

**ON FUNCTIONS NOT PRESERVING MAJORIZATION PRE-ORDERING
AND THEIR APPLICATIONS**

Running title: Functions not preserving majorization ordering

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ABSTRACT

It is well-known that Schur-convex functions $\phi : \mathbf{R} \rightarrow \mathbf{R}$ satisfy the inequalities $\phi(s/n, s/n, \dots, s/n) \leq \phi(y_1, y_2, \dots, y_n) \leq \phi(s, 0, \dots, 0)$ for all $y \in y_1, \dots, y_n \geq 0, \sum_{i=1}^n y_i = s$. This paper deals with the analysis of functions ϕ that are not Schur-convex but nevertheless satisfy the inequalities of the above form. We obtain sufficient conditions for a wide class of such functions to have extrema at the points $(s, 0, \dots, 0), (s/2, s/2, 0, \dots, 0), \dots, (s/(n-1), s/(n-1), \dots, s/(n-1), 0), (s/n, s/n, \dots, s/n)$. We further derive characterizations of a class of functions ϕ that preserve a pre-ordering implying majorization and thus satisfy the above bounds.

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1 Majorization pre-ordering and Schur-convex functions

Many mathematical inequalities can be represented in the form

$$\phi(s/n, s/n, \dots, s/n) \leq \phi(y_1, y_2, \dots, y_n) \leq \phi(s, 0, \dots, 0), \quad (1)$$

where $\phi : A \rightarrow \mathbf{R}$, $A \subseteq \mathbf{R}$, is some function and $s = \sum_{i=1}^n y_i$. Such relations are particularly important in the analysis of income inequality measures and value at risk models (see Section 13.F in Marshall & Olkin 1979, Ibragimov 2005, Ibragimov & Walden 2007, and references therein)

Powerful tools for obtaining inequalities (1) are given by majorization theory presented in detail in Marshall & Olkin (1979).

Definition 1 (Marshall & Olkin 1979) *A vector $x \in \mathbf{R}^N$ is said to be majorized by a vector $y \in \mathbf{R}^N$, written $x \prec y$, if $\sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]}$, $k = 1, \dots, N - 1$, and $\sum_{i=1}^N x_{[i]} = \sum_{i=1}^N y_{[i]}$, where $x_{[1]} \geq \dots \geq x_{[N]}$ and $y_{[1]} \geq \dots \geq y_{[N]}$ denote the components of x and y in non-increasing order.*

The majorization relation $x \prec y$ implies that the components of the vector x are less diverse than those of y (see Marshall & Olkin, 1979, and the discussion in Ibragimov, 2005). In this context, it is easy to see that the following relations hold:

$$(s/n, s/n, \dots, s/n) \prec (y_1, y_2, \dots, y_n) \prec (s, 0, \dots, 0) \quad (2)$$

whenever $y_i \geq 0$, $i = 1, \dots, n$, and $\sum_{i=1}^n y_i = s$.

Definition 2 (Marshall & Olkin 1979) *A function $\phi : A \rightarrow \mathbf{R}$ defined on $A \subseteq \mathbf{R}^N$ is called Schur-convex on A if ϕ preserves the majorization pre-ordering \prec on A , that is, $(x \prec y) \implies (\phi(x) \leq \phi(y))$ for all $x, y \in A$.*

Ch. 3 in Marshall & Olkin (1979) provides necessary and sufficient conditions for a function $\phi : A \rightarrow \mathbf{R}$ to be Schur-convex on A . From (2) it follows that Schur-convex functions ϕ satisfy inequalities (1) provided $(s/n, s/n, \dots, s/n), (s, 0, \dots, 0) \in A$.

However, there are a number of functions ϕ that do not preserve the majorization pre-ordering \prec but nevertheless satisfy (1). For instance, these properties are exhibited by the function

$$\phi(y_1, y_2, y_3) = 4y_1^3 + 4y_2^3 + 4y_3^3 + 15y_1y_2y_3.$$

Indeed, for all nonnegative y_1, y_2, y_3 such that $y_1 + y_2 + y_3 = s$ one has $\phi(s/3, s/3, s/3) \leq \phi(y_1, y_2, y_3) \leq \phi(s, 0, 0)$. At the same time, as is easy to check, the function ϕ does not preserve the pre-ordering \prec on the set $A = \{y \in \mathbf{R}^3 : y_1, y_2, y_3 \geq 0, y_1 + y_2 + y_3 = s\}$.

The aim of these note is to provide some methods for obtaining inequalities of type (1) for a wide class of functions ϕ that do not preserve the majorization relation \prec . The results obtained in the paper generalize characterizations of Schur-convex functions in Marshall & Olkin (1979).

