COPULA-BASED CHARACTERIZATIONS FOR HIGHER-ORDER MARKOV PROCESSES

By Rustam Ibragimov[1]

Department of Economics, Harvard University

Address for manuscript correspondence:

Rustam Ibragimov
Department of Economics
Harvard University
Littauer Center
1805 Cambridge St.
Cambridge, MA 02138
Email: ribragim@fas.harvard.edu
Phone: +1-617-496-4795
Fax: +1-617-495-7730

ABSTRACT

In this paper, we obtain characterizations of higher-order Markov processes in terms of copulas corresponding to their finite-dimensional distributions. The results are applied to establish necessary and sufficient conditions for Markov processes of a given order to exhibit $m-$dependence, $r-$independence or conditional symmetry. The paper also presents a study of applicability and limitations of different copula families in constructing higher-order Markov processes with the above dependence properties. We further introduce new classes of copulas that allow one to combine Markovness with $m-$dependence or $r-$independence in time series.

Key words and phrases: copulas, dependence, time series, Markov processes, $m-$dependence, $r-$independence, conditional symmetry, martingales, stochastic differential equations, Fourier copulas

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1 Introduction

In recent years, a number of studies in economics, finance and risk management have focused on the analysis of dependence measures and related concepts for time series. It was observed that the use of correlation as measure of dependence is problematic in many setups, including the departure from elliptic distributions that is common for real world risks and financial market data (see, among others, the discussion in Embrechts, McNeil and Straumann, 2002, Ch. 5 in McNeil, Frey and Embrechts, 2005, de la Peña, Ibragimov and Sharakhmetov, 2006, and references therein). Another problem with using correlation is that it is a bivariate measure of dependence and even using its time varying versions, at best, leads to only capturing the pairwise dependence in data sets, failing to measure more complicated dependence structures. Also, correlation is defined only in the case of data with finite second moments and its reliable estimation is problematic in the case of infinite fourth moments (see the discussion in Cont, 2001). However, as discussed in a number of studies (see, e.g., Loretan and Phillips, 1994, Cont, 2001, and Ibragimov, 2004, 2005*b*, and references therein), heavy-tailed behavior with infinite fourth moments is present in many financial and commodity market data sets and even first moments or variances are infinite for certain time series in finance and economics. Several approaches have been proposed recently to deal with the above problems. One of these approaches, which is becoming increasingly popular in dependence modeling and analysis is the one based on copulas. Copulas are functions that allow one, by a celebrated theorem due to Sklar (1959), to represent a joint distribution of random variables (r.v.'s) as a function of marginal distributions.[2] Copulas, therefore, capture dependence properties of the data generating process (more precisely, they reflect all the dependence properties that are invariant to increasing transformations of data). In recent year, copulas and related concepts have been applied to a wide range of problems in economics, finance and risk management (see, among others, Cherubini, Luciano and Vecchiato, 2004, and references therein, Patton, 2004, McNeil, Frey and Embrechts, 2005, Hu, 2006, the review in de la Peña, Ibragimov and Sharakhmetov, 2006, Granger, Teräsvirta and Patton, 2006, and Patton, 2006).

One should note that, so far, most of the studies have focused on the analysis of copulas and dependence measures only in the bivariate case and only a few papers have considered the problems in copula theory in the time series context. A drawback of the approach based on bivariate copulas and dependence measures is that, similar to the case of linear correlation, it can capture, at best, only pairwise dependence patterns and can not be used in the case of more complicated dependence structures. However, dependence characteristics of real data sets can be far more general than those determined by pairs of variables: for example, the behavior of financial indices across markets is interrelated and is affected by a number of factors common to all of the markets. In addition, estimation procedures for copulas developed in the context of independent observations of random vectors are not directly applicable in the analysis of time series dependence characteristics.

[2] The concept of copulas is closely related to the probability integral transformation (see Rosenblatt, 1952, Gouriéroux and Monfort, 1979, and Section 4 in Breymann, Dias and Embrechts, 2003) and to Fréchet classes of joint distributions (see Chapter 3 in Joe, 1997).

The problems of copula theory and its applications in the multivariate and time series context have been considered, in particular, in the following papers. Joe (1987, 1989) proposed multivariate extensions of Pearson's coefficient and the Kullback-Leibler and Shannon mutual information. Nelsen (1996) considered measures of multivariate association generalizing bivariate Spearman's rho and Kendall's tau. Focusing on finite-dimensional random vectors with dependent components, de la Peña, Ibragimov and Sharakhmetov (2006) obtained $U-$statistics-based representations for multivariate joint distributions and copulas. As a corollary of the results, de la Peña, Ibragimov and Sharakhmetov (2006) derived similar representations for multivariate dependence measures and obtained sharp complete decoupling moment and probability inequalities for dependent r.v.'s in terms of their dependence characteristics.

Darsow, Nguyen and Olsen (1992, hereafter DNO) obtained characterizations of first-order Markov processes in terms of copula functions corresponding to their two-dimensional distributions. Chen and Fan (2004, 2006) considered parametric copula estimation procedures for time-series based on bivariate copulas and applied the results in the problems of evaluating density forecasts. Fermanian, Radulović and Wegcamp (2004) established weak convergence of empirical copula processes in the case of independently observed vectors with dependent components. Doukhan, Fermanian and Lang (2004) focused on the analysis of the asymptotics of empirical copula processes for weakly dependent sequences of random vectors.[3] Ibragimov (2005a) discussed weak convergence of empirical copulas under $\beta-$mixing assumptions.

In this paper, we obtain characterizations of Markov processes of an arbitrary order in terms of copulas corresponding to their finite-dimensional distributions (Section 2). These results extend the characterizations of first-order Markov processes in terms of bivariate copulas in DNO. The results show that a Markov process of order k is fully determined by its $(k+1)-$dimensional copulas and one-dimensional marginal cdf's. The characterizations thus provide a justification for estimation of finite-dimensional copulas of time series with higher-order Markovian dependence structure. Using the results, we obtain necessary and sufficient conditions for higher-order Markov processes to exhibit several additional dependence properties, such as $m-$dependence, $r-$independence or conditional symmetry (Section 3). These conditions, in particular, generalize $U-$statistics-based characterizations of copulas of vectors with $r-$independent or $m-$dependent components in de la Peña, Ibragimov and Sharakhmetov (2006) to the case of dependent time series. They further show that dependence properties of copula-based time series provide additional non-trivial restrictions on the $U-$statistics characterizations of copulas for the processes in consideration that can be used in inference on their properties.

The results obtained in the paper provide a copula-based approach to the analysis of higher-order Markov processes which is alternative to the conventional one based on transition probabilities. The advantage of the approach based on copulas is that it allows one to separate the study of dependence properties (e.g., $r-$independence, $m-$dependence or conditional symmetry) of the stochastic processes

[3] I thank a referee for the reference to Doukhan, Fermanian and Lang (2004).

in consideration from the analysis of the effects of marginal distributions (say, unconditional heavy-tailedness or skewness). In particular, the results provide methods for construction of higher-order Markov processes with arbitrary one-dimensional margins that, possibly, satisfy additional dependence assumptions. These processes can be used, for instance, in the analysis of the robustness of statistical and econometric procedures to weak dependence. In addition, they provide examples of non-Markovian processes that nevertheless satisfy Chapman-Kolmogorov stochastic equations. Higher-order Markov processes with prescribed dependence properties can be constructed, for instance, using inversion of finite-dimensional cdf's of known examples of dependent time series (see the discussion at the end of Section 2).

In Sections 4 and 5, we present an analysis of applicability and limitations of different classes of copulas in constructing higher-order Markov processes with prescribed dependence properties. In Section 4, we focus on processes based on expansions by linear functions (bivariate and multivariate Eyraud-Farlie-Gumbel-Mongenstern copulas) as well as on more general copulas that involve products of nonlinear functions of the arguments, such as power copulas. We obtain impossibility/reduction-type results that show that time series based on such copulas that simultaneously exhibit Markovness and $m-$dependence or $r-$independence properties are, in fact, sequences of independent r.v.'s (Theorems 5 and 6 and Corollaries 2-4). In Section 5, we introduce a new class of copulas based on expansions by Fourier polynomials (Fourier copulas) that, in contrast to the copula families considered in Section 4, allow one to combine higher-order Markovness with $m-$dependence or $r-$independence. Section 6 of the paper makes some concluding remarks. Appendix A1 reviews the definition and discusses the main properties of copula functions, together with their examples. Appendix A2 contains the proofs of the results obtained in the paper.

2 Copula-based time series Markov characterizations

This section presents our main results on copula-based characterizations for Markov processes of an arbitrary order. In what follows, we use definitions and the results in copula theory reviewed in Appendix A1. In order to highlight the main concepts and ideas discussed, we assume throughout the paper that all copulas considered are absolutely continuous and the processes under study have continuous univariate cdf's, if not stated otherwise. These assumptions, in particular, imply that the copulas corresponding to the finite-dimensional distributions of the processes in consideration are unique (see Proposition 1 in Appendix A1). However, most of the results discussed in the paper can be extended to the case of not necessarily continuous marginal cdf's and not necessarily absolutely continuous copulas.[4]

[4]Analogues of the copula-based higher-order Markov characterizations in the paper for not necessarily continuous univariate cdf's can be obtained similar to the corresponding extensions in the case of first-order Markov processes in DNO, using linear (in each place) interpolation between the points at which the finite-dimensional copulas of the processes are determined uniquely. Extensions of other results in the paper can be obtained similar to their proofs in this work.

Let $m, n \geq k \geq 1$. Let A and B be, respectively, $m-$ and $n-$dimensional copulas such that

$$A(u_1, ..., u_{m-k}, \xi_1, ..., \xi_k)\Big|_{u_i=1, i=1,...,m-k} = B(\xi_1, ..., \xi_k, u_{k+1}, ..., u_n)\Big|_{u_i=1, i=k+1,...,n} = C(\xi_1, ..., \xi_k), \quad (1)$$

$\xi_i \in [0,1]$, $i = 1, ..., k$, where C is a $k-$dimensional copula (relation (1) means that a $k-$dimensional margin of the copula A is the same as a $k-$dimensional margin of the copula B).

