Heavy-tailedness and dependence: implications for economic decisions, risk management and financial markets

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Dependence vs. marginals

- **Fundamental problems** in **finance** & **risk management**: **Solution** is **affected** by **both**
  - Properties of **marginal** distributions (**heavy-tailedness**, **skewness**)
  - **Dependence** (**cross**- and **auto-covariances**, **mixing**, **long-range** dependence, **positive** or **negative** dependence, dependence **asymmetry**)

Heavy-tailedness in portfolio choice and value at risk theory

- **Portfolio choice** and **value at risk (VaR) theory**
    - **Marginal** effects under **independence**: degree of **heavy-tailedness** of risks or returns (not extremely heavy-tailed vs. extremely heavy-tailed) $\Longrightarrow$ **Opposite solutions**
    - Certain **dependent risks: diversification pays off** even under **extreme heavy-tailedness** in marginals
    - Different solutions: **Positive vs. negative** dependence

Value at risk (VaR)

- **VaR**
    - **Risk** X; **positive** values = **losses**
    - **Loss probability** q
    - $VaR_q(X) = z : P(X > z) = q$
- **Risks** $X_1, ..., X_n$
- $Z_w = \sum_{i=1}^n w_i X_i$: **return on portfolio** with weights $w = (w_1, ..., w_n)$
- **Problem** of interest:

$$\textbf{Minimize} VaR_q(Z_w)$$

s.t. $w_i \geq 0, \sum_{i=1}^n w_i = 1$

- When **diversification** $\Rightarrow$ **decrease** in **portfolio riskiness** (VaR)?

Problem: Minimize $VaR_q(Z_w)$ s.t. $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$

- $\underline{w} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$: **equal** weights; **most diversified**
- $\overline{w} = \left(1, 0, \ldots, 0\right)$: **one risk**; **least diversified**
- **Normal (light-tailed)** case: $X_1, \ldots, X_n \sim i.i.d.\ \mathcal{N}(0, \sigma)$
  - $Z_{\underline{w}} = \frac{1}{n} \sum_{i=1}^{n} X_i =_d \frac{1}{\sqrt{n}} X_1 = \frac{1}{\sqrt{n}} Z_{\overline{w}}$
  - $VaR_q(Z_{\underline{w}}) = \frac{1}{\sqrt{n}} VaR_q(Z_{\overline{w}}) < VaR_q(Z_{\overline{w}})$
  - $VaR_q(Z_{\underline{w}}) : \searrow$ as $n \nearrow$ (**Diversification** $\nearrow$)
- For all w and fixed n, $VaR_q(Z_{\underline{w}}) \leq VaR_q(Z_w) \leq VaR_q(Z_{\overline{w}})$
- **Optimal choice** is $\underline{w} = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ (**full diversification**)

- **Extremely heavy-tailed (Lévy)** case
- $X_1, \ldots, X_n \sim \; i.i.d.$ **Lévy distribution** with density $f(x) = \frac{\sigma}{\sqrt{2\pi}} x^{-3/2} \exp(-\frac{1}{2x})$ for $x > 0$; $f(x) = 0$ for $x \leq 0$
- **Lévy distribution**: $\alpha-$stable with $\alpha = 1/2$
- **Main properties** of stable distributions:
    - $X \sim S_\alpha(\sigma)$: symmetric **stable** distribution, $\alpha \in (0, 2]$
      CF: $E(e^{ixX}) = exp\Big\{ - \sigma^\alpha |x|^\alpha \Big\}$
        - **Normal** $\mathcal{N}(0, \sigma)$: $\alpha = 2$
        - **Cauchy**: $\alpha = 1$, $f(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$
        - **Lévy**: $\alpha = 1/2$, support $[0, \infty)$
    - **Power law** tails: $P(|X| > x) \approx \frac{C}{x^\alpha}$
      $\alpha \in (0, 2)$: **heavy-tailedness** index
    - **Moments** $E|X|^p$: **finite** iff $p < \alpha$
    - **Infinite first moment** for $\alpha < 1$; **infinite variances** for $\alpha < 2$

- **Extremely heavy-tailed (Lévy)** case

- $X_1, ..., X_n \sim \; i.i.d.$ **Lévy distribution** with density $f(x) = \frac{\sigma}{\sqrt{2\pi}} x^{-3/2} \exp(-\frac{1}{2x})$ for $x > 0$; $f(x) = 0$ for $x \leq 0$
    - $Z_{\underline{w}} = \frac{1}{n} \sum_{i=1}^{n} X_i =_d \left[\sum_{i=1}^{n} (\frac{1}{n})^{1/2} \right]^2 X_1 = nX_1 = nZ_{\overline{w}}$
    - $VaR_q(Z_{\underline{w}}) = nVaR_q(Z_{\overline{w}}) > VaR_q(Z_{\overline{w}})$
    - $VaR_q(Z_{\underline{w}}) : \nearrow$ as $n \nearrow$ (**Diversification** $\nearrow$)

- For all w and fixed n, $VaR_q(Z_{\overline{w}}) \leq VaR_q(Z_w) \leq VaR_q(Z_{\underline{w}})$

- **Optimal choice** is $\overline{w} = \big(1, 0, ..., 0\big) \Rightarrow$ invest **only into one risk**: **no diversification**

- **Heavy-tailedness (marginals)** matters: **diversification** $\Longrightarrow$ **opposite** effects on **portfolio riskiness**

Heavy-tailedness & portfolio diversification

- $X_1, ..., X_n \sim$ i.i.d. $\alpha-$stable with $\alpha > 1$:
  - **Power law** tails: $P(|X| > x) \approx \frac{C}{x^\alpha}$ with $\alpha \in (1, 2)$
  - **Finite first** moments: $E|X_i| < \infty$
- **Diversification** $\implies$ **Decrease** in **riskiness**
- $VaR_q(Z_{\underline{w}}) < VaR_q(Z_w) < VaR_q(Z_{\overline{w}})$
  - **Optimal** portfolio: $\underline{w}$: **most diversified**; **equal** weights
  - **Worst** portfolio: $\overline{w}$: **least diversified**; **one risk**

Heavy-tailedness & portfolio diversification

- $X_1, ..., X_n \sim$ i.i.d. $\alpha-$stable with $\alpha < 1$:
  - **Extremely heavy power law** tails: $P(|X| > x) \approx \frac{C}{x^{\alpha}}$ with $\alpha \in (0, 1)$
  - **Infinite first** moments: $E|X_i| = \infty$
- **Diversification** $\Longrightarrow$ **Increase** in **riskiness!**
- $VaR_q(Z_{\overline{w}}) < VaR_q(Z_w) < VaR_q(Z_{\underline{w}})$
  - **Optimal** portfolio: $\overline{w}$: **least diversified**; **one risk**
  - **Worst** portfolio: $\underline{w}$: **most diversified**; **equal** weights

What happens for intermediate heavy-tails?

