P(X_{mk} = \pm a_k) = 1/2(1 - \delta_m), \quad P(X_{mk} = \pm b_{mk}) = 1/2\delta_{mk}^*$$

$$\delta_{mk}^* = a_k^2 \delta_m / b_{mk}^2, \quad P(X_{mk} = 0) = \delta_m - \delta_{mk}^*$$

$$b_{mk} = ((b_k - a_k^t(1 - \delta_m)) / a_k^2 \delta_m)^{1/(t-2)}, \quad m = 1, 2, \dots$$

Then

$$b_{mk} \geq a_k, \quad 0 \leq \delta_{mk}^* \leq \delta_m,$$

$$EX_{mk}^2 = a_k^2, \quad E|X_{mk}|^t = b_k, \quad m = 1, 2, \dots \quad (26)$$

$$\delta_m \rightarrow 0, \quad b_{mk} \rightarrow \infty, \quad b_{mk}^t \delta_{mk}^* \rightarrow b_k - a_k^t, \quad m \rightarrow \infty$$

From (26) and the proof of Lemma 7.6 in Utev (1985) it follows that relations (24) are valid.

Let us prove that (25) is true. We have

$$\begin{aligned} & E \left| X_{mk} \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t \\ &= E \left| a_k U_k \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t (1 - \delta_m) \\ &+ E \left| \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t (\delta_m - \delta_{mk}^*) \\ &+ \left(E \left| b_{mk} U_k \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t \right. \\ &\left. - b_{mk}^t E \left| \sum_{j=1, j \neq k}^n X_j \right|^t \right) \delta_{mk}^* + b_{mk}^t \delta_{mk}^* E \left| \sum_{j=1, j \neq k}^n X_j \right|^t \end{aligned}$$

From (26) it follows that for the proof of (25) it suffices to check that

$$\begin{aligned} & \left(E \left| b_{mk} U_k \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j \right|^t \right. \\ &\left. - b_{mk}^t E \left| \sum_{j=1, j \neq k}^n X_j \right|^t \right) \delta_{mk}^* \rightarrow 0, \quad m \rightarrow \infty \end{aligned}$$

This follows from the fact that $b_{mk}^t \delta_{mk}^*$ converges and that, on the strength of the inequality $||x + y|^t - |x|^t| \leq 2^t t (|x|^{t-1} |y| + |y|^t)$, $x, y \in \mathbf{R}$, $t \geq 1$ (see Lemma 7.5 in Utev (1985)), and the dominated convergence principle,

$$\begin{aligned} \lim_{m \rightarrow \infty} E \left| U_k \left(\sum_{j=1, j \neq k}^n X_j \right) + \sum_{1 \leq i < j \leq n, i, j \neq k} X_i X_j / b_{mk} \right|^t \\ = E \left| U_k \left(\sum_{j=1, j \neq k}^n X_j \right) \right|^t = E \left| \sum_{j=1, j \neq k}^n X_j \right|^t \end{aligned}$$

Arguing analogously with the proof of Lemma 5, we easily obtain the following

Lemma 6. Let $1 \leq k \leq n$, $X_1, \dots, X_{k-1}, U_k, X_{k+1}, \dots, X_n, Y_1, \dots, Y_n$ be independent r.v.'s with $E |X_i|^t < \infty$, $i = 1, \dots, n$, $i \neq k$, $E |Y_i|^t < \infty$, $i = 1, \dots, n$, $a_k, b_k \geq 0$, $a_k^t \leq b_k$, $c_i \in \mathbf{R}$, $i = 1, \dots, k-1$, and let G_1 be the set of symmetric r.v.'s X_k independent of $X_1, \dots, X_{k-1}, X_{k+1}, \dots, X_n, Y_1, \dots, Y_n$ and satisfying the conditions $EX_k^2 \leq a_k^2$, $E |X_k|^t \leq b_k$, G_2 be the subset of G_1 consisting of r.v.'s X_k such that $EX_k^2 = a_k^2$, $E |X_k|^t = b_k$. If $2 < t < 4$, then

$$\begin{aligned} \sup_{X_k \in G_1} \left(\sum_{i=1}^n c_i E \left| \sum_{j=1, j \neq i}^n X_j \right|^t + E \left| \sum_{1 \leq i \neq j \leq n} X_i Y_j \right|^t \right) \\ = \sum_{i=1}^n c_i E \left| a_k U_k + \sum_{j=1, j \neq i, k}^n X_j \right|^t + \sum_{i=1}^n c_i (b_k - a_k^t) \\ + (b_k - a_k^t) E \left| \sum_{j=1, j \neq k}^n Y_j \right|^t \\ + E \left| a_k U_k \left(\sum_{j=1, j \neq k}^n Y_j \right) + \sum_{1 \leq i \neq j \leq n, i \neq k} X_i Y_j \right|^t, \quad l = 1, 2 \end{aligned}$$

