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A Characterization of Joint Distribution of Two-Valued Random Variables and Its Applications

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We obtain an explicit representation for joint distribution of two-valued random variables with given marginals and for a copula corresponding to such random variables. The results are applied to prove a characterization of r-independent two-valued random variables in terms of their mixed first moments. The characterization is used to obtain an exact estimate for the number of almost independent random variables that can be defined on a discrete probability space and necessary conditions for a sequence of r-independent random variables to be stationary.

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1. INTRODUCTION

In recent years, remarkable advances have been made in the field of probability distributions with given marginals (e.g., papers in [3, 11], monograph [24], and references therein). The main problem of the research area going back to M. Frechet [14, 15] is to determine a relationship between a multidimensional probability distribution and its lower dimensional marginals. Important contributions to the problem had been made in [8–10, 13, 16, 19, 29–34] and other works. In his ground-breaking paper [33], A. Sklar answered the question about the link between a multidimensional joint distribution and its one-dimensional margins in

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general by introducing the notion of a copula, that is, a function $C: [0, 1]^n \to [0, 1]$ such that $C(1, ..., 1, a_k, 1, ..., 1) = a_k$, $a_k \in [0, 1]$, $k = 1, ..., n$; $C(a_1, ..., a_{k-1}, 0, a_{k+1}, ..., a_n) = 0$, $a_i \in [0, 1]$, $i \neq k$, $k = 1, ..., n$; and the C-volume of any n-dimensional interval is non-negative. A. Sklar showed that if $X_1, ..., X_n$ are real random variables defined on a common probability space, with one-dimensional distribution functions $F_{X_k}(x_k) = P(X_k \leqslant x_k)$ and joint distribution function $F_{X_1, ..., X_n}(x_1, ..., x_n) = P(X_1 \leqslant x_1, ..., X_n \leqslant x_n)$, then there exists an n-dimensional copula $C_{X_1, ..., X_n}(u_1, ..., u_n)$ such that $F_{X_1, ..., X_n}(x_1, ..., x_n) = C_{X_1, ..., X_n}(F_{X_1}(x_1), ..., F_{X_n}(x_n))$ for all $x_k \in \mathbf{R}$, $k = 1, ..., n$. It is easy to see that if $F_{X_k}(x_k)$, $k = 1, ..., n$, are continuous, then $C_{X_1, ..., X_n}(u_1, ..., u_n) = F_{X_1, ..., X_n}(F^{-1}_{X_1}(u_1), ..., F^{-1}_{X_n}(u_n))$ where $F^{-1}_{X_k}(u_k) = \inf\{x_k \in \mathbf{R} : F_{X_k}(x_k) \geqslant u_k\}$, $k = 1, ..., n$ ($\inf \mathbf{R} = -\infty$, $\inf \phi = +\infty$). However, closed explicit expressions for copulas corresponding to a set of random variables with marginals from a certain class which do not involve the joint distributions of the random variables are unknown even for simple classes of the one-dimensional distributions.

In the present paper, we fill the above gap for random variables taking two values. More precisely, we show that $F: [0, 1]^n \to [0, 1]$ is a joint distribution function of n random variables X_k, $k = 1, ..., n$, taking values a_k, b_k, $k = 1, ..., n$, with one-dimensional distribution functions F_k, $k = 1, ..., n$, if and only if the subcopula corresponding to F has the form

$$C(u_1, ..., u_n) = \prod_{k=1}^{n} u_k \left(1 - \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1, ..., i_c} \prod_{k=1}^{c} |a_{i_k} - b_{i_k}| \, (1 - u_{i_k})\right),$$

$u_k \in \{0, F_k(\min(a_k, b_k)), 1\}$, $k = 1, ..., n$, where $\alpha_{i_1, ..., i_c} \in \mathbf{R}$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, ..., n$, satisfy the 2^n conditions

$$\sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1, ..., i_c} \prod_{k=1}^{c} (x_{i_k} - a_{i_k} p_{i_k} - b_{i_k} q_{i_k}) \geqslant -1,$$

$x_k \in \{a_k, b_k\}$, $k = 1, ..., n$, or, what is equivalent,

$$\sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1, ..., i_c} \prod_{k=1}^{c} |a_{i_k} - b_{i_k}| \, \varepsilon_{i_k} \leqslant 1,$$

$\varepsilon_k \in \{-F_k(\min(a_k, b_k)), 1 - F_k(\min(a_k, b_k))\}$, $k = 1, ..., n$. Therefore, the subcopula is a multivariate Eyraud–Farlie–Gumbel–Mongerstern (EFGM) subcopula (e.g., [6; 21; 24, p. 87]). In other words, F is a joint distribution function of the two-valued random variables if and only if there exists a multivariate Eyraud–Farlie–Gumbel–Mongerstern copula of the above form among copulas $C: [0, 1]^n \to [0, 1]$ corresponding to F (values of the copulas are uniquely defined on the set $\{0, F_1(\min(a_1, b_1)), 1\} \times \cdots \times \{0, F_n(\min(a_n, b_n)), 1\}$).

Equivalently, a function p: $\{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_n, b_n\} \to [0, 1]$ is a joint distribution of n random variables assuming values a_k and b_k, $k = 1, ..., n$, with probabilities p_k and $q_k = 1 - p_k$, $k = 1, ..., n$, respectively, if and only if it has the form

$$p(x_1, ..., x_n) = \prod_{k=1}^{n} p_k(x_k) \left(1 + \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1, ..., i_c} \prod_{k=1}^{c} (x_{i_k} - a_{i_k} p_{i_k} - b_{i_k} q_{i_k})\right),$$

$x_k \in \{a_k, b_k\}$, $p_k(a_k) = p_k$, $p_k(b_k) = q_k$, $k = 1, ..., n$, where $\alpha_{i_1, ..., i_c} \in \mathbf{R}$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, ..., n$, satisfy the 2^n conditions

$$\sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1, ..., i_c} \prod_{k=1}^{c} (x_{i_k} - a_{i_k} p_{i_k} - b_{i_k} q_{i_k}) \geqslant -1,$$
$$x_k \in \{a_k, b_k\}, \qquad k = 1, ..., n.$$

Moreover, if $p(x_1, ..., x_n)$ is the joint distribution of n random variables $X_1, ..., X_n$ assuming the values a_k and b_k, $k = 1, ..., n$, respectively, then $\alpha_{i_1, ..., i_c} = E \prod_{k=1}^{c} (X_{i_k} - EX_{i_k})/\mathrm{Var}\, X_{i_k}$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, ..., n$, and, therefore,

$$\begin{aligned} p(x_1, ..., x_n) &= P(X_1 = x_1, ..., X_n = x_n) \\ &= \prod_{k=1}^{n} P(X_k = x_k) \\ &\quad \times \left(1 + \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \prod_{k=1}^{c} E(X_{i_k} - EX_{i_k})(x_{i_k} - EX_{i_k})/\mathrm{Var}\, X_{i_k}\right) \\ &= \prod_{k=1}^{n} P(X_k = x_k)\, E \prod_{k=1}^{n} (1 + (X_k - EX_k)(x_k - EX_k)/\mathrm{Var}\, X_k), \end{aligned}$$

$x_k \in \{a_k, b_k\}$, $k = 1, ..., n$.