- A **very specific** and uninteresting case of $\alpha = 1$
- $X_1, ..., X_n$ i.i.d. **stable with $\alpha = 1$: Cauchy distribution**
  - **Density** $f(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$
  - **Heavy power law** tails: $P(|X| > x) \approx \frac{C}{x}$
  - **Infinite first moment**
- $Z_w = \sum_{i=1}^{n} w_i X_i =_d X_1 \; \forall w = (w_1, ..., w_n) : w_i \geq 0,$
- **Diversification**: **no effect** at all!

A tale of two tails

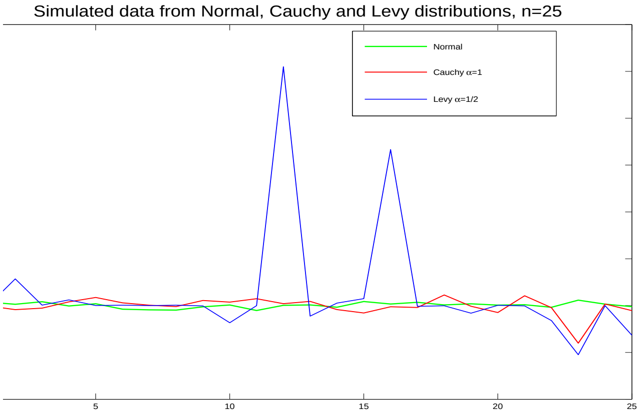


Figure: Heavy-tailed distributions: more extreme observations

A tale of two tails

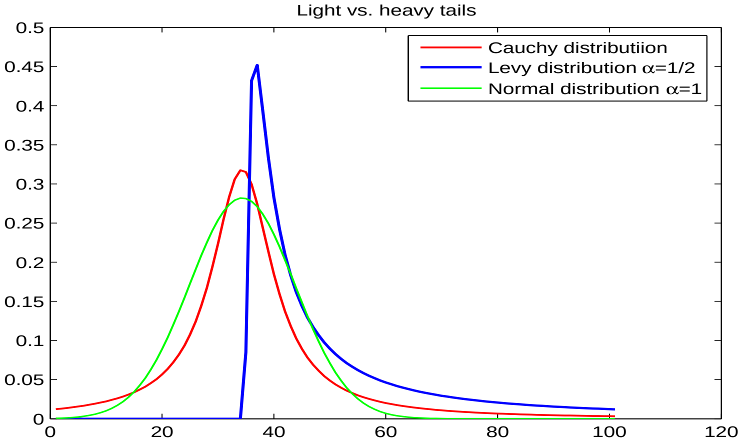


Figure: Tails of Cauchy distributions are heavier than those of normal distributions. Tails of Lévy distributions are heavier than those of Cauchy or normal distributions.

Summary so far: Heavy-tailedness and diversification

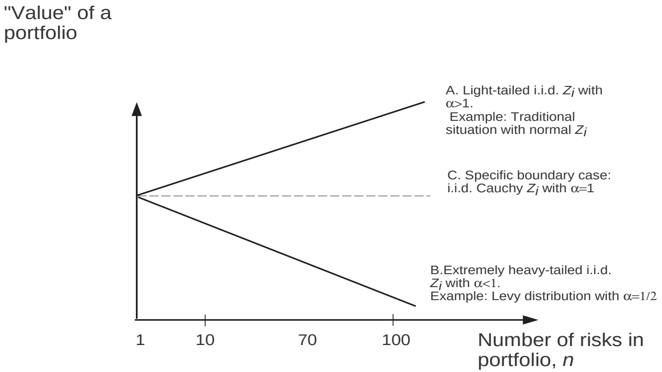


Figure: Value of diversification. A: Traditional case: Diversification pays for light-tailed risks. B: Extremely heavy-tailed risks: value always decreases with diversification. C: Cauchy risks with tail index $\alpha = 1$: diversification has no effect on the portfolio value at risk.

Empirical evidence

Many economic & financial **time series**: **heavy-tailed** distributions with **power law** tails: $P(|X| > x) \approx \frac{C}{x^{\alpha}}$

- **Returns** from **technological innovations**: $\alpha << 1$ (infinite first moment)
- **Loss** distributions of a number of **operational risks**: $\alpha < 1$
- **Economic losses** from **earthquakes**: $\alpha \in [0.6, 1.5]$, interval contains the threshold 1 (calculations based on seismic theory)
- **Economic losses** from **hurricanes**: $\alpha \approx 1.56$; $\alpha \approx 2.49$
- Profit in **motion pictures**: $1 < \alpha < 2$ (finite mean, infinite variance)
- **Firm sizes**, **sizes** of largest **mutual funds**, **city sizes**: $\alpha \approx 1$
- **Returns** on **many stocks** & **stock indices**: $\alpha \in (2, 4)$ (finite variance, infinite fourth moment)

Which world ($\alpha > 1$ or $\alpha < 1$) we are in?