Let $V_1, ..., V_m$ and $W_1, ..., W_n$ be r.v.'s with joint cdf's A and B (see Definition 5 in Appendix A1). Denote by $A_{1,...,m|m-k+1,...,m}(u_1, ..., u_{m-k}, \xi_1, ..., \xi_k) = P(V_1 \leq u_1, ..., V_{m-k} \leq u_{m-k}| \ V_{m-k+1} = \xi_1, ..., V_m = \xi_k)$ and $B_{1,...,n|1,...,k}(\xi_1, ..., \xi_k, u_{m+1}, ..., u_{m+n-k}) = P(W_{k+1} \leq u_{m+1}, ..., \ W_n \leq u_{m+n-k}| \ W_1 = \xi_1, ..., W_k = \xi_k)$ the conditional analogues of the copulas A and B. One has

$$A_{1,...,m|m-k+1,...,m}(u_1, ..., u_{m-k}, \xi_1, ..., \xi_k) = \frac{\partial^k A(u_1, ..., u_{m-k}, \xi_1, ..., \xi_k)}{\partial v_{m-k+1} ... \partial v_m} \Big/ \frac{\partial^k C(\xi_1, ..., \xi_k)}{\partial v_1 ... \partial v_k},$$
$$B_{1,...,n|1,...,k}(\xi_1, ..., \xi_k, u_{m+1}, ..., u_{m+n-k}) = \frac{\partial^k B(\xi_1, ..., \xi_k, u_{m+1}, ..., u_{m+n-k})}{\partial v_1 ... \partial v_k} \Big/ \frac{\partial^k C(\xi_1, ..., \xi_k)}{\partial v_1 ... \partial v_k}, \quad (2)$$

where $\partial^k A(v_1, ..., v_m)/\partial v_{m-k+1} ... \partial v_m$, $\partial^k B(v_1, ..., v_n)/\partial v_1 ... \partial v_k$ and $\partial^k C(v_1, ..., v_k)/\partial v_1 ... \partial v_k$ denote the partial derivatives of the copulas A, B and C.

Further, define the $\star^k-$product of the copulas A and B, $D = A \star^k B : \ [0,1]^{m+n-k} \to [0,1]$ via the relation

$$D(u_1, ..., u_{m+n-k}) = \int_0^{u_{m-k+1}} ... \int_0^{u_m} A_{1,...,m|m-k+1,...,m}(u_1, ..., u_{m-k}, \xi_1, ..., \xi_k) \times B_{1,...,n|1,...,k}(\xi_1, ..., \xi_k, u_{m+1}, ..., u_{m+n-k}) C(d\xi_1, ..., d\xi_k). \quad (3)$$

The $\star^k-$operator is a generalization of the star $\star-$operator considered in DNO; the $\star-$operator in DNO is a particular case of its above $\star^k-$ analogue with $k = 1$ (see Appendix A1). Similar to the case of $k = 1$ in DNO, one can show that the operator $\star^k$ is associative, distributive over convex combinations and continuous in each place (but not jointly continuous).

In terms of the densities $\frac{\partial^m A(v_1,...,v_m)}{\partial v_1 ... \partial v_m}$, $\frac{\partial^n B(v_1,...,v_n)}{\partial v_1 ... \partial v_n}$ and $\frac{\partial^{m+n-k} D(v_1,...,v_{m+n-k})}{\partial v_1 ... \partial v_{m+n-k}}$ of the copulas A, B and $D = A \star^k B$, relation (3) is equivalent to the following:

$$\frac{\partial^{m+n-k} D(u_1, ..., u_{m+n-k})}{\partial v_1 ... \partial v_{m+n-k}} = \frac{\partial^m A(u_1, ..., u_{m-k}, u_{m-k+1}, ..., u_m)}{\partial v_1 ... \partial v_m} \times \frac{\partial^n B(u_{m-k+1}, ..., u_m, u_{m+1}, ..., u_{m+n-k})}{\partial v_1 ... \partial v_n} \Big/ \frac{\partial^k C(u_{m-k+1}, ..., u_m)}{\partial v_1 ... \partial v_k},$$

or, equivalently,

$$\frac{\partial^{m+n-k} D(u_1, ..., u_{m+n-k})}{\partial v_1 ... \partial v_{m+n-k}} \cdot \frac{\partial^k C(u_{m-k+1}, ..., u_m)}{\partial v_1 ... \partial v_k} = \frac{\partial^m A(u_1, ..., u_{m-k}, u_{m-k+1}, ..., u_m)}{\partial v_1 ... \partial v_m} \cdot \frac{\partial^n B(u_{m-k+1}, ..., u_m, u_{m+1}, ..., u_{m+n-k})}{\partial v_1 ... \partial v_n}.$$

Let $T \subseteq \mathbf{R}$. The processes considered throughout the paper are assumed to be real-valued and continuous and to be defined on the same probability space $(\Omega, \Im, P)$. Stationarity refers to strict stationary. For a r.v. X on $(\Omega, \Im, P)$ and $x \in \mathbf{R}$, $I_{X<x}$ denotes the indicator of the event $\{X < x\}$. In addition, as usual, for r.v.'s $Y_1, ..., Y_s, X_1, ..., X_l$ on $(\Omega, \Im, P)$ and $x_1, ..., x_l \in \mathbf{R}$, $P(X_1 < x_1, ..., X_l < x_l | Y_1, ..., Y_s)$ stands for $E\big(I_{X_1<x_1} ... I_{X_l<x_l} \big| \, Y_1, ..., Y_s\big)$. For two r.v.'s X and Y on $(\Omega, \Im, P)$ we write $X = Y$ if $X = Y$ (a.s.).

Definition 1 *A process $\{X_t\}_{t \in T}$ is called a* Markov process of order $k \geq 1$ *if, for all $t, t_i \in T$, $i = 1, ..., n$, such that $t_1 < ... < t_{n-k} < t_{n-k+1} < ... < t_n < t$ and all $x \in \mathbf{R}$,*

$$P\big(X_t < x \big| X_{t_1}, ..., X_{t_{n-k}}, X_{t_{n-k+1}}, ..., X_{t_n}\big) = P\big(X_t < x \big| X_{t_{n-k+1}}, ..., X_{t_n}\big). \tag{4}$$

Throughout the rest of the section, $C_{t_1,...,t_k}$, $t_i \in T$, $i = 1, ..., k$, $t_1 < ... < t_k$, stand for copulas corresponding to the joint distribution of the r.v.'s $X_{t_1}, ..., X_{t_k}$ in the process $\{X_t\}_{t \in T}$ in consideration. In addition, throughout the paper, formulated equalities and inequalities for two functions f and g defined on $[a, b]^n \subseteq \mathbf{R}^n$ are understood to hold almost everywhere on $[a, b]^n$. That is, we write $f = g$ (or $f(u) = g(u)$) if f and g coincide almost everywhere on $[a, b]^n$: $f(u) = g(u)$ for all $u \in [a, b]^n \setminus \mathcal{A}$, where $\mathcal{A}$ is a subset of $[a, b]^n$ with the Lebesgue measure zero. The meaning of the inequalities $f \geq g$ and $f \leq g$ (or $f(u) \geq g(u)$ and $f(u) \leq g(u)$) is similar.

The following theorem provides a characterization of Markov processes of an arbitrary order in terms of their $(k+1)-$dimensional copulas.

Theorem 1 *A real-valued stochastic process $\{X_t\}_{t \in T}$, is a Markov process of order k, $k \geq 1$, if and only if for all $t_i \in T$, $i = 1, ..., n$, $n \geq k+1$, such that $t_1 < ... < t_n$,*

$$C_{t_1,...,t_n} = C_{t_1,...,t_{k+1}} \star^k C_{t_2,...,t_{k+2}} \star^k ... \star^k C_{t_{n-k},...,t_n}. \tag{5}$$

Let $n \geq k+1$ and $s \geq 1$. For an $n-$dimensional copula C denote by C^s the $s-$fold product $\star^k$ of C with itself.

Corollary 1 *A sequence of identically distributed r.v.'s $\{X_t\}_{t=1}^{\infty}$ is a stationary Markov process of order k, $k \geq 1$, if and only if for all $n \geq k+1$,*

$$C_{1,...,n}(u_1, ..., u_n) = \underbrace{C \star^k C \star^k ... \star^k C}_{n-k+1}(u_1, ..., u_n) = C^{n-k+1}(u_1, ..., u_n), \tag{6}$$

where C is a $k+1-$dimensional copula such that $C_{i_1+h,...,i_l+h} = C_{i_1,...,i_l}$, $1 \leq h \leq k+1-i_l$, $1 \leq i_1 < ... < i_l \leq k+1$, $l = 2, ..., k$, and $C_{j_1,...,j_l}$, $1 \leq j_1 < ... < j_l \leq k+1$, denote the corresponding marginals of C: $C_{j_1,...,j_l} = C|_{u_i=1, i \neq j_1,...,j_l}$.

Let, as above, $\{X_t\}_{t \in T}$ be a Markov process of order k with finite-dimensional copulas $C_{t_1,...,t_n}(u_1, ..., u_n)$, $t_i \in T$, $i = 1, ..., n$, $t_1 < ... < t_n$. For $t \in T$, denote by F_t the cdf of X_t. Let, for $t_1 < ... < t_k$, $x_1, ..., x_k \in \mathbf{R}$ and Borel sets $A \in \mathcal{B}(\mathbf{R})$, $p(t_1, ..., t_k, t_{k+1}, x_1, ..., x_k, A) = P(X_{t_{k+1}} \in A | X_{t_1} =$

$x_1, ..., X_{t_k} = x_k)$ stand for the transition probabilities of $\{X_t\}_{t \in T}$. Using (2), it is not difficult to see, similar to the case of first-order Markov processes in the proof of Theorem 3.2 in DNO, that, for all Borel sets $A \in \mathcal{B}(\mathbf{R})$ of the form $A = (-\infty, x_{k+1})$, $x_{k+1} \in \mathbf{R}$, one has

$$p(t_1, ..., t_k, t_{k+1}, x_1, ..., x_k, A) = \frac{\partial^k C_{t_1,...,t_k,t_{k+1}}\big(F_{t_1}(x_1), ..., F_{t_k}(x_k), F_{t_{k+1}}(x_{k+1})\big)}{\partial u_1 ... \partial u_k} \Big/ \frac{\partial^k C_{t_1,...,t_k}\big(F_{t_1}(x_1), ..., F_{t_k}(x_k)\big)}{\partial u_1 ... \partial u_k}, \tag{7}$$

where, as before, $\partial^k C_{t_1,...,t_k,t_{k+1}}(u_1, ..., u_k, u_{k+1})/\partial u_1 ... \partial u_k$ and $\partial^k C_{t_1,...,t_k}(u_1, ..., u_k)/\partial u_1 ... \partial u_k$ denote partial derivatives of the copulas $C_{t_1,...,t_k,t_{k+1}}(u_1, ..., u_k, u_{k+1})$ and $C_{t_1,...,t_k}(u_1, ..., u_k)$.