Existence of a multivariate EFGM copula among copulas corresponding to random variables $X_1, ..., X_n$ is important because its simplicity and the fact that the coefficients in the copula can be expressed as (scaled) mixed moments of X_k, $k = 1, ..., n$, allows one to immediately obtain a number of applications to the study of properties of discrete (dependent) random variables as demonstrated in the paper.

We prove the representation for joint distribution of random variables assuming two values and corresponding subcopula in Section 2 and use it in Section 3 to obtain a characterization of independent and, more generally, r-independent random variables assuming two values in terms of their first mixed moments. We then present the following applications of the above-mentioned results: in Section 4, we obtain the exact estimate for the number of almost independent random variables that can be defined on a probabilistic space of n points which complements the results obtained in [2, 22, 25]; in Section 5, we prove necessary conditions for a sequence of r-independent symmetric Bernoulli random variables to be strictly stationary which generalize and complement the results obtained in [27, 28].

2. JOINT DISTRIBUTION OF RANDOM VARIABLES ASSUMING TWO VALUES

Let $n \geqslant 2$, $0 \leqslant p_k \leqslant 1$, $q_k = 1 - p_k$, $k = 1, ..., n$, and let A_n be a set of numbers $\alpha_{i_1, ..., i_c} \in \mathbf{R}$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, ..., n$, satisfying the 2^n conditions

$$\sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1, ..., i_c} \prod_{k=1}^{c} (x_{i_k} - a_{i_k} p_{i_k} - b_{i_k} q_{i_k}) \geqslant -1,$$

$x_k \in \{a_k, b_k\}$, $k = 1, ..., n$.

The following theorem holds.

THEOREM 1. *A function $C: [0, 1]^n \to [0, 1]$ is a copula corresponding to some random variables $X_1, ..., X_n$ with one-dimensional distributions $P(X_k = a_k) = p_k$, $P(X_k = b_k) = q_k$, $k = 1, ..., n$, if and only if*

$$C(u_1, ..., u_n) = \prod_{k=1}^{n} u_k \left(1 - \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1, ..., i_c} \prod_{k=1}^{c} |a_{i_k} - b_{i_k}| (1 - u_{i_k})\right), \tag{1}$$

$u_k \in \{0, F_k(\min(a_k, b_k)), 1\}$, where $\alpha_{i_1, ..., i_c} \in A_n$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, ..., n$, or, what is equivalent,

$$\sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1, ..., i_c} \prod_{k=1}^{c} |a_{i_k} - b_{i_k}| \, \varepsilon_{i_k} \leqslant 1,$$

$\varepsilon_k \in \{-F_k(\min(a_k, b_k)), 1 - F_k(\min(a_k, b_k))\}$, $k = 1, ..., n$. Moreover, if (1) is the copula corresponding to the random variables $X_1, ..., X_n$ with one-dimensional distributions $P(X_k = a_k) = p_k$, $P(X_k = b_k) = q_k$, $k = 1, ..., n$, then

$$E \prod_{k=1}^{c} (X_{i_k} - EX_{i_k}) / \operatorname{Var} X_{i_k} = \alpha_{i_1, ..., i_c},$$

$$1 \leqslant i_1 < \cdots < i_c \leqslant n, \qquad c = 2, ..., n.$$

Equation (1) means that the joint distributions of random variables assuming two values are multivariate Eyraud–Farlie–Gumbel–Mongestern (EFGM) distributions (see, for example, [6; 7; 21; 24, p. 87; 35]). As S. Cambanis showed in [7], the most common dependence structures such as constant, exponential and m-dependence cannot be exhibited by stationary processes $\{X_n\}$ whose finite dimensional distributions are the following multivariate analogues of bivariate EFGM distributions:

$$P(X_{j_1}=x_{j_1}, \ldots, X_{j_n}=x_{j_n}) = \prod_{k=1}^{n} F_{j_k}(x_{j_k}) \left(1+\sum_{1 \leqslant l<m \leqslant n} \alpha_{lm}(1-F_{j_l}(x_{j_l}))(1-F_{j_m}(x_{j_m}))\right). \tag{2}$$

On the other hand, from the results obtained in [26, 27] it follows that there exist stationary processes of Bernoulli random variables that are pairwise independent and (1, 1)- or (1, 2)-independent simultaneously. These results and the above remark mean that the multivariate EFMG distributions of the form

$$P(X_{j_1}=x_{j_1}, \ldots, X_{j_n}=x_{j_n}) = \prod_{k=1}^{n} F_{j_k}(x_{j_k}) \left(1+\sum_{c=2}^{n} \sum_{i_1<\cdots<i_c \in \{j_1, \ldots, j_n\}} \alpha_{i_1, \ldots, i_c} \prod_{k=1}^{c} (1-F_{i_k}(x_{i_k}))\right)$$

(see the above-mentioned works [6; 21; 24, p. 87]) are more appropriate for being used as finite dimensional distributions of stationary processes with "good" dependence structures than the distributions of the form (2).

Theorem 1 immediately follows from Theorems 2 and 3 below.

THEOREM 2. *If $\alpha_{i_1, \ldots, i_c} \in A_n$, $1 \leqslant i_1<\cdots<i_c \leqslant n$, $c=2, \ldots, n$, $p(a_k)=p_k$, $p(b_k)=q_k$, $k=1, \ldots, n$, then the expression*

$$p(x_1, \ldots, x_n) = \prod_{k=1}^{n} p_k(x_k) \left(1+\sum_{c=2}^{n} \sum_{1 \leqslant i_1<\cdots<i_c \leqslant n} \alpha_{i_1, \ldots, i_c} \prod_{k=1}^{c} (x_{i_k}-a_{i_k} p_{i_k}-b_{i_k} q_{i_k})\right), \tag{3}$$

$x_k \in \{a_k, b_k\}$, $k=1, \ldots, n$, is the joint distribution of some random variables $X_1, \ldots, X_n$ with one-dimensional marginals $P(X_k=a_k)=p_k$, $P(X_k=b_k)=q_k$, $k=1, \ldots, n$.