- One may argue that all the economic, financial and insurance variables observed are **bounded**

- Framework of **s**table distributions: **robust** to the concerns (Ibragimov and Walden, 2007)

- Same VaR results: **valid** for **large** class of distributions with **bounded** support

- Diversification may fail if **the number of risks** available is small comparing to the **length** of support of risks

- **Contrast with unbounded** case:

  Number of risks available enters equation

Non-diversification for bounded risks

- Risk Z

- **Value at risk and safety-first**
  - Roy's (1952) **safety-first**:

$$P(Z > z) \rightarrow \min$$

  z: **disaster level**

  - **Value at risk**:

$$VaR_q[Z] \rightarrow \min$$

  q : **loss probability**

Non-diversification for bounded risks

- Start with, as before, i.i.d. **extremely heavy-tailed** X_i with Lévy distributions with $\alpha = 1/2$ or any stable distribution with $\alpha < 1$ (recall that **diversification fails** for X_i's)

- Define A-**truncated versions** of Z_i's:
  $Y_i(A) = X_i I(|X_i| \leq A)$

  - $Y_i(A) = X_i$ if $|X_i| \leq A$
  - $Y_i(A) = 0$ if $|X_i| > A$

- $Z_w = \sum_{i=1}^n w_i X_i$: return on portfolio of **unbounded risks** $X_1, ..., X_n$

- $Y_w(A) = \sum_{i=1}^n w_i Y_i(A)$: return on portfolio of **bounded risks** $Y_1(A), ..., Y_n(A)$

- Risk holder **problem**: $\min P\big(\sum_{i=1}^n w_i Y_i(A) > z\big)$ **(Bounded)** (**disaster probability**)

(Non-)diversification for bounded risks

- $\forall n \geq 2$ and disaster level $z > 0$

  $\exists$ sufficiently large threshold $A = A(n, z)$ (depends on n, z):

- Return on portfolio with **equal weights** $\underline{w} = (1/n, 1/n, ..., 1/n)$ (fully **diversified portfolio**)

  $$Y_{\underline{w}}(A) = \sum_{i=1}^{n} \frac{1}{n} Y_i(A) :$$

  more risky than of **only one** risk $Y_1(A)$

- Comparisons of **probabilities of disaster**:

$$\underbrace{P\Big(\sum_{i=1}^{n} \frac{1}{n} Y_i(A) > z\Big)}_{\text{Disaster probability for diversified portfolio}} \qquad > \qquad \underbrace{P\big(Y_1(A) > z\big)}_{\text{... for undiversified portfolio}}$$

(Non-)diversification for bounded risks

Simple limiting argument

- We know diversification fails for X_i :

$$\underbrace{P\Big(\sum_{i=1}^{n}\frac{1}{n}X_i > z\Big)}_{\text{Disaster probability for diversified portfolio}} \qquad > \qquad \underbrace{P\big(X_1 > z\big)}_{\text{... for undiversified portfolio}}$$

- $Y_i(A) = X_i I(|X_i| \leq A)$: truncations of X_i's
- $Y_i(A) \to_d X_i$ as $A \to \infty$
- Starting with some **threshold** A, the inequalities will hold for **bounded** $Y_i(A)$

$$\underbrace{P\Big(\sum_{i=1}^{n}\frac{1}{n}Y_i(A) > z\Big)}_{\text{Disaster probability for diversified portfolio}} \qquad > \qquad \underbrace{P\big(Y_1(A) > z\big)}_{\text{... for undiversified portfolio}}$$

Summary so far: Diversification for heavy-tailed and bounded distributions

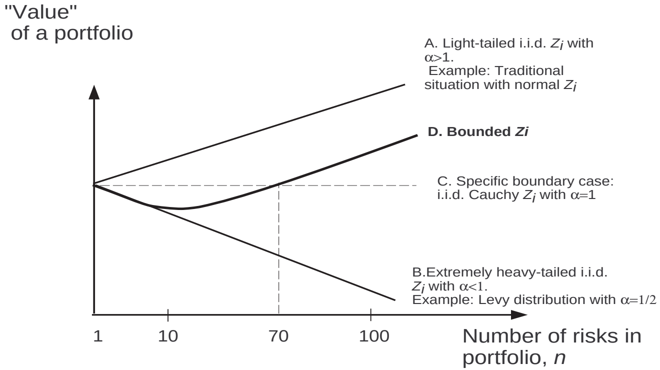


Figure: $N = 10$ risks/insurer; $M = 7$ insurers

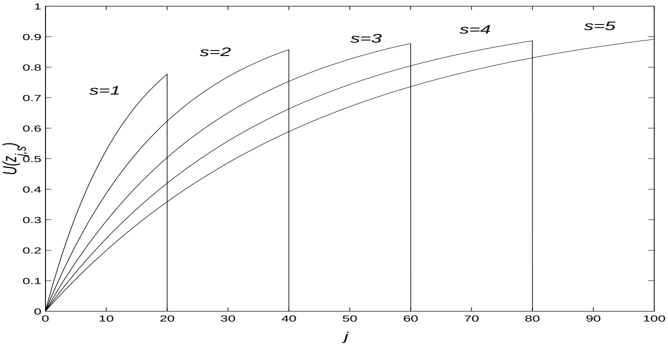
- D: Individual/non-diversification corners vs insurer and reinsurer equilibrium

Implications for markets for catastrophic risks

- **Equilibria in re-insurance markets for catastrophe risks** (Ibragimov, Jaffee and Walden, 2007)
    - **A diversification equilibrium** with **full risk pooling** for normally distributed (**light-tailed**) **risks**
    - **No risk pooling & no insurance or reinsurance activity** (**market collapse**) for **extremely heavy-tailed cat risks**
    - **Intermediate cases (heavy tails):** both
        - **Diversification equilibria**, in which **insurers offer catastrophe coverage** and **reinsure their risks**
        - **Non-diversification equilibria** with **no insurance or re-insurance**
        - **A coordination problem** must be solved to **shift from the bad to the good equilibrium**