2 Extrema of a class of functions defined on a simplex

Let $s > 0$, $n \geq 2$. Denote

$$A_n = \{y \in \mathbf{R}^n : \sum_{i=1}^n y_i = s, y_i \geq 0, i = 1, \dots, n\},$$

$$L_n = \{(s, 0, \dots, 0), (s/2, s/2, 0, \dots, 0), \dots, (s/(n-1), s/(n-1), \dots, s/(n-1), 0), (s/n, s/n, \dots, s/n)\}.$$

Further, denote by \mathcal{P}_n the set of $n \times n$ permutation matrices.

Definition 3 A function $\phi : \mathbf{R}^n \rightarrow \mathbf{R}$ is called symmetric if $\phi(y) = \phi(yP)$ for all $P \in \mathcal{P}_n$.

Theorem 1 allows one to reduce the problem of determining the extrema of a wide class of functions on the simplex A_n to the problem of comparing the values of these functions on the set L_n .

Theorem 1 Let $\phi : A_n \rightarrow \mathbf{R}$ be a continuous symmetric function such that

$$\max_{y_1 \in [0,1]} \phi(y_1, t - y_1, y_3, \dots, y_n) = \max\{\phi(t, 0, y_3, \dots, y_n), \phi(t/2, t/2, y_3, \dots, y_n)\} \quad (3)$$

for all fixed $t, y_3, \dots, y_n \geq 0$ with $t + \sum_{i=3}^n y_i = s$. Then

$$\max_{y \in A_n} \phi(y) = \max_{y \in L_n} \phi(y), \quad (4)$$

$$\min_{y \in A_n} \phi(y) = \min_{y \in L_n} \phi(y). \quad (5)$$

Remark 1 *It is not difficult to see that condition (3) can also be formulated as follows: Let l be a segment in A_n with the endpoints on the border of A_n that is parallel to the coordinate plane Oy_1y_2 . Then the maximum of $\phi(y)$ on l is attained either at the middle of the segment l or at one of its endpoints.*

Proof of Theorem 1. Let $A_n^0 = A_n$ and let, for $m = 1, \dots, n$, A_n^m be the projection of the simplex A_n on the hyperplane $x_m = 0$. Denote by L_{ij}^m the set of all segments in A_n^m with the endpoints on the border of A_n that are parallel to the coordinate plane Oy_iy_j , $m = 0, \dots, n$, $i, j = 1, \dots, n$, $i \neq j$. For a segment $l \subset \mathbf{R}^n$ with the endpoints a, b , set $M(l) = \{a, b, (a + b)/2\}$. Denote by D_n^m , $m = 0, \dots, n$, the sets

$$D_n^m = \bigcup_{i,j=1,\dots,n, i \neq j} \bigcup_{l \in L_{ij}^m} M(l).$$

The symmetry of the function $\phi(y)$ and condition (3), together with Remark 1, imply that

$$\max_{y \in A_n} \phi(y) = \max_{y \in D_n^0} \phi(y).$$

Let us show that

$$D_n^0 = \{yP : y \in L_n, P \in \mathcal{P}_n\} \quad (6)$$

using induction on n . Representation (6) is evident for $n = 2$. Let $n = k > 2$ and suppose that (6) holds for $n = k - 1$. It is not difficult to see that

$$D_k^0 = \bigcup_{m=1}^k D_k^m \cup \{1/k, \dots, 1/k\}. \quad (7)$$

From the representation $A_k^m = \{y \in \mathbf{R}^k : \sum_{i=1, i \neq m}^k y_i = s, y_m = 0, y_i \geq 0, i \neq m\}$, $m = 1, \dots, k$, and the induction hypothesis it follows that

$$D_k^m = \{(y_1, \dots, y_{m-1}, 0, y_{m+1}, \dots, y_k) : (y_1, \dots, y_{m-1}, y_{m+1}, \dots, y_k) = yP, y \in L_{k-1}, P \in \mathcal{P}_{k-1}\}. \quad (8)$$

Relations (7) and (8) together imply that (6) holds for $n = k$. Thus, (6) is indeed true and

$$\max_{y \in A_n} \phi(y) = \max_{y \in L_n, P \in \mathcal{P}_n} \phi(y).$$

Since the function $\phi(y)$ is symmetric, this implies that (4) indeed holds. Relation (5) follows in a similar way, by considering the function $-\phi$. ■

Theorem 1 immediately implies Corollaries 1 and 2.

Corollary 1 Let $\phi : A_n \rightarrow \mathbf{R}$ be a symmetric function such that the function $\psi(x_1) = \phi(x_1, t - x_1, x_3, \dots, x_n)$ is monotone in x_1 on the segment $t/2 \leq x_1 \leq t$ for all fixed $t, x_3, \dots, x_n \geq 0$ such that $t + \sum_{i=3}^n y_i = s$. Then ϕ satisfies (4) and (5).