If $t \geq 4$, then

$$\begin{aligned} \inf_{X_k \in G_2} \left(\sum_{i=1}^n c_i E \left| \sum_{j=1, j \neq i}^n X_j \right|^t + E \left| \sum_{1 \leq i \neq j \leq n} X_i Y_j \right|^t \right) \\ = \sum_{i=1}^n c_i E \left| a_k U_k + \sum_{j=1, j \neq i, k}^n X_j \right|^t + \sum_{i=1}^n c_i (b_k - a_k^t) \\ + (b_k - a_k^t) E \left| \sum_{j=1, j \neq k}^n Y_j \right|^t \\ + E \left| a_k U_k \left(\sum_{j=1, j \neq k}^n Y_j \right) + \sum_{1 \leq i \neq j \leq n, i \neq k} X_i Y_j \right|^t \end{aligned}$$

4. PROOFS OF THE THEOREMS

Proof of Theorem 3. Relations (4)–(7) easily follow from Lemmas 2 and 3 by induction. Let us prove (2). Let $2 < t < 4$, $1 \leq k \leq n$, $U_1, \dots, U_{k-1}, X_{k+1}, \dots, X_n$ be independent symmetric r.v.'s, $E|X_i|^t < \infty$, $i = k+1, \dots, n$, $a_i \geq 0$, $b_i \geq 0$, $a_i^t \leq b_i$, $i = 1, \dots, k$. Denote by H_1 the set of symmetric r.v.'s X_k independent of $U_1, \dots, U_{k-1}, X_{k+1}, \dots, X_n$ and satisfying the conditions $EX_k^2 \leq a_k^2$, $E|X_k|^t \leq b_k$, and by H_2 the subset of H_1 consisting of r.v.'s X_k such that $EX_k^2 = a_k^2$, $E|X_k|^t = b_k$. On the strength of Lemma 5 we have

$$\begin{aligned}
& \sup_{X_k \in H_1} \left(\sum_{1 \leq i < j \leq k-1} (b_i - a_i^t)(b_j - a_j^t) + \sum_{i=1}^{k-1} (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^{k-1} a_j U_j + \sum_{j=k}^n X_j \right|^t \right. \\
& \quad \left. + E \left| \sum_{i=1}^{k-1} a_i U_i \left(\sum_{j=i+1}^{k-1} a_j U_j + \sum_{j=k}^n X_j \right) + \sum_{i=k}^{n-1} X_i \left(\sum_{j=i+1}^n X_j \right) \right|^t \right) \\
& = \sum_{1 \leq i < j \leq k-1} (b_i - a_i^t)(b_j - a_j^t) \\
& \quad + \sum_{i=1}^{k-1} (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^k a_j U_j + \sum_{j=k+1}^n X_j \right|^t \\
& \quad + \sum_{i=1}^{k-1} (b_i - a_i^t)(b_k - a_k^t) + (b_k - a_k^t) E \left| \sum_{j=1}^{k-1} a_j U_j + \sum_{j=k+1}^n X_j \right|^t \\
& \quad + E \left| \sum_{i=1}^k a_i U_i \left(\sum_{j=i+1}^k a_j U_j + \sum_{j=k+1}^n X_j \right) + \sum_{i=k+1}^{n-1} X_i \left(\sum_{j=i+1}^n X_j \right) \right|^t \\
& = \sum_{1 \leq i < j \leq k} (b_i - a_i^t)(b_j - a_j^t) + \sum_{i=1}^k (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^k a_j U_j + \sum_{j=k+1}^n X_j \right|^t \\
& \quad + E \left| \sum_{i=1}^k a_i U_i \left(\sum_{j=i+1}^k a_j U_j + \sum_{j=k+1}^n X_j \right) \right. \\
& \quad \left. + \sum_{i=k+1}^{n-1} X_i \left(\sum_{j=i+1}^n X_j \right) \right|^t, \quad l = 1, 2 \tag{27}
\end{aligned}$$

Applying (27) n times we get (2).