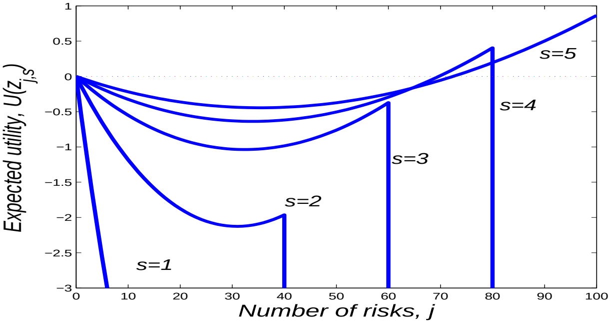
          Government regulations or **well functioning capital markets**

1st example: full risk pooling with normally distributed risks



Assume:	Results:
$1 \le s \le M$ $(= 5)$ insurers N $(= 20)$ risks/insurer $1 \le j \le Ns$ total risks i.i.d. normal X_i CARA utility, Unlimited liability $z_{j,s} = \left(\sum_{i=1}^{j} X_i\right)/s$	If $M - 1$ insurers are pooling, so will Mth If no insurers pool, each still has N risks

2nd example: Bernoulli-Lévy distribution with limited liability



Assume:

Limited liability:

maximum loss ($k = 80$)

$M = 5$ insurers

N $(= 20)$ max risks/insurer

$u(x) = (x + k)^{3/4}$

$z_{j,s} = \left(\sum_{i=1}^{j} X_i\right)/s$

Results:

If insurers can coordinate, they can reach

$MN = 100$ reinsurance equilibrium

But if not, each insurer

reverts to the $N = 0$ corner

Two-stage diversification game: assumptions

- **Stage 1** assumptions
  - There are at least $M = 2$ **insurers**
  - **Each insurer** can take on up to N i.i.d. **risks**
  - Insurers have **limited liability** up to k losses, $k \in (0, \infty]$
  - Insurers have **identical expected utility** (risk averse)
  - **Insurers taking on risks** can also **reinsure** (pool risks).
- **Stage 2** assumptions
  - All insurers **know** the **Stage 1 decisions**
  - **Insurers** with **no Stage 1 risks can** still **join reinsurance**
- The **two stages** represent the **coordination problem**

Two-stage diversification game: equilibrium classifications

- A **diversification equilibrium** is an equilibrium in which **insurance against all risks** in X is **offered**, i.e., Total number of risk insured $= NM$ (M insurers, N risks/insurer)

- A **diversification equilibrium** is **risk sharing** if **all risk insured** is **pooled** in the **reinsurance** market

- A **non-diversification equilibrium** is an equilibrium, in which **no insurance against risk** is offered, i.e., $N = 0$

- A **non-diversification trap** exists if both a **non-diversification** and a **risk sharing equilibria** exist

- A **genuine non-diversification trap** requires there exist a M_0, such that for all $M \geq M_0$, there is still a **non-diversification trap**

Results for markets for catastrophic risks

- **Absence** of **non-diversification traps**:
  - 1st example assumptions (light tails, normal) ⇒ no non-diversification trap
  - **Mean-variance** model ⇒ no non-diversification trap
  - **2nd moment finite**, **no limited liability** ⇒ **no genuine non-diversification traps**
- **Existence** of **non-diversification traps**:
  - 2nd example assumptions ⇒ **genuine diversification trap**
  - **Genuine non-diversification** traps can occur only with **fat tails** (infinite 2nd moments) and **limited liability**

Implications for markets for catastrophic risks

- **Catastrophic risks** have many **features favorable** to the **provision of insurance**
  - Generally **independent** over **risk types** and **geography**
  - **Few issues** of **asymmetric information** at the risk level
  - So a **complete failure of these markets** is **puzzling**
- We have shown that **market failures** (**non-diversification traps**) may arise when risks are **fat-tailed** and there is **limited liability**
  - **Diversification** may **not** be **beneficial** for the **single insurer**, although a **full reinsurance equilibrium may exist**.
  - **Government** programs (or **diversified equity** owners) may **allow** the **system to reach** the **full diversification** outcome

From independence to dependence through copulas

Common shock models and diversification

- $\alpha \in (0, 2]$: Models with **common shocks** $(X_1, ..., X_n) = (RU_1, ..., RU_n)$
  - i.i.d. U_i i.i.d. stable with heavy-tailedness index $\alpha \in (0, 2]$
  - $R \geq 0$: positive r.v. independent of U_i; **common shock** affecting all X_i (political, macroeconomic)
  - Exhibit both **dependence** & **heavy-tailedness** or skewness in **marginals**
    - $\alpha < 1$: **Infinite** marginal **first moments**
    - $\alpha > 1$: $E|X_i|^p < \infty$ if $ER^p < \infty$ and $E|U_i|^p < \infty$ $ER = \infty \implies$ **infinite first moments**

Common shock models and diversification

- $\alpha \in (0,2]$: Models with **common shocks** $(X_1, ..., X_n) = (RU_1, ..., RU_n)$
  - i.i.d. U_i i.i.d. stable with heavy-tailedness index $\alpha \in (0,2]$
  - $R \geq 0$: positive r.v. independent of U_i; **common shock** affecting all X_i (political, macroeconomic)
- $\alpha > 1$: **Diversification** $\Longrightarrow$ **decrease** in riskiness
- $\alpha < 1$: **Diversification** $\Longrightarrow$ **increase** in riskiness
- Both **dependence** & **heavy-tailedness** matter: **First moments** can be **infinite** in **both** worlds!

Dependence matters: Extreme examples

- **Minimize** $VaR_q(w_1X_1 + w_2X_2)$ s.t. $w_1, w_2 \geq 0,\ w_1 + w_2 = 1$
- **Independence**:
  - **Optimal** portfolio: $(\tilde{w}_1, \tilde{w}_2) = (\frac{1}{2}, \frac{1}{2})$ (**diversified**) if $\alpha > 1$ (**not extremely heavy-tailed**, **finite** means)
  - $(\tilde{w}_1, \tilde{w}_2) = (1, 0)$ (**not diversified**, **one risk**) if $\alpha < 1$ (**extremely heavy-tailed**, **infinite** means)

Dependence matters: Extreme examples

- Extreme **positive dependence**: $X_1 = X_2$ (a.s.) **comonotonic risks**
  - $VaR_q(w_1X_1 + w_2X_2) = VaR_q(X_1)\ \forall w$
  - **Diversification**: **no effect** at all (similar to Cauchy) **regardless** of **heavy-tailedness**
- Extreme **negative dependence** $X_1 = -X_2$ (a.s.) **countermonotonic risks**
  - $VaR_q(w_1X_1 + w_2X_2) = (w_1 - w_2)VaR_q(X_1)$
  - Optimal portfolio: $(\tilde{w}_1, \tilde{w}_2) = (1, 0)$ (**not diversified**, **one risk**) **regardless** of **heavy-tailedness**
- Optimal **portfolio choice**: affected by **both dependence** & properties of **marginals**

What is wrong with variances & correlations?