Theorem 1 and Corollary 1 provide copula-based characterizations of higher-order Markovian processes that is an alternative to the conventional characterization using their transition probabilities (and the initial distribution). One can specify a Markov process of order k by prescribing all one-dimensional marginal distributions and a family of $(k+1)-$dimensional copulas. Then one can generate the copulas of higher order and, thus, the finite-dimensional cdf's using (5). As discussed in the introduction, the advantage of the approach based on copulas is that it allows one to separate in the analysis the properties of time series determined by one-dimensional distributions from their dependence characteristics. Relations (7), on the other hand, allow one to recover one characterization of higher-order Markov processes given the other.

Corollary 1, together with the inversion method for constructing copulas described in Appendix A1, provide a device for obtaining new Markov processes of an arbitrary order that exhibit dependence properties similar to those of a given Markov process of the same order but have different marginals. Namely, let $\{X_t\}_{t=1}^{\infty}$ be a stationary Markov process of order $k \geq 1$ with $(k+1)-$dimensional cdf $\tilde{F}(x_1, ..., x_{k+1})$ and the one-dimensional marginal cdf F. Then the $(k+1)-$dimensional copula generating the process $\{X_t\}_{t=1}^{\infty}$ is, via formula (32) in Appendix A1, $C(u_1, ..., u_{k+1}) = \tilde{F}(F^{-1}(u_1), ..., F^{-1}(u_{k+1}))$. Given an arbitrary one-dimensional cdf G, the stationary $k-$th order Markov process that has the dependence structure similar to that of $\{X_t\}_{t=1}^{\infty}$ but a different one-dimensional marginal cdf G can be constructed via (6) by generating its copulas of an arbitrary order and substituting the new one-dimensional cdf to obtain its finite-dimensional cdf's. For instance, taking $\tilde{F}$ to be the $(k+1)-$dimensional normal cdf with a linear correlation matrix $R : \tilde{F} = \Phi_R^{k+1}(x_1, ..., x_{k+1})$ as in (36) with $n = k + 1$, one obtains a stationary $k-$th order Markov process based on the normal copula $C_R^{k+1}(u_1, ..., u_{k+1})$ in (37) with $n = k + 1$. In the case $k = 1$ and $T = \mathbf{R}$, the construction provides a first-order Markov process in Example 4.3 in DNO that is referred to as a Brownian motion with non-Gaussian marginal distributions therein. More generally, in the case $k \geq 1$, one obtains a process with an arbitrary one-dimensional marginal cdf's whose dependence structure is similar to that of a Gaussian autoregressive process of order k (see the discussion at the beginning of Section 4). In addition, using examples of (possibly higher-order) Markov processes that satisfy additional dependence assumptions available in the literature, such as examples of $k-$independent $k-$th order Markov processes or $m-$dependent Markov processes of the first order constructed in the works by Lévy, 1949, Rosenblatt and Slepian, 1962, Aaronson, Gilat and Keane, 1992, or Matúš, 1998 (see the discussion in the next section), one can use the above inversion procedure to construct Markov processes that exhibit similar dependence properties but have one-dimensional marginals different from those in the examples.

In what follows, we refer to the processes $\{X_t\}_{t=1}^{\infty}$ constructed via (6) as stationary $k-$th order Markov processes based on the ($(k+1)-$dimensional) copula C or as stationary $C-$based $k-$th order Markov processes for short.

Remark 1 *Let $f : \mathbf{R} \to \mathbf{R}$ be a strictly increasing function and let $\{X_t\}_{t=1}^{\infty}$ be a stationary $C-$based $k-$th order Markov process. Denote $Y_t = f(X_t)$, $t \geq 1$. Since copulas are invariant under strictly increasing transformations of r.v.'s (see Proposition 2 in Appendix A1) and Markov property is preserved by strictly monotone (hence one-to-one) functions of Markov processes, $\{Y_t\}_{t=1}^{\infty}$ is a stationary $k-$th order Markov process based on the same copula C.*[5] *More generally, it is easy to derive the expressions relating the copulas of the $k-$th order Markov process $\{X_t\}_{t=1}^{\infty}$ to those of the $k-$th order Markov process determined by $Y_t = f_t(X_t)$, $t \geq 1$, where $f_t : \mathbf{R} \to \mathbf{R}$ are strictly monotone functions. For instance, in the case of the first-order Markov processes X_t with bivariate copulas $C_{t_1t_2}$, the copulas $\tilde{C}_{t_1t_2}$ of the process $Y_t = f_t(X_t)$ are related to $C_{t_1t_2}$ as follows (see Theorem 2.4.4 in Nelsen, 1999):*

$\tilde{C}_{t_1,t_2}(u, v) = u - C_{t_1,t_2}(u, 1 - v)$, *if f_{t_1} is strictly increasing and f_{t_2} is strictly decreasing;*

$\tilde{C}_{t_1,t_2}(u, v) = v - C_{t_1,t_2}(1 - u, v)$, *if f_{t_1} is strictly decreasing and f_{t_2} is strictly increasing;*

$\tilde{C}_{t_1,t_2}(u, v) = u + v - 1 + C_{t_1,t_2}(1 - u, 1 - v)$, *if both f_{t_1} and f_{t_2} are strictly decreasing.*

3 Applications to combining higher-order Markovness with other dependence properties

From copula-based characterizations of Markov processes of an arbitrary order in Section 2 it follows, in particular, that any copulas and, thus, any $U-$statistics-based representations for copula functions in de la Peña, Ibragimov and Sharakhmetov (2006) (see Appendix A1 in this paper) can be used to construct higher-order Markov processes. The results in this and the next section show that additional dependence properties of such time series impose further restrictions on the $U-$statistics employed in the representations. These restrictions allow one to obtain, for instance, characterizations of Markov processes of an arbitrary order that satisfy additional assumptions of $r-$independence or $m-$dependence defined as follows.

Definition 2 *Let $r \geq 2$ and let $T \subseteq \mathbf{R}$ be an index set that contains at least $r+1$ elements. A process $\{X_t\}_{t\in T}$ is called $r-$independent if any r r.v.'s among X_t, $t \in T$, are jointly independent.*

Definition 3 *Let $m \geq 1$ and let $T \subseteq \mathbf{R}$ be an index set that contains at least $m+2$ elements. A process $\{X_t\}_{t\in T}$ is called $m-$dependent if, for all $1 \leq a \leq l-1$ and any indices $j_s \in T$, $s = 1, ..., l$, such that $1 \leq j_1 < ... < j_a < ... < j_l$ and $j_{a+1} - j_a \geq m+1$, the vectors $(X_{j_1}, X_{j_2}, ..., X_{j_{a-1}}, X_{j_a})$ and $(X_{j_{a+1}}, X_{j_{a+2}}, ..., X_{j_{l-1}}, X_{j_l})$ are independent.*

[5] In general, Markovness is not preserved by many-to-one transformations. Rosenblatt (1971, Ch. III) provides conditions under which general functions of Markov processes are still Markovian.

A number of studies have focused on problems of combining Markovian structures with other types of dependence. Lévy (1949) constructed a 2nd order Markov process consisting of pairwise independent uniformly distributed r.v.'s (a 2nd order pairwise independent Markov process). Motivated by applications in the study of the mechanism of human vision, Rosenblatt and Slepian (1962) constructed stationary $N-$th order Markov processes consisting of discrete r.v.'s such that every N variables of the process are independent while $N+1$ adjacent variables of the process are not independent (stationary $N-$th order $N-$independent Markov process). Rosenblatt and Slepian (1962) also obtained a result that is natural to refer to as an impossibility or a reduction property for Markov processes. This result shows that all $N-$th order $N-$independent Markov processes with two-valued X_t's are trivial in that they are processes of independent r.v.'s. Higher order Markov $r-$independent processes are important in testing empirically the sensitivity of statistical procedures developed on the independence assumption to weak dependence in the data generating process (see Rosenblatt and Slepian, 1962). In addition, such processes are of interest since they provide examples of processes which are not Markovian of first order but whose first order transition probabilities $P(s,x,t,A) = P(X_t \in A|X_s = x)$ nevertheless satisfy the Chapman-Kolmogorov stochastic equation

$$P(s,x,t,A) = \int_{-\infty}^{\infty} P(u,\xi,t,A)P(s,x,u,d\xi) \tag{8}$$

for all Borel sets A, all $s < t$ in T, $u \in (s,t) \cap T$ and for almost all $x \in \mathbf{R}$.[6]

Markov processes with 1-dependence appeared for the first time in Aaronson, Gilat and Keane (1992) and were considered, e.g., by Burton, Goulet and Meester (1993) and Matúš (1996), where the focus was on $1-$dependent Markov shifts and on the structure of block-factors. Matúš (1998) studied $m-$dependent Markov sequences consisting of discrete r.v.'s and showed, in particular, that generally no stationary sequence of r.v.'s which is Markov of order n but not of order $n-1$ and $m-$dependent but not $(m-1)-$dependent exists if the state space of the sequence has small cardinality (another type of an impossibility/reduction result for Markov processes). Matúš (1998) also showed that to ensure the existence of Markov processes of order $n=1$ that are $m-$dependent but not $(m-1)-$dependent the number of attainable states must be at least $m+2$ and this bound is tight.

The following result gives a characterization of stationary $k-$independent $k-$th order Markov processes. Below, $[x]$ stands for the integer part of $x \in \mathbf{R}$.