Let us show that (3) is valid. Let $2 < t < 4$, $1 \leq k \leq n$, $U_1, \dots, U_{k-1}, X_{k+1}, \dots, X_n, Y_1, \dots, Y_n$ be independent symmetric r.v.'s, $E|X_i|^t < \infty$, $i = k+1, \dots, n$, $E|Y_i|^t < \infty$, $i = 1, \dots, n$, $a_i \geq 0$, $b_i \geq 0$, $a_i^t \leq b_i$, $i = 1, \dots, k$. Denote by K_1 the set of symmetric r.v.'s X_k independent of $U_1, \dots, U_{k-1}, X_{k+1}, \dots, X_n, Y_1, \dots, Y_n$ and satisfying the conditions $EX_k^2 \leq a_k^2$, $E|X_k|^t \leq b_k$, and by K_2 the subset of K_1 consisting of r.v.'s X_k such that $EX_k^2 = a_k^2$, $E|X_k|^t = b_k$. From Lemma 6 with $c_i = 0$, $i = 1, \dots, n$, it follows that

$$\begin{aligned} & \sup_{X_k \in K_l} \left(\sum_{i=1}^{k-1} (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^n Y_j \right|^t \right. \\ & \quad \left. + E \left| \sum_{i=1}^{k-1} a_i U_i \left(\sum_{j=1, j \neq i}^n Y_j \right) + \sum_{i=k}^n X_i \left(\sum_{j=1, j \neq i}^n Y_j \right) \right|^t \right) \\ & = \sum_{i=1}^k (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^n Y_j \right|^t + E \left| \sum_{i=1}^k a_i U_i \left(\sum_{j=1, j \neq i}^n Y_j \right) \right|^t \\ & \quad + \sum_{i=k+1}^n X_i \left(\sum_{j=1, j \neq i}^n Y_j \right) \Big|^t, \quad l=1, 2 \end{aligned} \quad (28)$$

Using (28) n times we obtain

$$\begin{aligned} & \sup_{(X, n) \in M_k(n, a, b)} E \left| \sum_{1 \leq i \neq j \leq n} X_i Y_j \right|^t \\ & = \sum_{i=1}^n (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^n Y_j \right|^t + E \left| \sum_{1 \leq i \neq j \leq n} a_i U_i Y_j \right|^t, \quad k=1, 2 \end{aligned} \quad (29)$$

Let $1 \leq k \leq n$, $U_1, \dots, U_n, V_1, \dots, V_{k-1}, Y_{k+1}, \dots, Y_n$ be independent symmetric r.v.'s, $E|Y_i|^t < \infty$, $i = k+1, \dots, n$, $a_i \geq 0$, $b_i \geq 0$, $a_i^t \leq b_i$, $i = 1, \dots, n$, $c_i \geq 0$, $d_i \geq 0$, $c_i^t \leq d_i$, $i = 1, \dots, k$. Denote by B_1 the set of symmetric r.v.'s Y_k independent of $U_1, \dots, U_n, V_1, \dots, V_{k-1}, Y_{k+1}, \dots, Y_n$ and satisfying the conditions $EY_k^2 \leq c_k^2$, $E|Y_k|^t \leq d_k$, and by B_2 the subset of B_1 consisting of r.v.'s Y_k such that $EY_k^2 = c_k^2$, $E|Y_k|^t = d_k$.