Proof. Denote expression (3) by $p_\alpha(x_1, \ldots, x_n)$. It suffices to show that $p_\alpha(x_1, \ldots, x_n) \geqslant 0$ and $\sum_{x_1 \in \{a_1, b_1\}} \cdots \sum_{x_n \in \{a_n, b_n\}} p_\alpha(x_1, \ldots, x_n)=1$. The former relation is evident on the strength of the condition $\alpha_{i_1,\ldots,i_c} \in A_n$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, \ldots, n$. Since

$$\sum_{x_1 \in \{a_1, b_1\}} \cdots \sum_{x_n \in \{a_n, b_n\}} p_\alpha(x_1, \ldots, x_n) = \sum_{x_1 \in \{a_1, b_1\}} \cdots \sum_{x_n \in \{a_n, b_n\}} \prod_{k=1}^{n} p_k(x_k)$$
$$+ \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \alpha_{i_1,\ldots,i_c} \sum_{x_1 \in \{a_1, b_1\}} \cdots$$
$$\sum_{x_n \in \{a_n, b_n\}} \prod_{k=1}^{c} (x_{i_k} - a_{i_k} p_{i_k} - b_{i_k} q_{i_k}) \prod_{k=1}^{n} p_k(x_k),$$

then from the relations

$$\sum_{x_k \in \{a_k, b_k\}} p(x_k) = 1 \tag{4}$$

and

$$\sum_{x_k \in \{a_k, b_k\}} (x_k - a_k p_k - b_k q_k)\, p_k(x_k) = 0, \tag{5}$$

$k = 1, \ldots, n$, we obtain that $\sum_{x_1 \in \{a_1, b_1\}} \cdots \sum_{x_n \in \{a_n, b_n\}} p_\alpha(x_1, \ldots, x_n) = 1$. The proof is complete.

Note that relations (4) and (5) ensure that

$$\sum_{x_{i_1} \in \{a_{i_1}, b_{i_1}\}} \cdots \sum_{x_{i_k} \in \{a_{i_k}, b_{i_k}\}} p_\alpha(x_1, \ldots, x_n) = p_{\alpha, i_{k+1}, \ldots, i_n}(x_{i_{k+1}}, \ldots, x_{i_n}),$$

$x_{i_{k+1}} \in \{a_{i_{k+1}}, b_{i_{k+1}}\}, \ldots, x_{i_n} \in \{a_{i_n}, b_{i_n}\}$, $k = 1, \ldots, n-1$, $i_1 < \cdots < i_k \in \{1, \ldots, n\}$, $i_{k+1} < \cdots < i_n \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$, where

$$p_{\alpha, i_{k+1}, \ldots, i_n}(x_{i_{k+1}}, \ldots, x_{i_n})$$
$$= \prod_{s=k+1}^{n} p_s(x_s) \left(1 + \sum_{c=2}^{n-k} \sum_{k+1 \leqslant j_1 < \cdots < j_c \leqslant n} \alpha_{i_{j_1}, \ldots, i_{j_c}} \prod_{s=1}^{c} (x_{i_{j_s}} - a_{i_{j_s}} p_{i_{j_s}} - b_{i_{j_s}} q_{i_{j_s}})\right),$$

that is, the marginals of $p_\alpha(x_1, \ldots, x_n)$ are of the same type.

From (4), (5), and the evident equality

$$\sum_{x_k \in \{a_k, b_k\}} (x_k - a_k p_k - b_k q_k)^2\, p_k(x_k) = (a_k - b_k)^2\, p_k q_k$$

it also follows that if $X_1, \ldots, X_n$ are random variables with one-dimensional distributions $P(X_k = a_k) = p_k$, $P(X_k = b_k) = q_k$, $k = 1, \ldots, n$, and the joint distribution $p_\alpha(x_1, \ldots, x_n)$, then for any $1 \leqslant j_1 < \cdots < j_d \leqslant n$, $d = 2, \ldots, n$,

$$
\begin{aligned}
&E\prod_{k=1}^{d}(X_{j_k}-EX_{j_k})\\
&\quad=\sum_{x_1\in\{a_1,b_1\}}\cdots\sum_{x_n\in\{a_n,b_n\}}\prod_{k=1}^{d}(x_{j_k}-a_{j_k}p_{j_k}-b_{j_k}q_{j_k})\,p_\alpha(x_1,\ldots,x_n)\\
&\quad=\sum_{x_1\in\{a_1,b_1\}}\cdots\sum_{x_n\in\{a_n,b_n\}}\prod_{k=1}^{d}(x_{j_k}-a_{j_k}p_{j_k}-b_{j_k}q_{j_k})\prod_{k=1}^{n}p_k(x_k)\\
&\qquad\times\left(1+\sum_{c=2}^{n}\sum_{1\leqslant i_1<\cdots<i_c\leqslant n}\alpha_{i_1,\ldots,i_c}\prod_{k=1}^{c}(x_{i_k}-a_{i_k}p_{i_k}-b_{i_k}q_{i_k})\right)\\
&\quad=\alpha_{j_1,\ldots,j_d}\sum_{x_1\in\{a_1,b_1\}}\cdots\sum_{x_n\in\{a_n,b_n\}}\prod_{k=1}^{d}(x_{j_k}-a_{j_k}p_{j_k}-b_{j_k}q_{j_k})^2\prod_{k=1}^{n}p_k(x_k)\\
&\quad=\alpha_{j_1,\ldots,j_d}\prod_{k=1}^{d}(a_{j_k}-b_{j_k})^2\,p_{j_k}q_{j_k}=\alpha_{j_1,\ldots,j_d}\prod_{k=1}^{d}\operatorname{Var}X_{j_k}.
\end{aligned}
$$

Therefore,

$$
E\prod_{k=1}^{c}(X_{i_k}-EX_{i_k})/\operatorname{Var}X_{i_k}=\alpha_{i_1,\ldots,i_c},\tag{6}
$$

$1\leqslant i_1<\cdots<i_c\leqslant n$, $c=2,\ldots,n$, and from Theorem 2 it follows that if $\alpha_{i_1,\ldots,i_c}\in A_n$, $1\leqslant i_1<\cdots<i_c\leqslant n$, $c=2,\ldots,n$, then there exists a set of random variables $X_1,\ldots,X_n$ with one-dimensional marginals $P(X_k=a_k)=p_k$, $P(X_k=b_k)=q_k$, $k=1,\ldots,n$, and joint distribution $p_\alpha(x_1,\ldots,x_n)$ for which relations (6) hold.

It is natural to ask whether the distribution of certain random variables assuming the values a_k and b_k, $k=1,\ldots,n$, with probabilities p_k and $q_k=1-p_k$, $k=1,\ldots,n$, can have a form different from (3). The answer on this question is given by the following theorem.