Theorem 2 *Let C be a $(k+1)-$dimensional copula. A sequence of r.v.'s $\{X_t\}_{t=1}^{\infty}$, is a stationary $k-$independent $C-$based $k-$th order Markov process if and only if the density of C has the form*

$$\frac{\partial^{k+1} C(u_1,...,u_{k+1})}{\partial u_1 ... \partial u_{k+1}} = 1 + g(u_1,...,u_{k+1}), \tag{9}$$

where $g: [0,1]^{k+1} \to \mathbf{R}$ is a function satisfying the conditions

$$\int_0^1 ... \int_0^1 |g(u_1,...,u_{k+1})| du_1 ... du_{k+1} < \infty, \tag{10}$$

[6] Examples of non-Markovian processes for which Chapman-Kolmogorov equation is satisfied were also given, e.g., by Feller (1959) and Rosenblatt (1960).

$$\int_0^1 ... \int_0^1 \prod_{j=1}^{s} g(u_j, ..., u_{k+j}) du_{i_1} ... du_{i_s} =$$

$$\int_0^1 ... \int_0^1 g(u_1, ..., u_{k+1}) g(u_2, ..., u_{k+2}) ... g(u_s, ..., u_{k+s}) du_{i_1} ... du_{i_s} = 0 \tag{11}$$

for all $s \leq i_1 < ... < i_s \leq k+1,\ s = 1, 2, ..., \left[\frac{k+1}{2}\right]$, *and*

$$g(u_1, ..., u_{k+1}) \geq -1. \tag{12}$$

Remark 2 *Integration in condition (11) is with respect to all combinations of s variables among the arguments* $u_s, u_{s+1}, ..., u_{k+1}$ *that are common to all functions* $g(u_1, ..., u_{k+1}),\ g(u_2, ..., u_{k+2}),\ ...,\ g(u_s, ..., u_{k+s})$ *appearing in the integrand. These conditions ensure that all* $k-$*dimensional marginals of the copula of* $X_1, ..., X_{k+s},\ s \geq 1$, *are product copulas (35) with* $n = k$ *and thus the* $k-$*independence property is satisfied for the stationary* $k-th$ *order Markov process in consideration.*

The following theorem provides a characterization of Markov processes satisfying $m-$dependence properties.

Theorem 3 *Let* C *be a bivariate copula. A sequence of r.v.'s* $\{X_t\}_{t=1}^{\infty}$ *is a stationary* $m-$*dependent* $C-$*based first-order Markov process if and only if the density of* C *satisfies*

$$\frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} = 1 + g(u_1, u_2), \tag{13}$$

where $g : [0,1]^2 \to \mathbf{R}$ *is a function satisfying the conditions*

$$\int_0^1 \int_0^1 |g(u_1, u_2)| du_1 du_2 < \infty, \tag{14}$$

$$\int_0^1 g(u_1, u_2) du_i = 0,\ i = 1, 2, \tag{15}$$

$$g(u_1, u_2) \geq -1 \tag{16}$$

and such that

$$\int_0^1 ... \int_0^1 \prod_{i=1}^{m+1} g(u_i, u_{i+1}) du_2 du_3 ... du_{m+1} =$$

$$\int_0^1 ... \int_0^1 g(u_1, u_2) g(u_2, u_3) ... g(u_{m+1}, u_{m+2}) du_2 du_3 ... du_{m+1} = 0. \tag{17}$$

Remark 3 *Similar to (11), integration in condition (17) is with respect to the variables* $u_2, u_3, ..., u_{m+1}$ *that appear more than once among the arguments of the functions* $g(u_1, u_2),\ g(u_2, u_3),\ ...,\ g(u_{m+1}, u_{m+2})$. *This condition ensures that the r.v.'s* X_1 *and* X_{m+2} *are independent and, more generally, independence holds between the vectors* $(X_1, ..., X_n)$ *and* $(X_{m+n+1}, ..., X_{m+n+j}),\ n, j \geq 1$.

Remark 4 *Similar to the proof of Theorems 2 and 3, one can obtain necessary and sufficient conditions for a Markov process of an arbitrary order to be $m-$dependent and also exhibit certain additional dependence properties. For instance, let C be a $(k+1)-$dimensional copula and suppose that $\{X_t\}_{t=1}^{\infty}$ is a stationary $C-$based $k-$th order Markov process such that any k r.v.'s among $X_1, ..., X_{k+1}$ are independent (clearly, this assumption is weaker than $k-$independence of the process $\{X_t\}_{t=1}^{\infty}$ in Theorem 3). Using Propositions 3 and 4 and Corollary 1 as in the proof of Theorems 2 and 3, one can show that the process $\{X_t\}_{t=1}^{\infty}$ is $m-$dependent for some $m \geq k+1$ if and only if the density of C has form (9) with a function $g : [0,1]^{k+1} \to \mathbf{R}$ that satisfies conditions (10) and (12) and is such that*

$$\int_0^1 ... \int_0^1 \prod_{j=1}^{m-k+2} g(u_j, ..., u_{k+j}) du_2...du_{m+1} =$$

$$\int_0^1 ... \int_0^1 g(u_1, ..., u_{k+1}) g(u_2, ..., u_{k+2}) ... g(u_{m-k+2}, ..., u_{m+2}) du_2...du_{m+1} = 0.$$

In a number of applications, e.g., in finance, Markov and martingale properties hold simultaneously. The martingale property, in contrast to the Markov (first and higher order) properties is not determined by finite-dimensional copulas only and can be affected by changes in one-dimensional marginal distributions. Indeed, using (2) with $m = 2$ and $k = 1$ (or Theorem 3.1 in DNO), it is not difficult to see that a stationary process $\{X_t\}_{t=1}^{\infty}$ with bivariate copulas $C(u, v)$ and the univariate cdf $F(x)$ is a martingale difference sequence with respect to the natural filtration $\Im_t = \sigma(X_1, ..., X_t)$ if and only if $\int_{-\infty}^{\infty} x \frac{\partial C(F(x),F(y))}{\partial u \partial v} dF(x) = 0$. The martingale property is determined by copulas alone for the class of martingale differences that satisfy conditional symmetry assumptions.

Definition 4 *A sequence $\{X_t\}_{t=1}^{\infty}$ on a probability space $(\Omega, \Im, P)$ is a conditionally symmetric martingale difference with respect to an increasing sequence of $\sigma-$algebras $\Im_0 = (\Omega, \emptyset) \subseteq \Im_1 \subseteq \Im_2 \subseteq ... \subseteq \Im_n \subseteq \Im$ if, for all $t \geq 1$, the r.v. X_t is $\Im_t-$measurable and conditionally symmetric given $\Im_{t-1}$, that is, $P(X_t > x|\Im_{t-1}) = P(X_t < -x|\Im_{t-1})$, $x \geq 0$.*

Theorem 4 *Let C be a bivariate copula. A stationary $C-$based first-order Markov process $\{X_t\}_{t=1}^{\infty}$ consisting of symmetric r.v.'s is a conditionally symmetric martingale difference with respect to the natural filtration $\Im_0 = (\Omega, \emptyset)$, $\Im_t = \sigma(X_1, ..., X_t)$, $t \geq 1$, if and only if*

$$\frac{\partial C(u_1,\ 1/2-u)}{\partial u_1} + \frac{\partial C(u_1,\ 1/2+u)}{\partial u_1} = 1 \tag{18}$$

for all $u_1 \in [0,1]$, $u \in [0,1/2)$, or, equivalently, if the density of C satisfies

$$\frac{\partial C(u_1,\ 1/2-u)}{\partial u_1 \partial u_2} = \frac{\partial C(u_1,\ 1/2+u)}{\partial u_1 \partial u_2} \tag{19}$$

for all $u_1 \in [0,1]$, $u \in [0,1/2)$.

4 Reduction and impossibility theorems for Markov processes of an arbitrary order

For a Gaussian process, pairwise independence coincides with joint independence. Therefore, it is not surprising that a stationary higher-order Markov process based on a normal copula exhibits $r-$independence if and only if it is an i.i.d. sequence. Formally, let $C(u_1, ..., u_{k+1}) = C_R(u_1, ..., u_{k+1})$ be the normal copula with linear correlation matrix R defined in (37) with $n = k+1$ and let $\{X_t\}_{t=1}^{\infty}$ be a stationary $C-$based $k-$th order Markov process. Let $Y_t = \Phi^{-1}[F(X_t)]$, $t \geq 1$, where $\Phi(x)$ is the standard normal univariate cdf and $F(x)$ is the cdf of X_t. Since the random vector $\big(Y_1, ..., Y_{k+1}\big)$ has the multivariate normal distribution $\mathcal{N}(0, R)$ with correlation matrix R :

$$\big(Y_1, ..., Y_{k+1}\big) \sim \mathcal{N}(0, R), \tag{20}$$

we conclude that the process $\{X_t\}_{t=1}^{\infty}$ is $r-$independent for some $r \geq 2$ if and only if $R = I$, that is, if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.'s.

Let now $k = 1$ and let $C(u_1, u_2) = C_\rho(u_1, u_2)$ be the bivariate normal copula with correlation coefficient ρ (see (38)). Then (as in, e.g., Example 1 in Chen and Fan, 2004) we conclude that $\{Y_t\}_{t=1}^{\infty}$ is a Gaussian process and, thus, for $t \geq m+2$,

$$Y_t = \rho Y_{t-1} + \epsilon_t = \rho^{m+1} Y_{t-m-1} + \sum_{k=0}^{m} \rho^k \epsilon_{t-k}, \tag{21}$$

where ϵ_t has a normal distribution: $\epsilon_t \sim \mathcal{N}(0, 1-\rho^2)$. From (21) we conclude that Y_t and Y_{t-m-1} (and, thus, X_t and X_{t-m-1}) are independent if and only if $\rho = 0$. Thus, $\{X_t\}_{t=1}^{\infty}$ is a stationary $m-$dependent $C_\rho-$based Markov process (of the first order) if and only if $\rho = 0$, that is, if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.'s.

Theorems 2 and 3 imply several further reduction and impossibility results for Markov processes satisfying $m-$dependence and $r-$independence conditions that are similar in spirit to those in the case of normal copulas. The results show that a number of copula-based time series that simultaneously exhibit Markovness and $m-$dependence or $r-$independence properties are, in fact, sequences of independent r.v.'s.

Theorem 5 shows that a construction of non-trivial Markov processes of higher order that exhibit $r-$independence properties is impossible on the base of copulas whose densities C in Theorem 2 have functions g with a separable product form.