Remark 2 As follows from Proposition 3.A.2.b in Marshall & Olkin (1979), a function ϕ preserves the majorization pre-ordering \prec on A_n (that is, the function ϕ is Schur-convex on A_n) if and only if it is symmetric and the function $\psi(x_1)$ in Corollary 1 is non-decreasing in $t/2 \leq x_1 \leq t$.

Corollary 2 Let $\phi : A_n \rightarrow \mathbf{R}$ be a symmetric function that is continuous on A_n and is continuously differentiable in the interior of A_n . Set $\phi_{(k)}(y) = \partial\phi(y)/\partial y_k$. If the function $\phi_{(1)}(y_1, y_2, y_3^0, \dots, y_n^0) - \phi_{(2)}(y_1, y_2, y_3^0, \dots, y_n^0)$ preserves the sign in the domain $y_1 \geq y_2, y_1 + y_2 + \sum_{i=3}^n y_i^0 = s$ for all fixed $y_3^0, \dots, y_n^0 \geq 0$ such that $\sum_{i=3}^n y_i^0 \leq s$, then ϕ satisfies (4) and (5).

Corollary 2 allows one to obtain simple proofs of several new algebraic inequalities.

Corollary 3 Let $n \geq 2$. For all nonnegative y_1, y_2, \dots, y_n such that $\sum_{i=1}^n y_i = 1$ and $c \in \mathbf{R}$ the following inequalities hold:

$$\min(1/(n-1), 1/n + c/n^n) \leq \sum_{i=1}^n y_i^2 + c \prod_{i=1}^n y_i \leq \max(1, 1/n + c/n^n), \quad (9)$$

$$\min(1/(n-1)^2, 1/n^2 + c/n^n) \leq \sum_{i=1}^n y_i^3 + c \prod_{i=1}^n y_i \leq \max(1, 1/n^2 + c/n^n). \quad (10)$$

Proof of Corollary 3. Let $\phi(y_1, \dots, y_n) = \sum_{i=1}^n y_i^3 + c \prod_{i=1}^n y_i$. Then

$$\phi_{(1)}(y_1, y_2, y_3^0, \dots, y_n^0) - \phi_{(2)}(y_1, y_2, y_3^0, \dots, y_n^0) = (y_1 - y_2)(3(1 - \sum_{i=3}^n y_i^0) - c \prod_{i=3}^n y_i^0)$$

in the domain $y_1 + y_2 + \sum_{i=3}^n y_i^0 = 1$. Using Corollary 2, we obtain inequality (9). Inequality (10) follows in a similar way. ■

3 Characterization of functions preserving one vector pre-ordering

Relation (2) implies that the vectors $(s/n, s/n, \dots, s/n)$ and $(s, 0, \dots, 0)$ are, respectively, the minimal and maximal elements with respect to the pre-ordering \prec on the simplex A_n . However, one can define other pre-orderings \preceq on A_n for which

$$(s/n, s/n, \dots, s/n) \preceq (y_1, y_2, \dots, y_n) \preceq (s, 0, \dots, 0) \quad (11)$$

for all $(y_1, y_2, \dots, y_n) \in A_n$. Of particular interest are pre-orderings \preceq that have the property $(x \preceq y) \Rightarrow (x \prec y)$ for all $x, y \in A_n$. This is because the class of functions that preserve \preceq is wider than the class of Schur-convex functions preserving the majorization pre-ordering \prec .

This section considers an example of a pre-ordering defined in Marshall & Olkin (1979) that has the above property and obtains a characterization of functions that preserve it.

Definition 4 (Marshall & Olkin 1979, Section 5.B) For $x, y \in \mathbf{R}^n$ write $x \prec^1 y$ if the difference $y_{[i]} - x_{[i]}$ is non-increasing in $i = 1, \dots, n$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

Remark 3 As is shown in Proposition 5.B.1 in Marshall & Olkin (1979), $(x \preceq^1 y) \Rightarrow (x \prec y)$ for all $x, y \in \mathbf{R}^n$.

Remark 4 Let $A \subseteq \mathbf{R}^n$ be a symmetric set, that is, $x \in A$ implies $xP \in A$ for all $P \in \mathcal{P}_n$. Evidently, the pre-ordering \preceq^1 has the property that $x \preceq^1 xP \preceq^1 x$ for all $x \in A$ and $P \in \mathcal{P}_n$. Therefore, if a function ϕ preserves the pre-ordering \preceq^1 on a symmetric set A , then ϕ is symmetric on A . Thus, if a function ϕ is symmetric on a symmetric set A and it preserves the pre-ordering \preceq^1 on $A \cap D$, where $D = \{y_1 \geq \dots \geq y_n\}$, then ϕ also preserves \preceq^1 on A .