THEOREM 3. *If $X_1,\ldots,X_n$ is a set of random variables with one-dimensional distributions $P(X_k=a_k)=p_k$, $P(X_k=b_k)=q_k$, $k=1,\ldots,n$, and joint distribution $p(x_1,\ldots,x_n)=P(X_1=x_1,\ldots,X_n=x_n)$, $x_k\in\{a_k,b_k\}$, $k=1,\ldots,n$, then*

$$
\begin{aligned}
&p(x_1,\ldots,x_n)\\
&\quad=\prod_{k=1}^{n}P(X_k=x_k)\\
&\qquad\times\left(1+\sum_{c=2}^{n}\sum_{1\leqslant i_1<\cdots<i_c\leqslant n}E\prod_{k=1}^{c}(X_{i_k}-EX_{i_k})(x_{i_k}-EX_{i_k})/\operatorname{Var}X_{i_k}\right)\\
&\quad=\prod_{k=1}^{n}P(X_k=x_k)\,E\prod_{k=1}^{n}(1+(X_k-EX_k)(x_k-EX_k)/\operatorname{Var}X_k),
\end{aligned}\tag{7}
$$

$x_k\in\{a_k,b_k\}$, $k=1,\ldots,n$.

Proof. Since $1+(X_k-EX_k)(x_k-EX_k)/\text{Var}\, X_k \geqslant 0$ a.s., $k=1, ..., n$, then the right-hand side of (7) is non-negative. Hence,

$$\sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} E \prod_{k=1}^{c} (X_{i_k}-EX_{i_k})(x_{i_k}-EX_{i_k})/\text{Var}\, X_{i_k} \geqslant -1,$$

$x_k \in \{a_k, b_k\}$, $k=1, ..., n$, that is, $E\prod_{k=1}^{c} (X_{i_k}-EX_{i_k})/\text{Var}\, X_{i_k} \in A_n$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c=2, ..., n$.

Denote

$$p_k(x_k) = P(X_k = x_k), \qquad k=1, ..., n,$$

$$p_{i_1, ..., i_c}(x_{i_1}, ..., x_{i_c}) = P(X_{i_1} = x_{i_1}, ..., X_{i_c} = x_{i_c}),$$

$$g_{i_1, ..., i_c}(x_{i_1}, ..., x_{i_c}) = \sum_{k=2}^{c} (-1)^{c-k} \sum_{j_1 < \cdots < j_k \in \{i_1, ..., i_c\}} \times \left[p_{j_1, ..., j_k}(x_{j_1}, ..., x_{j_k}) \Big/ \left(\prod_{s=1}^{k} p_{j_s}(x_{j_s}) \right) - 1 \right],$$

$1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c=2, ..., n$. From the known inversion formula (see, for example, [5]) it follows that if $a_{i_1, ..., i_c}$, $b_{i_1, ..., i_c}$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c=2, ..., n$ are arbitrary numbers, then the relations

$$b_{i_1, ..., i_c} = \sum_{k=2}^{c} \sum_{j_1 < \cdots < j_k \in \{i_1, ..., i_c\}} a_{j_1, ..., j_k},$$

$1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c=2, ..., n$, and the relations

$$a_{i_1, ..., i_c} = \sum_{k=2}^{c} (-1)^{c-k} \sum_{j_1 < \cdots < j_k \in \{i_1, ..., i_c\}} b_{j_1, ..., j_k},$$

$1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c=2, ..., n$, are equivalent. Setting here

$$a_{i_1, ..., i_c} = g_{i_1, ..., i_c}(x_{i_1}, ..., x_{i_c}),$$

$$b_{i_1, ..., i_c} = p_{i_1, ..., i_c}(x_{i_1}, ..., x_{i_c}) \Big/ \left(\prod_{s=1}^{k} p_{i_s}(x_{i_s}) \right) - 1,$$

$1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c=2, ..., n$, we obtain that

$$p(x_1, ..., x_n) = \prod_{k=1}^{n} p_k(x_k) \left(1 + \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} g_{i_1, ..., i_c}(x_{i_1}, ..., x_{i_c}) \right).$$

Show that

$$g_{i_1,\ldots,i_c}(x_{i_1},\ldots,x_{i_{m-1}},a_{i_m},x_{i_{m+1}},\ldots,x_{i_c})\,p_{i_m} \\ = -g_{i_1,\ldots,i_c}(x_{i_1},\ldots,x_{i_{m-1}},b_{i_m},x_{i_{m+1}},\ldots,x_{i_c})\,q_{i_m}, \tag{8}$$

$1 \leqslant i_1 < \cdots < i_c \leqslant n$, $m = 1, \ldots, c$, $c = 2, \ldots, n$, $x_k \in \{a_k, b_k\}$, $k = 1, \ldots, n$. It suffices to consider the case $i_1 = 1, \ldots, i_c = c$, $m = 1$. We have that

$$\begin{aligned}
&g_{1,\ldots,c}(a_1,x_2,\ldots,x_c)\,p_1 + g_{1,\ldots,c}(b_1,x_2,\ldots,x_c)\,q_1 \\
&= \sum_{k=2}^{c}(-1)^{c-k}\Bigg\{\sum_{2\leqslant i_2<\cdots<i_k\leqslant c}\Bigg[p_{1,i_2,\ldots,i_k}(a_1,x_{i_2},\ldots,x_{i_k})\,p_1\Bigg/\Bigg(p_1\prod_{s=2}^{k}p_{i_s}(x_{i_s})\Bigg) \\
&\quad + p_{1,i_2,\ldots,i_k}(b_1,x_{i_2},\ldots,x_{i_k})\,q_1\Bigg/\Bigg(q_1\prod_{s=2}^{k}p_{i_s}(x_{i_s})\Bigg) - 1\Bigg] \\
&\quad + \sum_{2\leqslant i_1<\cdots<i_k\leqslant c}\Bigg[p_{i_1,i_2,\ldots,i_k}(x_{i_1},x_{i_2},\ldots,x_{i_k})\Bigg/\Bigg(\prod_{s=1}^{k}p_{i_s}(x_{i_s})\Bigg) - 1\Bigg]\Bigg\} \\
&= \sum_{k=2}^{c}(-1)^{c-k}\Bigg\{\sum_{2\leqslant i_2<\cdots<i_k\leqslant c}\Bigg[p_{i_2,\ldots,i_k}(x_{i_2},\ldots,x_{i_k})\Bigg/\Bigg(\prod_{s=2}^{k}p_{i_s}(x_{i_s})\Bigg) - 1\Bigg] \\
&\quad + \sum_{2\leqslant i_1<\cdots<i_k\leqslant c}\Bigg[p_{i_1,\ldots,i_k}(x_{i_1},\ldots,x_{i_k})\Bigg/\Bigg(\prod_{s=1}^{k}p_{i_s}(x_{i_s})\Bigg) - 1\Bigg]\Bigg\} = 0.
\end{aligned}$$

It is not difficult to obtain from (8) that $g_{i_1,\ldots,i_c}(x_{i_1},\ldots,x_{i_c}) = \alpha_{i_1,\ldots,i_c}\prod_{k=1}^{c}(x_{i_k} - EX_{i_k})$, where $\alpha_{i_1,\ldots,i_c} = g_{i_1,\ldots,i_c}(a_{i_1},\ldots,a_{i_c})/(\prod_{k=1}^{c}(a_{i_k} + b_{i_k})\,q_{i_k})$. Therefore, the joint distribution of the random variables $X_1, \ldots, X_n$ has the form (3) and, on the strength of (6), $\alpha_{i_1,\ldots,i_c} = E\prod_{k=1}^{c}(X_{i_k} - EX_{i_k})/\operatorname{Var} X_{i_k}$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, \ldots, n$. The proof is complete.