Theorem 5 *Let $k \geq 2$ and let C be a $(k+1)-$dimensional copula that has density (9), where $g(u_1, u_2, ..., u_{k+1}) = \alpha f(u_1) f(u_2) ... f(u_{k+1})$ for some $\alpha \in \mathbf{R}$ and some continuous function $f : [0, 1] \to \mathbf{R}$. A sequence of r.v.'s $\{X_t\}_{t=1}^{\infty}$ is a stationary $k-$independent $C-$based $k-$th order Markov process if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.'s.*

An example of copulas C in the separable product form of Theorem 5 is given by the special case of $(k+1)-$dimensional Eyraud-Farlie-Gumbel-Mongenstern copulas (41):

$$C(u_1, u_2, ..., u_{k+1}) = \prod_{i=1}^{k+1} u_i \Big(1 + \alpha(1 - u_1)(1 - u_2)...(1 - u_{k+1})\Big), \ -1 \le \alpha \le 1. \qquad (22)$$

These copulas have densities (9) with

$$g(u_1, u_2, ..., u_{k+1}) = \alpha(1 - 2u_1)(1 - 2u_2)...(1 - 2u_{k+1}). \qquad (23)$$

Corollary 2 *Let $k \ge 2$ and let C be a $(k+1)-$dimensional Eyraud-Farlie-Gumbel-Mongenstern copula (22) with density (9) where g is given by (23). A sequence of r.v.'s $\{X_t\}_{t=1}^{\infty}$ is a stationary $k-$independent $C-$based $k-$th order Markov process if and only if it is a sequence of i.i.d. r.v.'s.*

Corollary 3 is a generalization of Corollary (2) to the special case of $(k+1)-$dimensional power copulas (42) given by

$$C(u_1, u_2, ..., u_{k+1}) = \prod_{i=1}^{k+1} u_i \Big(1 + \alpha(u_1^l - u_1^{l+1})(u_2^l - u_2^{l+1})...(u_{k+1}^l - u_{k+1}^{l+1})\Big), \ -1 \le \alpha \le 1, \qquad (24)$$

where $l \in \{0, 1, 2, ...\}$ (copulas (24) reduce to those in (22) for $l = 0$). These copulas have density (9) in which

$$g(u_1, u_2, ..., u_{k+1}) = \alpha\Big((l+1)u_1^l - (l+2)u_1^{l+1}\Big)\Big((l+1)u_2^l - (l+2)u_2^{l+1}\Big)...\Big((l+1)u_{k+1}^l - (l+2)u_{k+1}^{l+1}\Big). \qquad (25)$$

Corollary 3 *Let $k \ge 2$ and let C be a $(k+1)-$dimensional power copula (24) with density (9), where g is given by (25). A sequence of r.v.'s $\{X_t\}_{t=1}^{\infty}$ is a stationary $k-$independent $C-$based $k-$th order Markov process if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.'s.*

Theorem 6 is an analogue of Theorem 5 that provides impossibility/reduction results for $m-$dependent Markov processes. Theorem 6 shows that construction of non-trivial examples (that is, those more general than sequences of i.i.d. r.v.'s) of stationary Markov processes exhibiting $m-$dependence is impossible on the base of bivariate copulas that have, similar, to Theorem 5, the function g in a separable product form.

Theorem 6 *Suppose that C is a bivariate copula that has the density $\partial^2 C(u_1, u_2)/\partial u_1 \partial u_2 = 1 + \alpha f(u_1) f(u_2)$ for some $\alpha \in \mathbf{R}$ and some continuous function $f : [0, 1] \to \mathbf{R}$. A sequence of r.v.'s $\{X_t\}_{t=1}^{\infty}$ is a stationary $m-$dependent $C-$based Markov process (of the first order) if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.'s.*

The following corollary is a specialization of Theorem 6 to the special case of bivariate Eyraud-Farlie-Gumbel-Mongenstern copulas (22) with $k = 1$:

$$C(u_1, u_2) = u_1 u_2 \Big(1 + \alpha(1 - u_1)(1 - u_2)\Big), \ -1 \le \alpha \le 1, \qquad (26)$$

that have density (13) with

$$g(u_1, u_2) = \alpha(1 - 2u_1)(1 - 2u_2). \qquad (27)$$

Corollary 4 *Let C be a bivariate copula Eyraud-Farlie-Gumbel-Mongenstern copula (26) with density (13), where g is given by (27). A sequence of r.v.'s $\{X_t\}_{t=1}^{\infty}$ is a stationary $m-$dependent $C-$based Markov process (of the first order) if and only if $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.'s.*

The results in this section that demonstrate that Markov processes with $m-$dependence and $r-$independent Markov processes of higher order cannot be constructed from Eyraud-Farlie-Gumbel-Mongenstern copulas and other separable copulas complement and substantially generalize the results of Cambanis (1991). Cambanis (1991) showed that the most common dependence structures such as constant, exponential and $m-$dependence cannot be exhibited by stationary processes $\{X_t\}$ whose finite-dimensional copulas are the following multivariate analogues of bivariate Eyraud-Farlie-Gumbel-Mongenstern copulas (41):

$$C_{j_1,...,j_n}(u_{j_1},...,u_{j_n}) = \prod_{s=1}^{n} u_{j_k}\Big(1 + \sum_{1\le l<m\le n} \alpha_{lm}(1-u_{j_l})(1-u_{j_m})\Big).$$

The results also complement the above-mentioned results by Rosenblatt and Slepian (1962) on non-existence of non-trivial $N-$th order $N-$independent Markov processes consisting of two-valued r.v.'s since, as follows from Sharakhmetov and Ibragimov (2002), the finite-dimensional copulas of sequences of r.v.'s concentrated on two points have multivariate Eyraud-Farlie-Gumbel-Mongenstern structure (41).

Remark 5 *Interestingly, in contrast to normal copulas, $t-$copulas cannot be used to construct (possibly higher-order) Markov processes that exhibit $r-$independence or $m-$dependence even if their correlation matrices are identity matrices corresponding to the case of uncorrelatedness. For instance, let $C(u_1,...,u_{k+1}) = C^t_{\nu,R}(u_1,...,u_{k+1})$ be a $t-$copula with correlation matrix R defined in (39) with $n = k+1$ and let $\{X_t\}_{t=1}^{\infty}$ be a stationary $C-$based $k-$th order Markov processes. Then relation (20) holds for $Y_t = (\sqrt{S}/\sqrt{\nu})\cdot t_\nu^{-1}[F(X_t)]$, where $S \sim \chi^2_\nu$ is a r.v. with chi-square distribution with ν degrees of freedom that is independent of $\{X_t\}_{t=1}^{\infty}$, $t_\nu(x)$ is the cdf of the univariate Student $t-$distribution with ν degrees of freedom and $F(x)$ is the cdf of X_t. As above, we obtain that $R = I$ and $\{Y_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. standard normal r.v.'s if $\{X_t\}_{t=1}^{\infty}$ is $r-$independent for some $r \ge 2$. Since the components of the random vector*

$$(F^{-1}[t_\nu(\sqrt{\nu}Y_{t-1}/\sqrt{S})], F^{-1}[t_\nu(\sqrt{\nu}Y_t/\sqrt{S})]) \tag{28}$$

are dependent if Y_{t-1} and Y_t are i.i.d. standard normal r.v.'s independent of $S \sim \chi^2_\nu$, we thus conclude that there does not exist a stationary $r-$independent $C^t_{\nu,R}-$based $k-$th order Markov process for any correlation matrix R.

Let $k = 1$ and let $C(u_1,u_2) = C^t_{\nu,\rho}(u_1,u_2)$ be a bivariate $t-$copula in (40). Then the process $Y_t = (\sqrt{S}/\sqrt{\nu})\cdot t_\nu^{-1}[F(X_t)]$ satisfies (21). We thus conclude that if $\{X_t\}_{t=1}^{\infty}$ is a stationary $m-$dependent $C^t_{\nu,\rho}-$based Markov process (of the first order), then $\rho = 0$ and $\{Y_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. standard normal r.v.'s. As before, this, together with dependence of the components of random vector (28) imply that there does not exist a stationary $m-$dependent $C^t_{\nu,\rho}-$based Markov process of the first order for any value of ρ.

5 Fourier copulas

The results on limitations of Eyraud-Farlie-Gumbel-Mongenstern, separable, normal and $t-$copulas presented in the previous section emphasize the substantial technical difficulty in constructing copula-based time series with flexible dependence structures. A class of copulas based on expansions by Fourier polynomials we introduce in this section allows one to overcome this difficulty.

It is not difficult to check that the conditions of Theorem 2 are satisfied for the following functions g :

$$g(u_1, ..., u_{k+1}) = \sum_{j=1}^{N} \big[\alpha_j sin(2\pi \sum_{i=1}^{k+1} \beta_i^j u_i) + \gamma_j cos(2\pi \sum_{i=1}^{k+1} \beta_i^j u_i)\big], \tag{29}$$

where $N \geq 1$, and $\alpha_j, \gamma_j \in \mathbf{R}$, and $\beta_i^j \in \mathbf{Z}$, $i = 1, ..., k+1$, $j = 1, ..., N$, are arbitrary numbers such that

$$\beta_1^{j_1} + \sum_{l=2}^{s} \delta_{l-1} \beta_l^{j_l} \neq 0,$$

for $j_1, ..., j_s \in \{1, ..., N\}$, $\delta_1, ..., \delta_{s-1} \in \{-1, 1\}$, $s = 2, ..., k+1$, and

$$1 + \sum_{j=1}^{N} \big[\alpha_j \delta_j + \gamma_j \delta_{j+N}\big] \geq 0$$

for $\delta_1, ..., \delta_{2N} \in \{-1, 1\}$. We refer to the copulas C corresponding to the functions g in such a way,

$$C(u_1, ..., u_{k+1}) = \int_0^{u_1} ... \int_0^{u_{k+1}} (1 + g(u_1, ..., u_{k+1})) du_1 ... du_{k+1},$$

as $(k+1)-$dimensional *Fourier* copulas. Each such copulas can thus be used to construct a stationary $k-$independent $k-$th order Markov process via (6).

Similarly, conditions of Theorem 3 are satisfied with $m = 1$ for the bivariate Fourier copulas corresponding to the functions g defined in (29) with $k = 1$, that is, for the Fourier copulas

$$C(u_1, u_2) = \int_0^{u_1} \int_0^{u_2} (1 + g(u_1, u_2)) du_1 du_2, \tag{30}$$

where

$$g(u_1, u_2) = \sum_{j=1}^{N} \big[\alpha_j sin(2\pi(\beta_1^j u_1 + \beta_2^j u_2)) + \gamma_j cos(2\pi(\beta_1^j u_1 + \beta_2^j u_2))\big], \tag{31}$$

$N \geq 1$, $\alpha_j, \gamma_j \in \mathbf{R}$, and $\beta_1^j, \beta_2^j \in \mathbf{Z}$, $j = 1, ..., N$, are arbitrary numbers such that $\beta_1^{j_1} + \beta_2^{j_2} \neq 0$ for $j_1, j_2 \in \{1, ..., N\}$ and $\beta_1^{j_1} - \beta_2^{j_2} \neq 0$, $1 + \sum_{j=1}^{N} \big[\alpha_j \delta_j + \gamma_j \delta_{j+N}\big] \geq 0$ for $\delta_1, ..., \delta_{2N} \in \{-1, 1\}$. The processes constructed from copulas (30) via (6) thus give examples of stationary $1-$dependent first-order Markov processes.