Applying Lemma 6 again with $c_i = b_i - a_i^t$ we obtain

$$\begin{aligned} & \sup_{Y_k \in B_l} \left(\sum_{i=1}^n (b_i - a_i^t) \left(\sum_{j=1, j \neq i}^{k-1} (d_j - c_j^t) \right) \right. \\ & \quad \left. + \sum_{i=1}^n (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^{k-1} c_j V_j + \sum_{j=k, j \neq i}^n Y_j \right|^t \right. \\ & \quad \left. + E \left| \sum_{j=1}^{k-1} c_j V_j \left(\sum_{i=1, i \neq j}^n a_i U_i \right) + \sum_{j=k}^n Y_j \left(\sum_{i=1, i \neq j}^n a_i U_i \right) \right|^t \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n (b_i - a_i^t) \left(\sum_{j=1, j \neq i}^k (d_j - c_j^t) \right) \\
&+ \sum_{i=1}^n (b_i - a_i^t) E \left| \sum_{j=1, j \neq i}^k c_j V_j + \sum_{j=k+1, j \neq i}^n Y_j \right|^t \\
&+ E \left| \sum_{j=1}^k c_j V_j \left(\sum_{i=1, i \neq j}^n a_i U_i \right) + \sum_{j=k+1}^n Y_j \left(\sum_{i=1, i \neq j}^n a_i U_i \right) \right|^t, \quad l=1, 2
\end{aligned} \tag{30}$$

Using (30) n times we get (3).

Relations (8) and (9) might be proven in the same way. \square

Proofs of Theorems 4–8. Let us prove (14). Let $2 < t < 4$, $D \geq 0$, and let $L(D)$ be a class of independent identically distributed r.v.'s X_1, \dots, X_n , for which

$$\max(C_n^2(E |X_1|^t)^2, (C_n^2)^{t/2} (EX_1^t)^t) = D$$

It is evident that

$$\begin{aligned}
&\sup_{(X, n) \in M_1(n, D^{1/2t}/(C_n^2)^{1/4}, D^{1/2}/(C_n^2)^{1/2})} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \\
&\leq \sup_{(X, n) \in L(D)} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \\
&\leq \sup_{(X, n) \in M_2(n, D^{1/2t}/(C_n^2)^{1/4}, D^{1/2}/(C_n^2)^{1/2})} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t
\end{aligned} \tag{31}$$

From relation (2) and its proof it follows that

$$\begin{aligned}
&\sup_{(X, n) \in M_k(n, D^{1/2t}/(C_n^2)^{1/4}, D^{1/2}/(C_n^2)^{1/2})} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t \\
&= \left(C_n^2 (1/(C_n^2)^{1/2} - 1/(C_n^2)^{t/2})^2 \right. \\
&\quad \left. + (1/(C_n^2)^{1/2} - 1/(C_n^2)^{t/2}) n / (C_n^2)^{t/4} E \left| \sum_{i=2}^n U_i \right|^t \right. \\
&\quad \left. + E \left| \sum_{1 \leq i < j \leq n} U_i U_j / (C_n^2)^{1/2} \right|^t \right) D, \quad k=1, 2
\end{aligned} \tag{32}$$

(14) now follows from (31), (32) and the equality

$$B_4^*(t, n) = \sup_{D > 0} \left(\sup_{(X, n) \in L(D)} E \left| \sum_{1 \leq i < j \leq n} X_i X_j \right|^t / D \right)$$

The remaining relations (15)–(21) might be proven in the similar way. \square

ACKNOWLEDGMENTS

We are grateful to an anonymous referee and Victor de la Peña for many useful suggestions, which led to the improvement of the paper. We would like also to thank Jakša Cvitanić, Victor de la Peña and other participants of the probability seminar in the Department of Statistics at Columbia University for an opportunity to present and discuss the results.