Let $n \geqslant 2$, $\bar{A}_n = \{\alpha_{i_1,\ldots,i_c} : \sum_{c=2}^{n}\sum_{1\leqslant i_1<\cdots<i_c\leqslant n}\alpha_{i_1,\ldots,i_c}x_{i_1}\cdots x_{i_c} \geqslant -1, x_k \in \{-1, 1\}\}$. Using Theorems 2 and 3, we obtain the following corollaries.

Corollary 1. *If* $\alpha_{i_1,\ldots,i_c} \in \bar{A}_n$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, \ldots, n$, *then the expression*

$$2^{-n}\Bigg(1 + \sum_{c=2}^{n}\sum_{1\leqslant i_1<\cdots<i_c\leqslant n}\alpha_{i_1,\ldots,i_c}x_{i_1}\cdots x_{i_c}\Bigg),$$

$x_k \in \{-1, 1\}$, $k = 1, \ldots, n$, *is the joint distribution of some random variables* $X_1, \ldots, X_n$ *with symmetric Bernoulli one-dimensional distributions* $P(X_k = \pm 1) = 1/2$, $k = 1, \ldots, n$, *and* $EX_{i_1}\cdots X_{i_c} = \alpha_{i_1,\ldots,i_c}$, $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, \ldots, n$.

COROLLARY 2. *If $X_1, \ldots, X_n$ is a set of symmetric Bernoulli random variables with joint distribution $p(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$, then*

$$p(x_1, \ldots, x_n) = 2^{-n}\left(1 + \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} EX_{i_1} \cdots X_{i_c} x_{i_1} \cdots x_{i_c}\right)$$

$$= 2^{-n} E \prod_{k=1}^{n} (1 + X_k x_k), \tag{9}$$

$x_k \in \{-1, 1\}$, $k = 1, \ldots, n$.

Equation (7) can be useful for calculating expectations of certain statistics in random variables assuming two values. It is easy to see that

$$\begin{aligned} &Ef(X_1, \ldots, X_n) \\ &\quad = \sum_{x_1 \in \{a_1, b_1\}} \cdots \sum_{x_n \in \{a_n, b_n\}} f(x_1, \ldots, x_n)\, p(x_1, \ldots, x_n) \\ &\quad = \sum_{x_1 \in \{a_1, b_1\}} \cdots \sum_{x_n \in \{a_n, b_n\}} f(x_1, \ldots, x_n) \prod_{k=1}^{n} p_k(x_k) \\ &\qquad \times \left(1 + \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} E \prod_{k=1}^{c} (X_{i_k} - EX_{i_k})(x_{i_k} - EX_{i_k})/\mathrm{Var}\, X_{i_k}\right) \\ &\quad = Ef(\xi_1, \ldots, \xi_n) + Ef(\xi_1, \ldots, \xi_n)\, S_n(\xi_1, \ldots, \xi_n), \end{aligned} \tag{10}$$

where $\xi_1, \ldots, \xi_n$ are independent copies of the random variables $X_1, \ldots, X_n$, $S_n(\xi_1, \ldots, \xi_n)$ is a linear combination of multilinear forms (polynomial chaos)

$$S_n(\xi_1, \ldots, \xi_n) = \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \prod_{k=1}^{c} ((\xi_{i_k} - EX_{i_k})\, E(X_{i_k} - EX_{i_k})/\mathrm{Var}\, X_{i_k}).$$

Relation (10) establishes a correspondence between arbitrary dependent random variables $X_1, \ldots, X_n$ assuming two values and linear combinations of multilinear forms in independent random variables $\xi_1, \ldots, \xi_n$ assuming two values.

Note that applying (10) for $f(x_1, \ldots, x_n) = \prod_{k=1}^{n} I(x_k \leqslant y_k)$, $x_k, y_k \in \{a_k, b_k\}$, $k = 1, \ldots, n$, where $I(x_k \leqslant y_k) = 1$ if $x_k \leqslant y_k$ and $I(x_k \leqslant y_k) = 0$ otherwise, and using the evident relations $E(\xi_k - E\xi_k)\, I(\xi_k \leqslant y_k) = -|a_k - b_k|\, F(y_k)(1 - F(y_k))$, $y_k \in \{a_k, b_k\}$, we obtain representation (1).

It is not difficult to show that if $X_1, \ldots, X_n$ are symmetric Bernoulli random variables then on the strength of (10)

$$E \exp\left(\frac{it}{\sqrt{n}}(X_1 + \cdots + X_n)\right)$$

$$= \prod_{k=1}^{n} E \exp\left(\frac{it}{\sqrt{n}}\xi_k\right) + \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} EX_{i_1} \cdots X_{i_c} \prod_{k=1}^{c} E\xi_{i_k}$$

$$\times \exp\left(\frac{it}{\sqrt{n}}\xi_{i_k}\right) \prod_{\substack{k=1,\\ k \neq i_1, \ldots, i_c}}^{n} E \exp\left(\frac{it}{\sqrt{n}}\xi_k\right)$$

$$= \left(\cos\frac{t}{\sqrt{n}}\right)^n + \sum_{c=2}^{n} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n} EX_{i_1} \cdots X_{i_c} \left(i \sin\frac{t}{\sqrt{n}}\right)^c \left(\cos\frac{t}{\sqrt{n}}\right)^{n-c}$$

$$= \left(\cos\frac{t}{\sqrt{n}}\right)^n E \prod_{k=1}^{n} \left(1 + i \tan\left(\frac{t}{\sqrt{n}}\right) X_k\right).$$

This implies the following

COROLLARY 3. *The central limit theorem holds for a sequence of symmetric Bernoulli random variables* $X_1, \ldots, X_n$ *if and only if* $E \prod_{k=1}^{n} (1 + i \tan(t/\sqrt{n}\,)\, X_k) \to 1$ *as* $n \to \infty$.

Some other applications of the obtained representation for the joint distribution of random variables assuming two values are presented in Sections 3–5.

3. CHARACTERIZATION OF CERTAIN CLASSES OF DEPENDENT RANDOM VARIABLES ASSUMING TWO VALUES

In the present section, we apply Theorems 1–3 to characterize certain classes of dependent random variables assuming two values.