6 Conclusion

The results in this paper provide a justification for estimation of $(k+1)-$dimensional copulas for stationary time series with k-th order Markovian dependence structure. For instance, the results imply that all finite-dimensional copulas and, thus, all copula-based multivariate dependence measures and properties of such time series (such as, for instance, multivariate analogues of Spearman's rho or relative entropy for finite-dimensional distributions and $\beta-$mixing properties) are determined by and can be recovered from their $(k+1)-$dimensional copulas. The paper also shows how dependence properties of time series place additional non-trivial restrictions on copulas of the processes in consideration that can be applied in inference on the properties of the processes. These restrictions also allow one to construct time series with prescribed dependence structures that can be used, for instance, in the analysis of the robustness of statistical and econometric procedures to dependence. The results further provide an approach to obtaining approximations of higher-order Markov-processes and their functionals, including those that arise in contingent claim pricing and other applications in economics, finance and risk management (see Cherubini, Luciano and Vecchiato, 2004, and McNeil, Frey and Embrechts, 2005), using approximations to their copulas of a given order. This can be accomplished, for instance, using expansions of the copulas by orthogonal functions or degenerate $U-$statistic kernels as in de la Peña, Ibragimov and Sharakhmetov (2006) and Proposition 3 in this paper, Bernstein polynomials as in Sancetta and Satchell (2004), Hermite polynomials as in Gram-Charlier copulas discussed in Ibragimov (2005a), or Fourier polynomials similar to the constructions discussed in Section 5.[7] The study of the above problems is left for future research.

Appendix A1. Copulas and their properties. $U-$statistics-based copula characterizations

We begin with the definition of copulas and formulation of Sklar's theorem referred to in the introduction (see, e.g., Nelsen, 1999, Embrechts, McNeil and Straumann, 2002, and Embrechts, Lindskog and McNeil, 2003).

Definition 5 *A function $C : [0,1]^n \rightarrow [0,1]$ is called a $n-$dimensional* copula *if it satisfies the following conditions:*

1. *$C(u_1, ..., u_n)$ is increasing in each component u_i.*
2. *$C(u_1, ..., u_{k-1}, 0, u_{k+1}, ..., u_n) = 0$ for all $u_i \in [0,1]$, $i \neq k$, $k = 1, ..., n$.*
3. *$C(1, ..., 1, u_i, 1, ..., 1) = u_i$ for all $u_i \in [0,1]$, $i = 1, ..., n$.*
4. *For all $(a_1, ..., a_n)$, $(b_1, ..., b_n) \in [0,1]^n$ with $a_i \leq b_i$,*

$$\sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+...+i_n} C(x_{1i_1}, ..., x_{ni_n}) \geq 0,$$

[7]See also the recent work by Lowin, 2007, for the analysis of properties and economic and financial applications of Fourier copulas introduced in Section 5 and their generalizations.

where $x_{j1} = a_j$ *and* $x_{j2} = b_j$ *for all* $j \in \{1, ..., n\}$. *Equivalently,* C *is a* $n-$*dimensional copula if it is a joint cdf of* n *r.v.'s* $U_1, ..., U_n$ *each of which is uniformly distributed on* $[0, 1]$.

Definition 6 *A copula* $C : [0, 1]^n \to [0, 1]$ *is called* absolutely continuous *if, when considered as a joint cdf, it has a joint density given by* $\partial C^n(u_1, ..., u_n)/\partial u_1...\partial u_n$.

Proposition 1 *(Sklar, 1959). If* $X_1, ..., X_n$ *are r.v.'s defined on a common probability space, with the one-dimensional cdf's* $F_{X_k}(x_k) = P(X_k \leq x_k)$ *and the joint cdf* $F_{X_1,...,X_n}(x_1, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n)$, *then there exists an* $n-$*dimensional copula* $C_{X_1,...,X_n}(u_1, ..., u_n)$ *such that* $F_{X_1,...,X_n}(x_1, ..., x_n) = C_{X_1,...,X_n}(F_{X_1}(x_1), ..., F_{X_n}(x_n))$ *for all* $x_k \in \mathbf{R}$, $k = 1, ..., n$. *If the univariate marginal cdf's* $F_{X_1}, ..., F_{X_n}$ *are all continuous, then the copula is unique and can be obtained via* inversion *method:*

$$C_{X_1,...,X_n}(u_1, ..., u_n) = F_{X_1,...,X_n}(F_{X_1}^{-1}(u_1), ..., F_{X_n}^{-1}(u_n)), \qquad (32)$$

where $F_{X_k}^{-1}(u_k) = \inf\{x : F_{X_k}(x) \geq u_k\}$. *Otherwise, the copula is uniquely determined at points* $(u_1, ..., u_n)$, *where* u_k *is in the range of* F_k, $k = 1, ..., n$.

Copulas are invariant under strictly increasing transformations of r.v.'s with continuous univariate cdf's.

Proposition 2 *Let* X_k, $1 \leq k \leq n$, *be r.v.'s with continuous univariate marginal cdf's* F_{X_k} *and a copula* C. *If* $f_k : \mathbf{R} \to \mathbf{R}$, $1 \leq k \leq n$, *are strictly increasing functions, then the r.v.'s* $Y_k = f_k(X_k)$ *have the same copula* C.

In a more general case when f_k are strictly monotone (either strictly increasing or decreasing) functions there exist simple relations expressing the copula of $Y_k = f(X_k)$, $k = 1, ..., n$, in terms of the copula of X_k, $k = 1, ..., n$ (see Theorem 2.4.4 in Nelsen, 1999, and Remark 1 for the case $n = 2$).

The following proposition provides $U-$statistics-based characterizations of copulas.

Proposition 3 *(de la Peña, Ibragimov and Sharakhmetov, 2006). A function* $C : [0, 1]^n \to [0, 1]$ *is an absolutely continuous* $n-$*dimensional copula if and only if there exist functions* $g_{i_1,...,i_c} : [0, 1]^c \to \mathbf{R}$, $1 \leq i_1 < ... < i_c \leq n$, $c = 2, ..., n$, *satisfying the conditions*

A1 (integrability):

$$\int_0^1 ... \int_0^1 |g_{i_1,...,i_c}(t_{i_1}, ..., t_{i_c})| dt_{i_1}...dt_{i_c} < \infty,$$

A2 (degeneracy):

$$E(g_{i_1,...,i_c}(u_{i_1}, ..., u_{i_{k-1}}, u_{i_k}, u_{i_{k+1}}, ..., u_{i_c})|u_{i_1}, ..., u_{i_{k-1}}, u_{i_{k+1}}, ..., u_{i_c}) =$$

$$\int_0^1 g_{i_1,...,i_c}(u_{i_1}, ..., u_{i_{k-1}}, t_{i_k}, u_{i_{k+1}}, ..., u_{i_c}) dt_{i_k} = 0,$$

$1 \leq i_1 < ... < i_c \leq n$, $k = 1, 2, ..., c$, $c = 2, ..., n$, *A3 (positive definiteness):*

$$\sum_{c=2}^{n} \sum_{1 \le i_1 < ... < i_c \le n} g_{i_1,...,i_c}(u_{i_1}, ..., u_{i_c}) \ge -1,$$

and such that

$$C(u_1, ..., u_n) = \int_0^{u_1} ... \int_0^{u_n} \left(1 + \sum_{c=2}^{n} \sum_{1 \le i_1 < ... < i_c \le n} g_{i_1,...,i_c}(t_{i_1}, ..., t_{i_c})\right) \prod_{i=1}^{n} dt_i, \tag{33}$$

or, equivalently, such that the density of C satisfies

$$\frac{\partial^n C(u_1, ..., u_n)}{\partial u_1 ... \partial u_n} = 1 + \sum_{c=2}^{n} \sum_{1 \le i_1 < ... < i_c \le n} g_{i_1,...,i_c}(u_{i_1}, ..., u_{i_c}). \tag{34}$$

R.v.'s $X_1, ..., X_n$ with copula $C(u_1, ..., u_n)$ are jointly independent if and only if C is the product copula:

$$C(u_1, ..., u_n) = u_1 ... u_n. \tag{35}$$

Well-studied examples of copulas are given by, for example, Clayton, Gumbel and Frank copulas (see Joe, 1997, and Nelsen, 1999). Taking in (32) F to be the $n-$dimensional normal cdf with linear correlation matrix R:

$$F(x) = \Phi_R^n(x) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} \phi_{n,R}(x) dx, \tag{36}$$

with $\phi_{n,R}(x) = 1/((2\pi)^{n/2}|R|^{1/2}) \exp(-\frac{1}{2} x R^{-1} x)$, one obtains the well-known normal, or Gaussian, copula $C_R^n(u_1, ..., u_n)$:

$$C_R^n(u_1, ..., u_n) = \Phi_R^n(\Phi^{-1}(u_1), ..., \Phi^{-1}(u_n)), \tag{37}$$

where, as in Section 4, $\Phi(x)$ denotes the standard normal univariate cdf. In the bivariate case, the normal copula can be written as

$$C_\rho(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big(-\frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{2(1-\rho^2)}\Big) du_1 du_2, \tag{38}$$

where ρ is the linear correlation coefficient of the corresponding bivariate normal distribution.[8]

Let $\nu > 0$ and let F be the $n-$dimensional Student $t-$cdf $t_{\nu,R}^n$ with ν degrees of freedom, the linear correlation matrix R and the location parameter $0 \in \mathbf{R}^n$. That is, $F = t_{\nu,R}^n$ is the joint cdf of the random vector $\sqrt{\nu} Y / \sqrt{S}$, where $Y \sim \mathcal{N}_n(0, R)$ has the $n-$dimensional normal distribution with the

[8]Using approximations to multinormal cdf's via multidimensional Hermite polynomials (see Slepian, 1972), one can obtain a class of approximations to normal copulas (37) by so-called Gram-Charlier copulas discussed in Ibragimov, 2005*a*.

correlation matrix R and $S \sim \chi^2(\nu)$ is a chi-square r.v. with ν degrees of freedom that is independent of Y. Formula (32) then gives $n-$dimensional $t-$copulas with correlation matrix R:

$$C^t_{\nu,R}(u_1, ..., u_n) = t^n_{\nu,R}(t^{-1}_\nu(u_1), ..., t^{-1}_\nu(u_n)), \tag{39}$$

where $t_\nu(x)$ denotes the cdf of the univariate Student t distribution with ν degrees of freedom.

In the bivariate case, $t-$copulas take the form

$$C^t_{\nu,\rho}(u_1, u_2) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \int_{-\infty}^{t_\nu^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \Big(1 + \frac{u_1^2 - 2\rho u_1 u_2 + u_2^2}{\nu(1-\rho^2)}\Big)^{-(\nu+2)/2} du_1 du_2, \tag{40}$$

where $\rho \in (-1, 1)$.