REFERENCES

- Bonami, A. (1970). Etude des coefficients de Fourier des fonctions de $L_p(G)$. *Ann. Inst. Fourier* **20**, 335–402.
- Borovskikh, Yu. V., and Korolyuk, V. S. (1997). *Martingale Approximation*, VSP, Utrecht, 322 pp.
- Cambanis, S., Rosiński, J., and Woyczyński, W. A. (1985). Convergence of quadratic forms in p -stable random variables and θ_p -radonifying operators. *Ann. Probab.* **13**, 885–897.
- Cecen, A. A., and Erkal, C. (1996a). Distinguishing between stochastic and deterministic behavior in high frequency foreign exchange rate returns: Can non-linear dynamics help forecasting? *International J. Forecast.* **12**, 465–473.
- Cecen, A. A., and Erkal, C. (1996b). Distinguishing between stochastic and deterministic behavior in high frequency exchange rate returns: Further evidence. *Economics Letters* **51**, 323–329.
- De la Peña, V. H. (1992). Decoupling and Khintchine's inequalities for U -statistics. *Ann. Probab.* **20**, 1877–1892.
- De la Peña, V. H., and Klass, M. J. (1994). Order of magnitude bounds for expectations involving quadratic forms. *Ann. Probab.* **22**, 1044–77.
- De la Peña, V. H., and Montgomery-Smith (1995). Decoupling inequalities for the tail probabilities of multivariate U -statistics. *Ann. Probab.* **23**, 806–816.
- Figiel, T., Hitczenko, P., Johnson, W. B., Schechtman, G., and Zinn, J. (1997). Extremal properties of Rademacher functions with applications to the Khintchine and Rosenthal inequalities. *Trans. Amer. Math. Soc.* **349**, 997–1027.
- Giné, E., Latała, R., and Zinn, J. (2000). Exponential and moment inequalities for U -statistics. *High Dimensional Probability II—Progress in Probability*, Birkhäuser, pp. 13–35.
- Hitczenko, P. (1990). Best constants in martingale version of Rosenthal's inequality. *Ann. Probab.* **18**, 1656–1668.
- Hitczenko, P. (1994). On a domination of sums of random variables by sums of conditionally independent ones. *Ann. Probab.* **22**, 453–468.
- Ibragimov, R. (1997). *Estimates for the Moments of Symmetric Statistics*, Ph.D. Dissertation, Institute of Mathematics of Uzbek Academy of Sciences, Tashkent, 127 pp. (in Russian).

- Ibragimov, R., and Sharakhmetov, Sh. (1995). On the best constant in Rosenthal's inequality. In *Theses of Reports of the Conference on Probability Theory and Mathematical Statistics Dedicated to the 75th Anniversary of Academician S. Kh. Sirajdinov (Fergana, Uzbekistan)*, Tashkent, pp. 43–44 (in Russian).
- Ibragimov, R., and Sharakhmetov, Sh. (1996a). Bounds for the moments of symmetric statistics. Submitted to "Studia Scientiarum Mathematicarum Hungarica."
- Ibragimov, R., and Sharakhmetov, Sh. (1996b). Some extremal problems in moment inequalities. To be published in *Theory Probab. Appl.*
- Ibragimov, R., and Sharakhmetov, Sh. (1997). On an exact constant for the Rosenthal inequality. *Teor. Veroyatnost. i Primen.* **42**, 341–350 (translation in *Theory Probab. Appl.* **42** (1997), 294–302 (1998)).
- Ibragimov, R., and Sharakhmetov, Sh. (1998a). Exact bounds on the moments of symmetric statistics. *7th Vilnius Conference on Probability Theory and Mathematical Statistics. 22nd European Meeting of Statisticians. Abstracts of Communications*, Vilnius, Lithuania, pp. 243–244.
- Ibragimov, R., and Sharakhmetov, Sh. (1998b). The best constant in Rosenthal's inequality for random variables with zero mean. To be published in *Theory Probab. Appl.*
- Ibragimov, R., and Sharakhmetov, Sh. (1999). Analogues of Khintchine, Marcinkiewicz–Zygmund and Rosenthal inequalities for symmetric statistics. *Scand. J. Statist.* **26**, 621–633.
- Ibragimov, R., and Sharakhmetov, Sh. (2000). Moment inequalities for symmetric statistics. In *Modern Problems of Probability Theory and Mathematical Statistics. Proceedings of the 4th Fergana International Colloquium on Probability Theory and Mathematical Statistics held 27–29 September, 1995*, Tashkent, pp. 184–193 (in Russian); also available at <http://front.math.ucdavis.edu/math.PR/0005004>.
- Johnson, W. B., Schechtman, G., and Zinn, J. (1985). Best constants in moment inequalities for linear combinations of independent and exchangeable random variables. *Ann. Probab.* **13**, 234–253.
- Khintchine, A. (1923). Über dyadische Brüche. *Math. Z.* **18**, 109–116.
- Klass, M. J., and Nowicki, K. (1997a). Order of magnitude bounds for expectations of A_2 -functions of generalized bilinear forms. To appear in *Probab. Theory Related Fields*.
- Klass, M. J., and Nowicki, K. (1997b). Order of magnitude bounds for expectations of A_2 -functions of nonnegative random bilinear forms and generalized U -statistics. *Ann. Probab.* **25**, 1471–1501.
- Koroljuk, V. S., and Borovskich, Yu. V. (1994). *Theory of U -statistics*, Mathematics and its Applications, Vol. 273, Kluwer Academic Publishers Group, Dordrecht, 552 pp.
- Krakowiak, W., and Szulga, J. (1986). Random multilinear forms. *Ann. Probab.* **14**, 955–973.
- Kwapień, S., and Szulga, J. (1991). Hypercontraction methods in moment inequalities for series of independent random variables in normed spaces. *Ann. Probab.* **19**, 1–8.
- Kwapień, S., and Woyczyński, W. (1992). *Random Series and Stochastic Integrals: Single and Multiple*, Birkhäuser, Boston, 360 pp.
- Marcinkiewicz, J., and Zygmund, A. (1937). Sur les fonction independantes. *Fund. Math.* **29**, 60–90.
- McConnell, T. R., and Taqqu, M. (1986). Decoupling inequalities for multilinear forms in independent symmetric random variables. *Ann. Probab.* **14**, 943–954.
- Nagaev, S. V. (1990). On a new approach to the study of the distribution of a norm of a random element in Hilbert space. *Probability Theory and Mathematical Statistics, Proceedings of the Fifth Vilnius Conference*, Mosklas/VSP, Vilnius/Utrecht, pp. 214–226.
- Nagaev, S. V. (1998). Some refinements of probabilistic and moment inequalities. *Theory Probab. Appl.* **42**, 707–713.
- Nagaev, S. V., and Pinelis, I. F. (1977). Some inequalities for the distributions of sums of independent random variables. *Theory of Probab. Appl.* **22**, 248–256.