Let $X_1, \ldots, X_n$ be random variables defined on a probability space $(\Omega, \mathfrak{I}, P)$.

Remind the following definitions.

DEFINITION 1. Random variables $X_1, \ldots, X_n$ are said to be a multiplicative system (of order 1) if $E|X_k| < \infty$, $k = 1, \ldots, n$, and $E \prod_{k=1}^{n} X_k^{\alpha_k} = \prod_{k=1}^{n} EX_k^{\alpha_k}$ for all $\alpha_k \in \{0, 1\}$, $k = 1, \ldots, n$.

A definition of multiplicative systems was introduced by G. Alexits [1] for the purpose of studying of orthogonal functions. Examples of multiplicative systems are given, besides of independent random variables, by a sequence of martingale-differences, lacunary trigonometric systems $\{\cos 2\pi n_k x, \sin 2\pi n_k x, k = 1, 2, \ldots\}$ on the interval $[0, 1]$ with Lebesgue measure for $n_{k+1}/n_k \geqslant 2$ and ε-independent and asymptotically independent random variables introduced by V. M. Zolotarev in [36].

DEFINITION 2. Random variables $X_1, ..., X_n$ are said to be r-independent ($2 \leqslant r \leqslant n$) if any r of them are mutually independent. $(n-1)$-independent random variables $X_1, ..., X_n$ are said to be almost independent.

DEFINITION 3. Random variables $X_1, ..., X_n$ are said to be strictly r-independent ($2 \leqslant r \leqslant n-1$) if they are r-independent but not $(r+1)$-independent.

Several authors have focused on studying of properties of r-independent random variables. S. Bernstein [4] constructed an example of strictly pairwise independent (2-independent) random variables. P. Levy [23] constructed an example of stationary pairwise independent process consisting of identically distributed on the interval $[0, 1]$ random variables. M. Rosenblatt and D. Slepian [28] constructed examples of strictly r-independent rth order stationary Markov processes with discrete distributions and also proved that such processes must have at least three points in their probability space. J. B. Robertson and J. M. Womack [27] and J. B. Robertson [26] brought examples of stationary pairwise independent processes consisting of symmetric Bernoulli random variables and obtained a number of necessary conditions for a sequence of such random variables to be stationary. H. O. Lancaster [22] obtained an exact estimate for the number of pairwise independent random variables that can be defined on a probability space of n points. G. L. O'Brien [25] solved a similar problem with the additional requirement that each random variable assumes M distinct values for some $M > 1$. A. Joffe [20] constructed a set of r-independent random variables, each uniformly distributed over the set $\{0, 1, ..., p-1\}$ where p is a prime number. B. V. Gladkov [17] proved the existence of r-independent random variables with given one-dimensional distributions and proved central limit theorem for sums of r-independent random variables. Y. H. Wang [35] constructed examples of joint distributions of r-independent random variables with given marginals.

The following theorem shows that for random variables assuming two values, the properties of multiplicativity and mutual independence are equivalent.

THEOREM 4. *Random variables $X_1, ..., X_n$ with one-dimensional distributions $P(X_k = a_k) = p_k$, $P(X_k = b_k) = q_k$, $k = 1, ..., n$, are mutually independent if and only if $X_1, ..., X_n$ form a multiplicative system.*

Proof. It is evident that if the random variables $X_1, \ldots, X_n$ are mutually independent then they form a multiplicative system. On the other hand, if $X_1, \ldots, X_n$ form a multiplicative system, then it is easy to see that $E\prod_{k=1}^{c}(X_{i_k} - EX_{i_k}) = 0$ for all $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, \ldots, n$, and, therefore, on the strength of Theorem 2,

$$P(X_1 = x_1, \ldots, X_n = x_n) = \prod_{k=1}^{n} P(X_k = x_k),$$

$x_k \in \{a_k, b_k\}$, $k = 1, \ldots, n$. This means that the random variables $X_1, \ldots, X_n$ are mutually independent.

Theorem 4 implies the following corollary.

COROLLARY 4. *A sequence of random variables $\{X_n\}$ on a probability space $(\Omega, \mathfrak{I}, P)$ assuming two values is a martingale-difference with respect to an increasing sequence of σ-algebras $\mathfrak{I}_0 = (\Omega, \varnothing) \subseteq \mathfrak{I}_1 \subseteq \cdots \subseteq \mathfrak{I}$ if and only if the random variables $\{X_n\}$ are mutually independent.*

Using Theorem 4, one can also show (e.g., [12]) that if $X_1, \ldots, X_n$ are symmetric Bernoulli r.v.'s and $1 \leqslant h \leqslant n-1$, then the r.v.'s $X_i X_{i+h}$, $i = 1, \ldots, n-h$, are mutually independent; the property of mutual independence holds even for more general products of the r.v.'s X_i, $i = 1, \ldots, n$. From this it follows (see [12]) that if X_i, $i = 1, \ldots, n$, are independent symmetric r.v.'s with $E|X_i|^t < \infty$, $i = 1, \ldots, n$, $t > 0$, then the exact constants in Khintchine's inequality for generalized sample autocovariance

$$C_1(t)\left(\sum_{i=1}^{n-h} c_i^2 X_i^2 X_{i+h}^2\right)^{t/2} \leqslant E\left|\sum_{i=1}^{n-h} c_i X_i X_{i+h}\right|^t \leqslant C_2(t)\left(\sum_{i=1}^{n-h} c_i^2 X_i^2 X_{i+h}^2\right)^{t/2}$$

and in its analogues for moving averages of arbitrary order are the same as in Khintchine's inequality for Rademacher functions (concerning the constants in the latter inequality see [18]).

Using Theorem 2 and the definitions of r-independence and strict r-independence, we obtain the following characterizations of r-independent and strictly r-independent random variables assuming two values.

THEOREM 5. *Random variables $X_1, \ldots, X_n$ with one-dimensional distributions $P(X_k = a_k) = p_k$, $P(X_k = b_k) = q_k$, $k = 1, \ldots, n$, are r-independent $(2 \leqslant r \leqslant n)$ if and only if $E\prod_{k=1}^{c} X_{i_k} = \prod_{k=1}^{c} EX_{i_k}$ for all $1 \leqslant i_1 < \cdots < i_c \leqslant n$, $c = 2, \ldots, r$, and, therefore, the joint distribution of $X_1, \ldots, X_n$ has the form*

$$P(X_1=x_1, \ldots, X_n=x_n)$$
$$=\prod_{k=1}^{n} P(X_k=x_k)\left(1+\sum_{c=r+1}^{n} \sum_{1 \leqslant i_1<\cdots<i_c \leqslant n} E\right.$$
$$\left.\times \prod_{k=1}^{c}(X_{i_k}-EX_{i_k})(x_{i_k}-EX_{i_k})/\operatorname{Var} X_{i_k}\right),$$

$x_k \in \{a_k, b_k\}$, $k=1, \ldots, n$.