Let $\alpha_{i_1,...,i_c} \in \mathbf{R}$ be constants such that $\sum_{c=2}^n \sum_{1\le i_1<...<i_c\le n} \alpha_{i_1,...,i_c}\delta_{i_1}...\delta_{i_c} \ge -1$ for all $\delta_i \in \{-1, 1\}$, $i = 1, ..., n$. Taking in Proposition 3 $g_{i_1,...,i_c}(t_{i_1}, ..., t_{i_c}) = \alpha_{i_1,...,i_c}(1-2t_{i_1})(1-2t_{i_2})...(1-2t_{i_c})$, $1 \le i_1 < ... < i_c \le n$, $c = 2, ..., n$, we obtain the following generalized multivariate Eyraud-Farlie-Gumbel-Morgenstern copulas (see Johnson and Kotz, 1975, and Cambanis, 1977):

$$C(u_1, ..., u_n) = \prod_{k=1}^n u_k \left(1 + \sum_{c=2}^n \sum_{1\le i_1<...<i_c\le n} \alpha_{i_1,...,i_c}(1-u_{i_k})\right). \tag{41}$$

In the bivariate case these copulas have form (26). As shown in Sharakhmetov and Ibragimov (2002), the generalized Eyraud-Farlie-Gumbel-Morgenstern copulas and corresponding cdf's completely characterize joint distributions of two-valued r.v.'s.

Let now $\alpha_{i_1,...,i_c} \in \mathbf{R}$ be such that $\sum_{c=2}^n \sum_{1\le i_1<...<i_c\le n} |\alpha_{i_1,...,i_c}| \le 1$ (it is easy to see that this condition is satisfied if $\alpha_{i_1,...,i_c} = \lambda_{i_1}...\lambda_{i_c}$, where $\sum_{i=1}^n |\lambda_i| \le 1$). Taking in Proposition 3

$$g_{i_1,...,i_c}(t_{i_1}, ..., t_{i_c}) = \alpha_{i_1,...,i_c}\left((l+1)t^l_{i_1} - (l+2)t^{l+1}_{i_1}\right)...\left((l+1)t^l_{i_c} - (l+2)t^{l+1}_{i_c}\right),$$

where $l \in \{0, 1, 2, ...\}$, we get the following extensions of Eyraud-Farlie-Gumbel-Morgenstern copulas (41) that are natural to call power copulas:

$$C(u_1, ..., u_n) = \prod_{i=1}^n u_i \left(1 + \sum_{c=2}^n \sum_{1\le i_1<...<i_c\le n} \alpha_{i_1,...,i_c}(u^l_{i_1} - u^{l+1}_{i_1})...(u^l_{i_c} - u^{l+1}_{i_c})\right). \tag{42}$$

Proposition 3 implies the following characterizations of r.v.'s satisfying $m-$dependence or $r-$independence.

Proposition 4 *(de la Peña, Ibragimov and Sharakhmetov, 2006, Theorem 13) Let $2 \le r < n$. R.v.'s $X_1, ..., X_n$ are $r-$independent if and only if the functions $g_{i_1,...,i_c}$ in representation (33) satisfy the conditions $g_{i_1,...,i_c}(u_{i_1}, ..., u_{i_c}) = 0$, $1 \le i_1 < ... < i_c \le n$, $c = 2, ..., r$.*

Proposition 5 *(de la Peña, Ibragimov and Sharakhmetov, 2006, Theorem 11). Let $1 \le m \le n-2$. R.v.'s $X_1, ..., X_n$ are $m-$dependent if and only if the functions g in representation (33) satisfy the conditions $g_{i_1,...,i_k,i_{k+1},...,i_c}(u_{i_1}, ..., u_{i_k}, u_{i_{k+1}}, ..., u_{i_c}) = g_{i_1,...,i_k}(u_{i_1}, ..., u_{i_k})g_{i_{k+1},...,i_c}(u_{i_{k+1}}, ..., u_{i_c})$ for all $1 \le i_1 < ... < i_k < i_{k+1}... < i_c \le n$, $i_{k+1} - i_k \ge m+1$, $k = 1, ..., c-1$, $c = 2, ..., n$.*

DNO obtained the following necessary and sufficient conditions for a time series process based on bivariate copulas to be first-order Markov. For copulas $A, B : [0,1]^2 \to [0,1]$, set

$$(A * B)(x,y) = \int_0^1 \frac{\partial A(x,t)}{\partial t} \cdot \frac{\partial B(t,y)}{\partial t} dt.$$

Further, for copulas $A : [0,1]^m \to [0,1]$ and $B : [0,1]^n \to [0,1]$, define their $\star-$product $A \star B : [0,1]^{m+n-1} \to [0,1]$ via

$$A \star B(x_1, ..., x_{m+n-1}) = \int_0^{x_m} \frac{\partial A(x_1, ..., x_{m-1}, \xi)}{\partial \xi} \cdot \frac{\partial B(\xi, x_{m+1}, ..., x_{m+n-1})}{\partial \xi} d\xi.$$

As shown in DNO, the operators $*$ and $\star$ on copulas are distributive over convex combinations, associative and continuous in each place, but not jointly continuous.

DNO proved that the transition probabilities $P(s, x, t, A) = P(X_t \in A | X_s = x)$ of a real-valued stochastic process $\{X_t\}_{t \in T}$, $T \subseteq \mathbf{R}$, satisfy Chapman-Kolmogorov equations (8) if and only if the copulas corresponding to bivariate distributions of X_t are such that

$$C_{st} = C_{su} * C_{ut} \tag{43}$$

for all $s, u, t \in T$ such that $s < u < t$. DNO also showed that a real-valued stochastic process $\{X_t\}_{t \in T}$ is a first-order Markov process if and only if the copulas corresponding to the finite-dimensional distributions of $\{X_t\}$ satisfy the conditions

$$C_{t_1, ..., t_n} = C_{t_1 t_2} \star C_{t_2 t_3} \star ... \star C_{t_{n-1} t_n}$$

for all $t_1, ..., t_n \in T$ such that $t_k < t_{k+1}$, $k = 1, ..., n-1$.

Appendix A2. Proofs

As before, for a r.v. X_t in the process $\{X_t\}_{t \in T}$, we denote by F_t its cdf. As usual, for a Borel set $A \in \Im$, the notation $X^{-1}(A)$ will stand for the event $\{\omega \in \Omega : X(\omega) \in A\}$.

Proof of Theorem 1. Clearly, it suffices to consider the case $T = \mathbf{N}$. Let $n \geq k+1$. Let us show that the Markovian (order k) property (4) holds for $t_1 = 1, ..., t_n = n$ and $t = n+1$ if and only if

$$P\big(X_1 < x_1, ..., X_{n-k} < x_{n-k}, X_{n+1} < x_{n+1} \big| X_{n-k+1}, ..., X_n\big) =$$
$$P\big(X_1 < x_1, ..., X_{n-k} < x_{n-k} \big| X_{n-k+1}, ..., X_n\big) P\big(X_{n+1} < x_{n+1} \big| X_{n-k+1}, ..., X_n\big). \tag{44}$$

Indeed, suppose that (4) holds for $t_1 = 1, ..., t_n = n$ and $t = n + 1$. Then we have

$$\begin{aligned}
P\big(X_1 < x_1, ..., X_{n-k} < x_{n-k}, X_{n+1} < x_{n+1}\big|X_{n-k+1}, ..., X_n\big) &= \\
E\big(I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}I_{X_{n+1}<x_{n+1}}\big|X_{n-k+1}, ..., X_n\big) &= \\
E\Big\{E\big(I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}I_{X_{n+1}<x_{n+1}}\big|X_1, ..., X_n\big)\Big|X_{n-k+1}, ..., X_n\Big\} &= \\
E\Big\{I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}E\big(I_{X_{n+1}<x_{n+1}}\big|X_1, ..., X_n\big)\Big|X_{n-k+1}, ..., X_n\Big\} &= \\
E\Big\{I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}E\big(I_{X_{n+1}<x_{n+1}}\big|X_{n-k+1}, ..., X_n\big)\Big|X_{n-k+1}, ..., X_n\Big\} &= \\
E\big(I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}\big|X_{n-k+1}, ..., X_n\big)P\big(I_{X_{n+1}<x_{n+1}}\big|X_{n-k+1}, ..., X_n\big) &= \\
P\big(X_1 < x_1, ..., X_{n-k} < x_{n-k}\big|X_{n-k+1}, ..., X_n\big)P\big(X_{n+1} < x_{n+1}\big|X_{n-k+1}, ..., X_n\big)&,
\end{aligned}$$

that is, (44) holds. Conversely, if (44) holds, then from the above chain of equalities read in the opposite order it follows that

$$\begin{aligned}
E\Big\{I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}E\big(I_{X_{n+1}<x_{n+1}}\big|X_1, ..., X_n\big)\Big|X_{n-k+1}, ..., X_n\Big\} &= \\
E\Big\{I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}E\big(I_{X_{n+1}<x_{n+1}}\big|X_{n-k+1}, ..., X_n\big)\Big|X_{n-k+1}, ..., X_n\Big\}&,
\end{aligned}$$

that is, for all Borel subsets $B_{n-k+1}, ..., B_n$ of $\mathbf{R}$,

$$\begin{aligned}
E\Big\{I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}I_{X_{n-k+1}\in B_{n-k+1}}...I_{X_n\in B_n}E\big(I_{X_{n+1}<x_{n+1}}\big|X_1, ..., X_n\big)\Big\} &= \\
E\Big\{I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}I_{X_{n-k+1}\in B_{n-k+1}}...I_{X_n\in B_n}E\big(I_{X_{n+1}<x_{n+1}}\big|X_{n-k+1}, ..., X_n\big)\Big\}&.
\end{aligned}$$

This relation means that (4) holds for $t_1 = 1, ..., t_n = n$ and $t = n + 1$.