- Peshkir, G., and Shiryaev, A. N. (1995). Khintchine's inequalities and a martingale extension of the area of their action. *Uspekhi Mat. Nauk* **50**(5), 3–62 (in Russian).
- Pinelis, I. F. (1980). Estimates for moments of infinite-dimensional martingales. *Math. Notes* **27**, 459–462.
- Pinelis, I. (1994). Extremal probabilistic problems and Hotteling's T^2 test under a symmetry condition. *Ann. Probab.* **22**, 357–368.
- Pinelis, I. F., and Utev, S. A. (1984). Estimates of moments of sums of independent random variables. *Theory Probab. Appl.* **29**, 574–577.
- Prokhorov, Yu. V. (1962). Extremal problems in limit theorems. In *Proc. VI All-Union Conference on Probability Theory and Mathematical Statistics*, Vilnius, pp. 77–84 (in Russian).
- Rosenthal, H. P. (1970). On the subspaces of L^p ($p > 2$) spanned by sequences of independent random variables. *Israel J. Math.* **8**, 273–303.
- Rosiński, J., and Szulga, J. (1982). *Product Random Measures and Double Stochastic Integrals*, Lecture Notes in Math., Vol. 939, pp. 181–199, Springer, Berlin/New York.
- Rosiński, J., and Woyczyński, W. A. (1984). *Products of Random Measures, Multilinear Forms and Multiple Stochastic Integrals*, Lecture Notes in Math., Vol. 1089, pp. 294–315, Springer, Berlin/New York.
- Rosiński, J., and Woyczyński, W. A. (1986). On Ito stochastic integration with respect to p -stable motion: Inner clock, integrability of sample paths, double and multiple integrals. *Ann. Probab.* **14**, 271–286.
- Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*, Wiley, New York, 371 pp.
- Sharakhmetov, Sh. (1995). Estimates for moments of symmetric statistics. In *Theses of Reports of the Conference on Probability Theory and Mathematical Statistics Dedicated to the 75th Anniversary of Academician S. Kh. Sirajdinov (Fergana, Uzbekistan)*, Tashkent, p. 119 (in Russian).
- Sharakhmetov, Sh. (1997). *General Representations for a Joint Distribution of Random Variables and Their Applications*, Doctor of Sciences Dissertation. Institute of Mathematics of Uzbek Academy of Sciences, 229 pp. (in Russian).
- Sjörögen, P. (1982). On the convergence of bilinear and quadratic forms in independent random variables. *Studia Math.* **71**, 285–296.
- Talagrand (1989). Isoperimetry and integrability of the sum of independent Banach-space valued random variables. *Ann. Probab.* **17**, 1546–1570.
- Utev, S. A. (1985). Extremal problems in moment inequalities. In *Proc. Mathematical Institute of the Siberian Branch of the USSR Academy of Sciences*, Vol. 5, pp. 56–75 (in Russian).