- **Problematic** in a number of real-world settings

- Work well under **multivariate normality** and, more generally, **elliptic** distributions (affine transformations of **spherical**)

  Departures from **normality**: **typical** for financial & insurance data
- **Heavy-tailedness** problem
  - **Defined** for risks and returns with **finite second** moments: $Er_t^2 < \infty$
  - A number of time series in finance & insurance: **infinite variances** and even **means**!
  - Reliable **estimation**: **problematic** if **fourth moments** are **infinite**

    Many **risks** & **returns** on financial assets: **tail index** $\alpha \in (2, 4) \Longrightarrow$ Finite variances but **infinite fourth moments**

What is wrong with variances & correlations?

- **Symmetric measures** of **dependence & risk**
    - $\rho(X, Y) = \rho \Longrightarrow$ **Cannot tell** whether **markets** X & Y are more **likely** to **crash together** or to **boom together**

Dependence in financial & insurance markets: **typically asymmetric**

- **Crashes** are more **likely** to **occur together**
- **Returns & exchange rates** typically exhibit **greater correlation & dependence** during market **downturns** than market **upturns**
- **Good** story: **individual**; **bad** story: **worldwide**

- **Correlation**: measure of **linear dependence**
  - $\rho(X, Y) = 0 \nRightarrow$ **independence**
  - **Not invariant** under **transformations** of risks: $\rho(X, Y) \neq \rho(f(X), f(Y))$ for nonlinear increasing f
    - **Returns** in financial markets: **uncorrelated** $Corr(r_t, r_{t+h}) \approx 0$
    - **Long-range dependence** in simple **nonlinear functions** of r_t: $Corr(r_t^2, r_{t+h}^2) \approx \frac{C}{h^\beta}$, $\beta \in [0.2, 0.4]$; **decay** to zero **slowly** as $h \longrightarrow \infty$
    - **Similar** patterns: **Other powers**: $Corr[|r_t|^p, |r_{t+h}|^p]$ & **nonlinear functions**: $Corr[\log(|r_t|), \log(|r_{t+h}|)]$
    - **Volatility clustering**: **large** price **variations** are likely to be **followed** by **large** price **variations**
    - Simple **uncorrelatedness does not account** for nonlinear **dependence**
- **Bivariate measure** of **dependence**
  - **Pairwise** independence $\nRightarrow$ **Joint** independence

Copulas and dependence

- **Main idea**: **separate** effects of **dependence** from effects of **marginals**
    - What **matters** more in **portfolio choice**: **heavy-tailedness** & **skewness** or (positive or negative) **dependence**?
- **Copulas**: **functions** that **join together marginal** cdf's to form **multidimensional** cdf

Copulas and dependence

- **Sklar's theorem**
- **Risks** X, Y:
  - **Joint cdf** $H_{XY}(x,y) = P(X \leq x, Y \leq y)$: affected by **dependence** and by **marginal** cdf's $F_X(x) = P(X \leq x)$ and $G_Y(x) = P(Y \leq y)$
  - $C_{XY}(u,v)$: **copula** of X, Y:

$$H_{XY}(x,y) = \underbrace{C_{XY}}_{\textbf{dependence}} \left(\underbrace{F_X(x), G_Y(y)}_{\textbf{marginals}} \right)$$

- **Similar** definition: **arbitrary number** of **risks** $X_1, \ldots, X_n$
- F_X, G_Y : properties of **marginals**: **heavy-tailedness**, **skewness**, **moments**, **range**
- C_{XY}: captures **all dependence** between risks X and Y

Copulas: Main properties

Advantages:

- **Exists** for **any risks** (**correlation**: finiteness of **second moments**)
- Characterizes **all dependence** properties
- **Flexibility** in **dependence modeling**
  - **Asymmetric** dependence: **Crashes** vs. **booms**
  - **Positive** vs. **negative** dependence
  - **Independence**: **Nested** as a particular case: **Product** copula, particular values of **parameter(s)**
  - **Extreme** dependence: $X = Y$ or $X = -Y \Leftrightarrow$ **extreme copulas**; **dependence** in C_{XY} varies in **between**

Copulas: Main properties

- F_X, G_Y : **continuous** $\Rightarrow$ C_{XY} : **unique**
  - $C_{XY}(u,v) = H_{XY}\big(F_X^{-1}(u), G_Y^{-1}(v)\big)$
  - $F^{-1}(u) = \inf\{t : F(t) \geq u\}$
    (F_X, G_Y : nondecreasing but may be constant on some intervals)
  - F_X^{-1}, G_Y^{-1} : usual **inverses** for strictly increasing F_X & G_Y

Copulas: cdf's with uniform marginals

- $C(u,v)$: **copula** iff $C =$ **cdf** of two (dependent) $\mathcal{U}nif(0,1)$ r.v.'s: $C(u,v) = P(U \leq u, V \leq v)$
- **Equivalent** definition: $C(u,v)$: **copula** if
  - $C(u,v)$: nondecreasing in u & v
  - $C(1,v) = v$, $C(u,1) = u$
  - $u_1 \leq u_2$, $v_1 \leq v_2 \Longrightarrow$

    $$C(u_2,v_2) + C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) \geq 0$$

    $C-$**volume** of any rectangle: **nonnegative**
    C induces a **probability measure** on $[0,1]^2$