THEOREM 6. *Random variables $X_1, \ldots, X_n$ assuming two values are strictly r-independent ($2 \leqslant r \leqslant n-1$) if and only if $E\prod_{k=1}^{c} X_{i_k}=\prod_{k=1}^{c} EX_{i_k}$ for all $1 \leqslant i_1<\cdots<i_c \leqslant n$, $c=2, \ldots, r$, but $E\prod_{k=1}^{r+1} X_{j_k} \neq \prod_{k=1}^{r+1} EX_{j_k}$ for some $1 \leqslant j_1<\cdots<j_{r+1} \leqslant n$.*

Applying Theorem 5, we obtain the following

COROLLARY 5. *If $X_1, \ldots, X_n$ are symmetric Bernoulli random variables, then the random variables $\prod_{k=1}^{c} X_{i_k}$, $1 \leqslant i_1<\cdots<i_c \leqslant n$, $c=1, \ldots, n$, are pairwise independent and the random variables $X_1, \ldots, X_n$ and $\prod_{k=1}^{n} X_k$ are almost independent.*

4. THE NUMBER OF ALMOST INDEPENDENT RANDOM VARIABLES ON A DISCRETE PROBABILITY SPACE

It is well known that one can define not more than $[\log_2 N]$ mutually independent nonconstant random variables on a probability space of N points (see, for example, [2])). In [22], H. O. Lancaster showed that at most $N-1$ pairwise independent random variables with nondegenerate distributions can be defined on such a probability space.

Consider a problem closely related to the mentioned ones. Namely, how many nonconstant almost independent random variables can be defined on a probability space of N points?

The answer on this question is given by the following

THEOREM 7. *At most $[\log_2 N]+1$ almost independent random variables with nondegenerate distributions can be defined on a probability space $(\Omega, \mathfrak{I}, P)$ consisting of N points. A maximal set can be obtained for any $N=2^m$, $m=1, 2, \ldots$.*

Proof. Let $X_1, \ldots, X_n$ be almost independent random variables defined on a probability space of N points. Since by definition random variables $X_1, \ldots, X_{n-1}$ are mutually independent, we have that $n-1 \leqslant [\log_2 N]$. Therefore, the estimate given by the theorem is true.

Let us bring an example which shows that the bound $[\log_2 N]+1$ is accessible. Let $N=2^m$, $\Omega=\{(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m): \varepsilon_k = \pm 1, k=1, ..., m\}$, $\mathfrak{I}=2^{\Omega}$, $P\{(\varepsilon_1, \varepsilon_2, ..., \varepsilon_m)\}=2^{-m}$ and let $X_k=2I(\varepsilon_k=1)-1$, $k=1, ..., m$, be a set of m independent symmetric Bernoulli random variables defined on $(\Omega, \mathfrak{I}, P)$. On the strength of Corollary 6, the random variables $X_1, ..., X_m$ and $X_{m+1}=X_1 \cdots X_m$ are almost independent. The proof is complete.

In the case $N=4$ the maximal set of three pairwise independent random variables is given by the random variables X_1, X_2 and X_1X_2 where X_1 and X_2 are independent symmetric Bernoulli random variables and is equivalent to the well known S. Bernstein's example [4] showing that pairwise independence of random variables is not sufficient for their mutual independence.

It is interesting to note that the exact estimate for N given by Theorem 7 is a simple consequence of the definition of almost independence and the estimate for the number of mutually independent random variables on a probability space of N points.

Problem and Hypothesis. What is the maximal number n of $r(n)$-independent random variables with nondegenerate distributions which can be defined on a probability space of N points? According to the results presented above, $n=N-1$ for $r(n)=2$, $n=[\log_2 N]$ for $r(n)=n$, and $n=[\log_2 N]+1$ for $r(n)=n-1$. It seems to be plausible that ($C_m^s = m!/(s!\,(m-s)!)$ for $s \leqslant m$; $C_m^s=0$ for $s>m$)

$$n=\max\left\{m: \sum_{k=0}^{m} C_m^{(m-r(m)+1)k} \leqslant N\right\}.$$

5. NECESSARY CONDITIONS FOR STATIONARITY OF SEQUENCES OF SYMMETRIC BERNOULLI RANDOM VARIABLES

J. B. Robertson and J. M. Womack proved in [27] the following results.

THEOREM 8 [27]. *Suppose that $\{X_n\}$ is a stationary process with the following properties: $\{X_n\}$ is strictly stationary and pairwise independent; $P(X_n=\pm 1)=1/2$, $n=1, 2, ...$. Then $|EX_1X_2X_3| \leqslant 1/2$.*

THEOREM 9 [27]. *If, in addition to the assumptions of Theorem 8, $EX_1X_2X_3=1/2$, then the distribution of X_1, X_2, X_3, X_4 is given by*

$$P(X_1=x_1, X_2=x_2, X_3=x_3, X_4=x_4)=\tfrac{1}{16}\left(1+\tfrac{1}{2}(x_1x_2x_3+x_2x_3x_4)\right),$$

$x_k \in \{-1, 1\}$, $k=1, 2, 3, 4$.

The following theorem holds for an arbitrary stationary process of symmetric Bernoulli random variables.

THEOREM 10. *Suppose that* $\{X_n\}$ *is a stochastic process with the following properties:*

(a) $\{X_n\}$ *is strictly stationary;*

(b) $P(X_n = \pm 1) = 1/2$, $n = 1, 2, \ldots$.

Then $|EX_1 X_2 \cdots X_n + 1/2 EX_1 X_{n+1}| \leqslant 1/2$, $n = 1, 2, \ldots$.

Proof. On the strength of (7) we have that

$$\sum_{c=2}^{n+1} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n+1} EX_{i_1} \cdots X_{i_c} x_{i_1} \cdots x_{i_c} \geqslant -1, \tag{11}$$

$x_k \in \{-1, 1\}$, $k = 1, \ldots, n+1$. Relation (11) implies that

$$\sum_{\substack{x_1, \ldots, x_{n+1} \in \{-1, 1\}, \\ x_1 \cdots x_{n+1} = x_1, \\ x_1 \cdots x_{n+1} = x_{n+1}}} \sum_{c=2}^{n+1} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n+1} EX_{i_1} \cdots X_{i_c} x_{i_1} \cdots x_{i_c} \geqslant -2^{n-1}.$$

But it is not difficult to see that, on the strength of stationarity of $\{X_n\}$,

$$\begin{aligned} &\sum_{\substack{x_1, \ldots, x_{n+1} \in \{-1, 1\}, \\ x_1 \cdots x_{n+1} = x_1, \\ x_1 \cdots x_{n+1} = x_{n+1}}} \sum_{c=2}^{n+1} \sum_{1 \leqslant i_1 < \cdots < i_c \leqslant n+1} EX_{i_1} \cdots X_{i_c} x_{i_1} \cdots x_{i_c} \\ &\quad = 2^{n-1} EX_1 X_{n+1} + 2^{n-1} EX_1 \cdots X_n + 2^{n-1} EX_2 \cdots X_{n+1} \\ &\quad = 2^{n-1} EX_1 X_{n+1} + 2^n EX_1 \cdots X_n. \end{aligned} \tag{12}$$

From (11) and (12) it follows that $EX_1 X_2 \cdots X_n + 1/2 EX_1 X_{n+1} \geqslant -1/2$. The inequality $EX_1 X_2 \cdots X_n + 1/2 EX_1 X_{n+1} \leqslant 1/2$ might be proven in a similar way. ∎

Corollary 6 and Theorem 11 below generalize Theorems 8 and 9 in the case of r-independent symmetric Bernoulli random variables.