Remark 5 Let $A \subseteq \mathbf{R}^n$. A pre-ordering \preceq on A is called a cone pre-ordering induced by a convex cone K if $x \preceq^1 y$ holds if and only if $y - x \in K$. It is not difficult to see that the pre-ordering \preceq^1 is a cone pre-ordering on the set $D = \{y_1 \geq \dots \geq y_n\}$. The cone that induces the pre-ordering \preceq^1

is given by $K = \{y \in \mathbf{R}^n : 0 \preceq^1 y\} = \{y \in \mathbf{R}^n : yM \geq 0\}$, where

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ -1 & 1 & \dots & 0 & 1 \\ 0 & -1 & \dots & 0 & 1 \\ \dots & & \dots & & \dots \\ 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

From Remark 4 it follows that it suffices to characterize functions that preserve the pre-ordering \preceq^1 on the set D . Let $\{e_i\}_{i=1}^n$ be the set of unit vectors in \mathbf{R}^n .

Theorem 2 *Let $\phi : D \rightarrow \mathbf{R}$ be a function that is continuous on the boundary of D and satisfies the conditions*

$$\phi\left(\sum_{i=1}^k (y_i + \lambda(n-k))e_i + \sum_{i=k+1}^n (y_i - \lambda k)e_i\right) \geq \phi(y), \quad k = 1, \dots, n-1,$$

$$\phi(y_1 + \lambda, \dots, y_n + \lambda) \geq \phi(y)$$

for all $\lambda > 0$ and $y \in D$ such that $\sum_{i=1}^k (y_i + \lambda(n-k))e_i + \sum_{i=k+1}^n (y_i - \lambda k)e_i \in D$. Then ϕ preserves the pre-ordering \preceq^1 on D .

For the proof of Theorem 2 we need the following

Lemma 1 (Marshall, Walkup & Wets 1967) *Let a convex cone $K \in \mathbf{R}^n$ be positively spanned by the set $F \subseteq K$, that is, each point in K can be represented as a linear combination of a finite number of points in F with nonnegative coefficients. Further, let A be a convex set with a nonempty interior. Then a function ϕ preserves the conic pre-ordering induced by the cone K on A if ϕ is continuous on the boundary of A and $\phi(y + \lambda t) - \phi(y) \geq 0$ for all $\lambda > 0$ and $t \in F$ such that $y + \lambda t \in A$.*

Proof of Theorem 2. It is not difficult to show (see Marshall & Olkin 1979) that if the convex cone K has the form $K = \{y : yM' \geq 0\}$, where $M' \in \mathbf{R}^{n \times n}$ is a nonsingular matrix, then K is

positively spanned by the set of vectors given by rows of the inverse matrix M'^{-1} . The inverse of the matrix M in Remark 5 is given by

$$M = \begin{pmatrix} n-1 & -1 & -1 & \dots & -1 & -11 \\ n-2 & n-2 & -2 & \dots & -2 & -2 \\ n-3 & n-3 & n-3 & \dots & -3 & -3 \\ \dots & & & \dots & & \dots \\ 1 & 1 & 1 & \dots & 1 & -(n-1) \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}.$$

This, together with Lemma 1 implies the conclusion of the theorem. ■

Theorem 3 *Let a function ϕ be continuous on D and continuously differentiable in the interior of D . Further, let $\phi_{(k)}(y) = \partial\phi(y)/\partial y_k$. The function ϕ preserves the pre-ordering \preceq^1 on D if and only if*

$$(n-k) \sum_{i=1}^k \phi_{(i)}(y) \geq k \sum_{i=k+1}^n \phi_{(i)}(y), \quad k = 1, \dots, n-1,$$

$$\sum_{i=1}^n \phi_{(i)}(y) \geq 0$$

for all $y \in D$.

Proof of Theorem 3. Theorem 3 immediately follows from Theorem 2 and the following Lemma 2.

Lemma 2 *(Marshall & Olkin 1979) Let ϕ be a continuous function defined on a convex set $A \subseteq \mathbf{R}^n$ with gradient $\nabla\phi(y) = (\partial\phi(y)/\partial y_1, \dots, \partial\phi(y)/\partial y_n)$ at each point in the nonempty interior of A . Further, let \preceq be a pre-ordering on A induced by a convex cone K that is positively spanned by a set F . The function ϕ preserves the pre-ordering \preceq on A if and only if $\nabla\phi(y) \cdot t \geq 0$ for all $t \in F$ and all y in the interior of A .*

■

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