Copulas: Main properties

- X, Y: **independent** $\Leftrightarrow H_{XY}(x,y) = F_X(x)G_Y(y) \Leftrightarrow C_{UV}(u,v) = uv$
  - **Minimal** dependence
- **Invariant** under **increasing transformations**:
  - $f(X), g(Y)$, **increasing** f, g: **same copula** C_{XY}
  - **Not true** for **correlation**: $\rho\big[f(X), g(Y)\big] \neq \rho(X,Y)$
  - Take $f(x) = F_X^{-1}(x)$, $g(x) = G_Y^{-1}(x)$
    - $f(X), g(Y) \sim \mathcal{U}nif(0,1)$ & **same copula** $C_{XY}(u,v)$
    - But **all moments** of $f(X), g(Y)$ **exist**: **solution** to **heavy-tailedness** problem!
    - Contrast to **correlation**: can deal with dependence under **infinite variances**

Copulas: Examples

- **Product** (**independence**) copula: $C(u,v) = uv$
- **Clayton:** $C^{Clayton}(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \ \ \theta > 0$
- **Gumbel:** $C^{Gumbel}(u,v) = \exp\big(-\big[(-ln\ u)^{\theta} + (-ln\ v)^{\theta}\big]\big)^{-1/\theta}$
- **Frank:** $C^{Frank}(u,v) = -\frac{1}{\theta} ln\Big(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\Big)$
- **Eyraud-Farlie-Gumbel-Morgenstern** (**EFGM**) copulas: $C(u,v) = uv(1 + \theta(1-u)(1-v)), -1 \le \theta \le 1$
  - θ : **dependence parameter**

Copulas: construction

Inversion method
$C(u, v) = H\left[F_X^{-1}(u), G_Y^{-1}(v)\right]$

- **Gaussian** copula
    - $H(x, y) = \Phi_\rho(x, y)$: **bivariate normal**, correlation ρ
    - $F_X, G_Y \sim F_{\mathcal{N}}$: standard **univariate normal** cdf
    - $C^{Gaussian}(u, v, \rho) = \Phi_\rho\left[F_{\mathcal{N}}^{-1}(u), F_{\mathcal{N}}^{-1}(u)\right]$
- **Student** t copula
    - $F = T_{\rho,\nu}(x, y)$: **bivariate Student** t, correlation ρ, n degree of freedom
    - $F_X, G_Y \sim t_n$: **univariate Student** cdf with n d.f.'s
    - $C^{Student}(u, v, \rho, n) = T_{\rho,\nu}\left[t_\nu^{-1}(u), t_\nu^{-1}(v)\right]$

Copulas: construction

Archimedean copulas: $C(u, v, \phi) = \phi^{-1}\Big[\phi(u) + \phi(v)\Big]$

ϕ: continuous strictly decreasing function (**generator**)

- $C^{Clayton} : \phi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$
- $C^{Gumbel} : \phi_\theta(t) = (-\ln t)^\theta$
- $C^{Frank} : \phi_\theta(t) = -\ln(\frac{e^{-\theta t}-1}{e^{-\theta}-1})$

Application: Probability of “perfect storm”

- **Important**: Are **crashes** (**booms**) in markets X & Y **likely** to **occur together**?
  - $x = VaR_q(X), y = VaR_q(Y)$: **disaster** levels in markets X & Y
  - **How likely** is the **event** $\{X > x\} \bigcap \{Y > y\}$ (**“perfect storm”**)
  - Need to **characterize** $P(X > x, Y > y)$: **Joint survival** function
  - Note $P(X > x, Y > y) = P(U > q, V > q) = \overline{C}_{XY}(q, q)$
    - $U, V \sim \mathcal{U}(0, 1)$, **joint cdf** $C_{XY}(U, V)$
    - $\overline{C}_{XY}(u, v) = P(U > u, V > v)$: **joint survival** function of U, V
    - **Easy to find** if **copula is known**: $\overline{C}_{XY}(u, v) = 1 - u - v + C(u, v)$

Hoeffding-Fréchet bounds: best & worst dependence

When the **probability** of “**perfect storm**” $P(X > x, Y > y) = \overline{C}_{XY}(q, q)$ **maximal** or **minimal**?

- Answer: **Hoeffding-Fréchet** bounds
  - **Maximal** under **perfect positive** dependence
  - **Minimal** under **perfect negative** dependence
- $C_L(u, v) = \max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v) = C_U(u, v)$
- Same ordering: **Survival functions**; **joint crash** probabilities $(u, v = q)$
  - $\overline{C}_L(u, v) = \max(1 - u - v, 0) \leq \overline{C}(u, v) \leq \min(1 - u, 1 - v) = \overline{C}_U(u, v)$
  - $\overline{C}_L(q, q) = \max(1 - 2q, 0) \leq \overline{C}(q, q) \leq 1 - q = \overline{C}_U(q, q)$

Hoeffding-Fréchet bounds: best & worst dependence

- C_L and C_U: **perfect dependence** copulas
- $(X, Y) \sim C_U(u, v) \Leftrightarrow (X, Y) = (f(Z), g(Z))$, $f \nearrow$, $g \nearrow$, X, Y: **comonotonic**
- $(X, Y) \sim C_L(u, v) \Leftrightarrow (X, Y) = (f(Z), g(Z))$, $f \nearrow$, $g \searrow$, X, Y: **countermonotonic**

Bounds in multivariate case

- $C_L(u_1, u_2, ..., u_n) = \max(u_1 + ... + u_n + 1 - n, 0) \leq C(u_1, ..., u_n) \leq \min(u_1, u_2, ..., u_n) = C_U(u_1, u_2, ..., u_n)$
  - C_U : **copula** $\forall n \geq 2$
  - C_L : **no longer** copula for $n > 2$
  - However, the **lower bound** is **sharp**