COROLLARY 6. *If* $\{X_n\}$ *is a stochastic process satisfying conditions* (a) *and* (b) *and for some* $m > 1$ *the random variables* X_1 *and* X_{m+1} *are independent, then*

$$|EX_1 X_2 \cdots X_m| \leqslant 1/2.$$

THEOREM 11. *If* $\{X_n\}$ *is an r-independent stochastic process satisfying conditions* (a), (b), *and*

(c) $EX_1X_2\cdots X_{r+1}=\gamma\in\{-1/2, 1/2\}$,

then the joint distribution of the random variables $X_1, X_2, \ldots, X_{r+2}$ *is given by*

$$P(X_1=x_1, \ldots, X_{r+2}=x_{r+2})=2^{-(r+2)}(1+\gamma x_1x_2\cdots x_{r+1}+\gamma x_2x_3\cdots x_{r+2}).$$

Proof. From Theorem 5 it follows that

$$P(X_1=x_1, \ldots, X_{r+2}=x_{r+2})$$
$$=2^{-(r+2)}\left(1+\gamma x_1x_2\cdots x_{r+1}+\gamma x_2x_3\cdots x_{r+2}\right.$$
$$\left.+\sum_{k=2}^{r+1}\alpha_{1,\ldots,k-1,k+1,\ldots,r+2}(x_1\cdots x_{r+2}/x_k)+\alpha_{1,\ldots,r+2}x_1\cdots x_{r+2}\right), \tag{13}$$

$x_k\in\{-1, 1\}$, $k=1, \ldots, n$, where

$$1+\gamma x_1x_2\cdots x_{r+1}+\gamma x_2x_3\cdots x_{r+2}+\sum_{k=2}^{r+1}\alpha_{1,\ldots,k-1,k+1,\ldots,r+2}(x_1\cdots x_{r+2}/x_k)$$
$$+\alpha_{1,\ldots,r+2}x_1\cdots x_{r+2}\geqslant 0,$$

$x_k\in\{-1, 1\}$, $k=1, \ldots, n$, and, therefore,

$$\sum_{\substack{x_1,\ldots,x_{r+2}\in\{-1,1\},\\ x_1\cdots x_{r+2}=1,\\ x_1\cdots x_{r+1}=-\operatorname{sign}\gamma\\ x_2\cdots x_{r+2}=-\operatorname{sign}\gamma}}\left(1+\gamma x_1x_2\cdots x_{r+1}+\gamma x_2x_3\cdots x_{r+2}\right.$$
$$\left.+\sum_{k=2}^{r+1}\alpha_{1,\ldots,k-1,k+1,\ldots,r+2}(x_1\cdots x_{r+2}/x_k)+\alpha_{1,\ldots,r+2}x_1\cdots x_{r+2}\right)$$
$$=\sum_{\substack{x_1=x_{r+2}=-\operatorname{sign}\gamma\\ x_2,\ldots,x_{r+1}\in\{-1,1\}\\ x_2\cdots x_{r+1}=1}}\left(1+\gamma x_1x_2, \ldots, x_{r+1}+\gamma x_2x_3\cdots x_{r+2}\right.$$
$$\left.+\sum_{k=2}^{r+1}\alpha_{1,\ldots,k-1,k+1,\ldots,r+2}(x_1\cdots x_{r+2}/x_k)+\alpha_{1,\ldots,r+2}x_1\cdots x_{r+2}\right)$$
$$=2^{r-1}\alpha_{1,\ldots,r+2}\geqslant 0.$$

Therefore,

$$\alpha_{1,\ldots,r+2}\geqslant 0. \tag{14}$$

Similarly,

$$\sum_{\substack{x_1, \ldots, x_{r+2} \in \{-1, 1\}, \\ x_1 \cdots x_{r+2} = -1, \\ x_1 \cdots x_{r+1} = -\operatorname{sign} \gamma \\ x_2 \cdots x_{r+2} = -\operatorname{sign} \gamma}} \Bigg(1 + \gamma x_1 x_2 \cdots x_{r+1} + \gamma x_2 x_3 \cdots x_{r+2}$$

$$+ \sum_{k=2}^{r+1} \alpha_{1, \ldots, k-1, k+1, \ldots, r+2} (x_1 \cdots x_{r+2}/x_k) + \alpha_{1, \ldots, r+2} x_1 \cdots x_{r+2} \Bigg)$$

$$= \sum_{\substack{x_1 = x_{r+2} = \operatorname{sign} \gamma, \\ x_2, \ldots, x_{r+1} \in \{-1, 1\} \\ x_2 \cdots x_{r+1} = -1}} \Bigg(1 + \gamma x_1 x_2 \cdots x_{r+1} + \gamma x_2 x_3 \cdots x_{r+2}$$

$$+ \sum_{k=2}^{r+1} \alpha_{1, \ldots, k-1, k+1, \ldots, r+2} (x_1 \cdots x_{r+2}/x_k) + \alpha_{1, \ldots, r+2} x_1 \cdots x_{r+2} \Bigg)$$

$$= -2^{r-1} \alpha_{1, \ldots, r+2} \geqslant 0,$$

and, therefore,

$$\alpha_{1, \ldots, r+2} \leqslant 0. \tag{15}$$

From (14) and (15) it follows that $\alpha_{1, \ldots, r+2} = 0$. Similarly, one can show that $\alpha_{1, \ldots, k-1, k+1, \ldots, r+2} = 0$ for all $k = 2, \ldots, r+1$. This and (13) imply that

$$P(X_1 = x_1, \ldots, X_{r+2} = x_{r+2}) = 2^{-(r+2)}(1 + \gamma x_1 x_2 \cdots x_{r+1} + \gamma x_2 x_3 \cdots x_{r+2}),$$

$x_k \in \{-1, 1\}$, $k = 1, \ldots, n$,

The proof is complete.

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