Bounds in multivariate case

- Similar bounds: **Survival function (SF)**
  $\overline{C}(u_1, ..., u_n) = P(U_1 > u_1, ..., U_n > u_n)$
  - $x_i = VaR_q(X_i)$: **disaster levels** in markets $i = 1, ..., n$
  - $P_{Crash} = P(U_1 > q, ..., U_n > q) = \overline{C}(q, ..., q)$: **Probability** that **crash occurs** at the same **in all n markets**
  - Probability of a "**perfect storm**"

$$\overline{C}_L(u_1, ..., u_n) = \max(1 - u_1 - ... - u_n, 0) \leq P_{Crash} \leq$$
$$\overline{C}_U(u_1, ..., u_n) = \min(1 - u_1, ..., 1 - u_n)$$

  - $\overline{C}_L$: **SF** of $U_1, ..., U_n \sim C_L$
  - $\overline{C}_U$: **SF** of $U_1, ..., U_n \sim C_U$

Tail dependence

- **Will crash** (**boom**) in market X lead to a **crash** (**boom**) in market Y?
- Are **extreme values** of X & Y (**very large** losses) **likely** to **occur together**?
- **How likely** is $X \nearrow (X \searrow) \Longrightarrow Y \nearrow (Y \searrow)$?
- **Loss probability** $q \to 1- \Longrightarrow VaR_q(X), VaR_q(Y) \to \infty$: **extremely large** values of X & Y
  - Coefficient of **upper tail** dependence: $\lambda_U = \lim_{q\to 1-} P[Y > VaR_q(Y) | X > VaR_q(X)]$
- **Loss probability** $q \to 0+ \Longrightarrow VaR_q(X), VaR_q(Y) \to -\infty$: extremely small values of X & Y
  - Coefficient of **lower tail** dependence: $\lambda_L = \lim_{q\to 0+} P[Y < VaR_q(Y) | X < VaR_q(X)]$

- **Copula** representations $(U, V) \sim C_{XY}(u, v)$
  $\overline{C}_{XY}(u, v) = P(U > u, V > v) = C_{XY}(1-u, 1-v) =$
  $1 - u - v + C_{XY}(u, v)$
  - $\lambda_U = \lim_{q \to 1-} \frac{P(U>q, V>q)}{1-q} = \lim_{q \to 1-} \frac{\overline{C}(q,q)}{1-q}$
  - $\lambda_L = \lim_{q \to 0+} \frac{P(U \leq q, V \leq q)}{q} = \lim_{q \to 0+} \frac{C(q,q)}{q}$
- λ_U, λ_L : **Measures** of **asymptotic** dependence

Different copulas $\Longrightarrow$ different tail dependence

- $C^{Gaussian}(u, v, \rho) : \lambda_U = \lambda_L = 0$ iff $|\rho| < 1$
  Asymptotic independence $\forall \rho : \ |\rho| < 1$

- $C^{Gumbel}(u, v, \theta)$
  $\lambda_L = 0$, $\lambda_U = 2 - 2^{1/\theta}$: **upper tail dependence** provided $\theta \neq 1$

- $C^{Clayton}(u, v, \theta)$
  $\lambda_L = 2^{-1/\theta}$ for $\theta > 0$, $\lambda_U = 0$ for $\theta > 0$: **lower tail dependence** provided $\theta \neq 1$

  - C^{Gumbel}, $C^{Clayton}$: models for **asymmetric** dependence
  - **More dependence** in market **upturns or downturns**

Different copulas $\Longrightarrow$ different tail dependence

- $C^{Student}(u, v, n, \rho)$
  $\lambda_U = \lambda_L = 2P(t_{n+1} > \frac{\sqrt{n+1}\sqrt{1-\rho}}{\sqrt{1+\rho}})$:
  Tail dependence for all $\rho > -1$
  Asymptotic dependence in tails even for zero correlations

- **Archimedean copulas**: $\phi'(0) \neq 0 \Longrightarrow \lambda_U = 0$
  $\phi'(0) = 0 \Longrightarrow \lambda_U = 2 - 2\lim_{s \to 0+} \frac{\phi'(s)}{\phi'(2s)}$
  $\lambda_L = 2\lim_{s \to +\infty} \frac{\phi'(s)}{\phi'(2s)}$

Further applications Bounds on portfolio VaR & expected payoffs & fair prices of contingent claims

- Fix weights $w_1, w_2 \geq 0,\ w_1 + w_2 = 1$
- $X \sim F_X(x),\ Y \sim G_Y(y)$
- Determine **extrema** $VaR_q(Z_w) = VaR_q(w_1 X + w_2 Y)$ over all possible **dependence** structures for X, Y
- What is the **best** (**worst**) VaR portfolio scenario?

Further applications: Bounds on portfolio VaR & expected payoffs & fair prices of contingent claims

- Key: **comonotonic** & **countermonotonic** copulas C_U & C_L
  - $VaR_q(Z_w) \leq VaR_q(L)$
  - Risk L: cdf

$$P(L \leq z) = \sup_{w_1 x + w_2 y = z} C_L\left[F_X(x), G_Y(y)\right] =$$

$$\sup_{w_1 x + w_2 y = z} \max\left[F_X(x) + G_Y(y) - 1, 0\right]$$

Contrasting VaR & correlation

- Note: **Worst case** portfolio VaR $\not\Leftrightarrow$ **comonotonic** $X, Y \sim C_U$ (**maximal correlation** $\rho(X, Y)$)!
- Crucial **difference** from **variance** as measure of risk:
  - $\sigma^2(w_1X + w_2Y) = w_1^2\sigma^2(X) + w_2^2\sigma^2(Y) + 2w_1w_2\rho(X, Y)\sigma^2(X)\sigma^2(Y)$
  - **Maximal** when $\rho(X, Y)$: **maximal**
  - **Breaks down** when **VaR** is used as **measure of riskiness** of portfolio return Z_w!
  - Corollary: **option & contingent claim bounds**
    - $EU(Z_w) \leq EU(L)\ \forall$ increasing U
    - $U(x) = \max(x - K, 0)$: **bounds** on **expected payoffs** & fair **prices** of European call **option**
    - Arbitrary **contingent claims**; **utility** bounds!

Empirical evidence on copula models

- Patton (2004): Parametric estimation of **conditional copula** models for **exchange rates** (Deutsche mark & yen)
- Significant evidence that **dependence** between DM-USD & Yen-USD exchange rates is **asymmetric**
- Strong evidence of a **structural break** in conditional **copula** following the **introduction of euro** in Jan 1999
    - **Tail dependence**: **decreases**
    - Significant **upper tail** dependence **changes** to weak **lower tail** dependence

Empirical evidence on copula models
Patton (2004)

- **Symmetrized Joe-Clayton** copula

$$C^{SJC}(u,v) = \frac{1}{2}\Big(C^{JC}(u,v) + C^{JC}(1-u,1-v) + u + v - 1\Big)$$

$$C^{JC}(u,v) = 1 - \Big[1 - \big([1-(1-u)^{\kappa}]^{-\gamma} +$$

$$[1-(1-v)^{\kappa}]^{-\gamma} - 1\big)^{-1/\gamma}\Big]^{1/\kappa}$$

$$\kappa = 1/\log_2(2-\tau^{U}), \;\; \gamma = -1/\log_2(\tau^{L})$$

  - **Extension** of $C^{Clayton}(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$
  - **Tail dependence** parameters λ_U and λ_L **determine copula**
  - **Symmetric iff** $\lambda_U = \lambda_L$

Empirical evidence on copula models

Patton (2004)

- Parameter estimates (**MLE**):
  - **Pre-Euro**: $\tau^U = 0.359$ (s.e: 0.025), $\tau_L = 0.294$ (s.e.: 0.027)
  - **Post-Euro**: $\tau_U = 0$ (imposed), $\tau_L = 0.093$ (s.e.: 0.038)
- **Structural break**; **reversal** of the **dependence direction** following the **introduction of Euro**

Empirical evidence on copula models

- Hu (2004): **modeling & estimation** of **copula models** for international financial markets using **mixed copulas**
- **Semiparametric** estimation
  - **First stage**: **Nonparametric** estimation of **marginals**
  - **Second stage**: Estimation of **mixed copulas**

Empirical evidence on copula models Hu (2004):

- **Mixed copula** model
  - Idea: account for **asymmetry** in **tail dependence** using a **mixture** of $C^{Gaussian}$ (**no tail dependence** $\lambda_U = \lambda_L = 0$), C^{Gumbel} (**positive** λ^U) and Gumbel survival copula (**positive** λ_L)
  - $C(u, v) = w_1 C^{Gaussian}(u, v, \rho) + w_2 C^{Gumbel}(u, v, \theta_1) + w_3 C^{SGumbel}(u, v, \theta_2)$
    - $C^{Gumbel}(u, v, \theta_1) = \exp\big(- \left[(-ln\ u)^{1/\theta_1} + (-ln\ v)^{1/\theta_1}\right]\big)^{-\theta_1}$
    - $C^{SGumbel}(u, v, \theta_2) = u + v - 1 + \exp\big(- \left[(-ln\ (1-u))^{1/\theta_2} + (-ln\ v)^{1/\theta_2}\right]\big)^{-\theta_2}$

Empirical evidence on copula models
Hu (2004)

- Four **stock market indices**: S&P 500 (US), FTSE 100 (UK), Nikkei 225 (Japan), Hang Seng (HK)
  - Need to **estimate**: one-dimensional **marginals**, individual copula **parameters** (ρ, θ_1, θ_2) and mixture **weights** w_i
  - **Parameters** of C: $\rho, \theta_1, \theta_2, w_1, w_2, w_3$
  - Problem: financial data is **not i.i.d.** over time
    GARCH filter is applied to data on stock indices before empirical distributions are computed

    Alternative: apply **asymptotic theory** for **empirical processes** for **dependent time series** or **copula-based time series**

Hu (2004)

- Estimation results:

  For **all pairs** of stock indices: $w_2 = 0$: **no upper tail** dependence

  - (S& P, FTSE): $w_1 = 0.16$, $w_3 = 0.84$ (some **weight** on **symmetric** $C^{Gaussian}$)
  - (FTSE, Nikkei): $w_1 = 1$, $w_2 = w_3 = 0$ (**symmetric dependence**)
  - **All remaining pairs** (S& P, Nikkei), (S& P, Hang Seng), (FTSE, Hang Seng), (Nikkei, Hang Seng): $w_3 = 1$, $w_1 = w_2 = 0$ (**positive lower tail** dependence!)

    Good story is **individual**, while **bad story** is **worldwide**

Conclusion

- **Fundamental problems** in **economics, finance** & **risk management**:
    - **Properties** of **marginal** distributions
    - **Dependence**

- **Portfolio choice** in **value at risk** theory: **Key factors**:
    - Index of heavy-tailedness α, **index** of $\alpha-$symmetric distributions under **dependence**
        - $\alpha > 1$: diversification typically **preferable**
        - $\alpha < 1$: diversification typically **fails**
    - **Length** of distribution **support** and number of risks available **Diversification: Suboptimal** if **support large** compared to **number** of risks

Conclusion

Implications for re-insurance markets for catastrophe risks

- **A diversification equilibrium** with **full risk pooling** for normally distributed (**light-tailed**) **risks**
- **No risk pooling & no insurance or reinsurance activity** (**market collapse**) for **extremely heavy-tailed cat risks**
- **Intermediate cases (heavy tails):** both
  - **Diversification equilibria**, in which **insurers offer catastrophe coverage** and **reinsure their risks**
  - **Non-diversification equilibria** with **no insurance or re-insurance**
  - **A coordination problem** must be solved to **shift from the bad to the good equilibrium**

    Government regulations or **well functioning capital markets**

Conclusion

- **Copulas**: **convenient tool** to **account** for **both effects** and to **separate** one from the other
- **Separation** of **marginal effects** from **dependence**: **Key** to
    - **Reduction** of problems under **dependence** to **independent case**
    - **Sharp bounds** on **value at risk** for financial portfolios & contingent claim (option) **prices**
    - **Estimation** of **co-movements** in financial & insurance markets