Suppose now that $\{X_t\}_{t=1}^{\infty}$, is a Markov process of order k. Integrating (44) over $X_{n-k+1}^{-1}((-\infty, x_{n-k+1})) \times ... \times X_n^{-1}((-\infty, x_n))$, we get

$$\begin{aligned}
C_{1,2,...,n+1}(F_1(x_1), ..., F_{n+1}(x_{n+1})) = F_{1,2,...,n+1}(x_1, ..., x_{n+1}) &= \\
\int_{-\infty}^{x_{n-k+1}} ... \int_{-\infty}^{x_n} C_{1,2,...,n|n-k+1,...,n}(F_1(x_1), ..., F_{n-k}(x_{n-k}), F_{n-k+1}(\eta_{n-k+1}), ..., F_n(\eta_n)) &\times \\
C_{n-k+1,...,n,n+1|n-k+1,...,n}(F_{n-k+1}(\eta_{n-k+1}), ..., F_n(\eta_n), F_{n+1}(x_{n+1})) &\times \\
dF_{n-k+1,...,n}(\eta_{n-k+1}, ..., \eta_n) &= \\
\int_0^{F_{n-k+1}(x_{n-k+1})} ... \int_0^{F_n(x_n)} C_{1,2,...,n|n-k+1,...,n}(F_1(x_1), ..., F_{n-k}(x_{n-k}), \xi_{n-k+1}, ..., \xi_n) &\times \\
C_{n-k+1,...,n,n+1|n-k+1,...,n}(\xi_{n-k+1}, ..., \xi_n, F_{n+1}(x_{n+1}))C_{n-k+1,...,n}(d\xi_{n-k+1}, ..., d\xi_n) &= \\
C_{1,2,...,n} \star^k C_{n-k+1,...,n+1}(F_1(x_1), ..., F_{n+1}(x_{n+1}))&.
\end{aligned}$$

By induction, this implies that (5) holds.

Conversely, suppose that (5) holds. Then we have

$$EI_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}I_{X_{n-k+1}<x_{n-k+1}}...I_{X_n<x_n}I_{X_{n+1}<x_{n+1}} =$$

$$C_{1,2,...,n+1}(F_1(x_1),...,F_{n-k}(x_{n-k}),F_{n-k+1}(x_{n-k+1}),...,F_n(x_n),F_{n+1}(x_{n+1})) =$$

$$\int_0^{F_{n-k+1}(x_{n-k+1})}...\int_0^{F_n(x_n)} C_{1,2,...,n|n-k+1,...,n}(F_1(x_1),...,F_{n-k}(x_{n-k}),\xi_{n-k+1},...,\xi_n)\times$$

$$C_{n-k+1,...,n,n+1|n-k+1,...,n}(\xi_{n-k+1},...,\xi_n,F_{n+1}(x_{n+1}))C_{n-k+1,...,n}(d\xi_{n-k+1},...,d\xi_n) =$$

$$E\big(E(I_{X_1<x_1}...I_{X_{n-k}<x_{n-k}}|X_{n-k+1},...,X_n)I_{X_{n-k+1}<x_{n-k+1}}...I_{X_n<x_n}E(X_{n+1}|X_{n-k+1},...,X_n)\big).$$

This implies that (44) holds. ■

Proof of Theorem 2. Let C be a $(k+1)-$dimensional copula and let $\{X_t\}_{t=1}^{\infty}$ be a stationary $C-$based $k-$th order Markov process. Using Propositions 3 and 4, we obtain that if the process $\{X_t\}_{t=1}^{\infty}$ is $k-$independent, then the density of the copula C has form (34) with $n=k+1$ and the functions g such that $g_{i_1,...,i_c}(u_{i_1},...,u_{i_c})=0$, $1\le i_1<...<i_c\le n$, $c=2,...,k$, that is, (9) holds with $g(u_1,...,u_{k+1}) = g_{1,...,k+1}(u_1,...,u_{k+1})$. In addition, by the same propositions, the above function g satisfies conditions (10) and (12) and is such that

$$\int_0^1 g(u_1,...,u_{k+1})du_j = 0, \;\; j=1,2,...,k+1. \tag{45}$$

Further, from Corollary 1 it follows that the density of the copula $C_{1,2,...,k+1,k+2}$ of the r.v.'s $X_1, X_2, ...,$ X_{k+1}, X_{k+2} is given by

$$\frac{\partial^{k+2}C_{1,2,...,k+1,k+2}(u_1,u_2,...,u_{k+1},u_{k+2})}{\partial u_1\partial u_2...\partial u_{k+1}\partial u_{k+2}} = (1+g(u_1,....,u_{k+1}))(1+g(u_2,....,u_{k+2})) =$$

$$1+g(u_1,....,u_{k+1})+g(u_2,....,u_{k+2})+g(u_1,....,u_{k+1})g(u_2,....,u_{k+2}). \tag{46}$$

Using (45) and (46) we get that, for $2\le i_1<i_2\le k+1$, the density of the copula of the r.v.'s X_j, $j\in\{1,2,...,k+2\}\setminus\{i_1,i_2\}$ is given by

$$1+\int_0^1\int_0^1 g(u_1,....,u_{k+1})g(u_2,....,u_{k+2})du_{i_1}du_{i_2}.$$

This and $k-$independence of $\{X_t\}$ imply that

$$\int_0^1\int_0^1 g(u_1,....,u_{k+1})g(u_2,....,u_{k+2})du_{i_1}du_{i_2} = 0, \;\; 2\le i_1<i_2\le k+1. \tag{47}$$

In complete similarity, by considering the $k-$dimensional marginal copulas of the r.v.'s $X_1, X_2, ...,$ X_{k+2}, X_{k+3} and using (45), (46) and (47), we obtain

$$\int_0^1\int_0^1\int_0^1 g(u_1,....,u_{k+1})g(u_2,....,u_{k+2})g(u_3,....,u_{k+3})du_{i_1}du_{i_2}du_{i_3} = 0,$$

$3\le i_1<i_2<i_3\le k+1$. Continuing in the same fashion, we get that the property that $\{X_t\}_{t=1}^{\infty}$ is a stationary $k-$independent $C-$based $k-$th order Markov process implies that (11) holds for all $s\le u_{i_1}<...<u_{i_s}\le k+1$, $s=1,2,...,\left[\frac{k+1}{2}\right]$.

Suppose now that relation (11) holds for all $s \leq u_{i_1} < ... < u_{i_s} \leq k+1$, $s = 1, 2, ..., \left[\frac{k+1}{2}\right]$. One then easily gets that (11) also holds for all $1 \leq u_{i_1} < ... < u_{i_s} \leq k+s$, $s \geq 1$, and the product $g(u_1, ..., u_{k+1})g(u_2, ..., u_{k+2})...g(u_s, ..., u_{k+s})$ that appears in the density $\frac{\partial^{k+s} C_{1,...,k+s}(u_1,...,u_{k+s})}{\partial u_1 ... \partial u_{k+s}} = \prod_{j=1}^{s}(1+g(u_j, ..., u_{k+j}))$ of the copula of $X_1, X_2, ..., X_{k+s}$. It is not difficult to see, similar to the above analysis, that this implies that the copula $C_{1,2,...,k+s}$ has $k-$dimensional marginal copulas in product form (35) with $n = k$. Thus, the r.v.'s $X_1, X_2, ..., X_{k+s}$ are $k-$independent for all $s \geq 1$. ■

Proof of Theorem 3. Let C be a bivariate copula and let $\{X_t\}_{t=1}^{\infty}$ be a stationary $C-$based first-order Markov process. By Proposition 3, the density of the copula C is given by (13) with the function $g(u_1, u_2)$ satisfying conditions (14)-(16). In addition, from Corollary 1 it follows that the density of the copula $C_{1,2,...,m+1,m+2}$ of the r.v.'s $X_1, X_2, ..., X_{m+1}, X_{m+2}$ has the form

$$\frac{\partial^{m+2} C_{1,2,...,m+1,m+2}(u_1, u_2, ..., u_{m+1}, u_{m+2})}{\partial u_1 \partial u_2 ... \partial u_{m+1} \partial u_{m+2}} =$$

$$(1 + g(u_1, u_2))(1 + g(u_2, u_3))...(1 + g(u_{m+1}, u_{m+2})) = \prod_{s=1}^{m+1}(1 + g(u_s, u_{s+1})).$$

Using relations (15) we thus get that the copula $C_{1,m+2}$ of the r.v.'s X_1 and X_{m+2} is given by

$$C_{1,m+2}(u_1, u_{m+2}) = \int_0^1 ... \int_0^1 \prod_{s=1}^{m+1}(1 + g(u_s, u_{s+1}))du_2...du_{m+1} =$$

$$1 + \int_0^1 ... \int_0^1 \prod_{s=1}^{m+1} g(u_s, u_{s+1})du_2...du_{m+1}.$$

Thus, the copula $C_{1,m+2}$ is the product copula: $C_{1,m+2}(u_1, u_{m+2}) = u_1 u_{m+2}$ if and only if condition (17) is satisfied. It is not difficult to see that if (17) holds, then one also has

$$\int_0^1 ... \int_0^1 \prod_{s=1}^{m+n+j} g(u_s, u_{s+1})du_{n+1}...du_{m+n} = 0$$

for all $n \geq 1$, $j \geq 0$. Similar to the above analysis, this relation, together with (15), implies that the random vectors $(X_1, ..., X_n)$ and $(X_{m+n+1}, ..., X_{m+n+j+1})$ in the stationary $C-$based Markov process $\{X_t\}_{t=1}^{\infty}$ are independent for all $n \geq 1$, $j \geq 0$. ■

Proof of Theorem 4. By Markov property, $P(X_t > x|\Im_{t-1}) = P(X_t < -x|\Im_{t-1})$, $x \geq 0$, if and only if $P(X_t > x|X_{t-1}) = P(X_t < -x|X_{t-1})$, $x \geq 0$. The latter inequality, in turn, is equivalent to $P(V_n > 1/2+u|V_{n-1}) = P(V_n < 1/2-u|V_{n-1})$, $u \in [0, 1/2)$, where $V_t = F(X_t)$ and, by stationarity, to $P(V_2 > 1/2+u|V_1) = P(V_2 < 1/2-u|V_1)$, $u \in [0, 1/2)$. We have therefore, that $\{X_t\}$ is a conditionally symmetric martingale difference if and only if $\frac{\partial C(V_1, 1/2-u)}{\partial u_1} = 1 - \frac{\partial C(V_1, 1/2+u)}{\partial u_1}$, or, equivalently, if and only if (18) and (19) hold. ■

Proof of Theorem 5. Using relations (11) in Theorem 2 with $s = 2$ and $i_1 = 2$, $i_2 = 3$, we get that if, under the conditions of Theorem 5, $\{X_t\}_{t=1}^{\infty}$ is a stationary $k-$independent $C-$based $k-$th order Markov process, then

$$\int_0^1 \int_0^1 g(u_1, u_2, ..., u_{k+1}) g(u_2, u_3, ..., u_{k+2}) du_2 du_3 =$$

$$\int_0^1 \int_0^1 \alpha^2 f(u_1) f^2(u_2) f^2(u_3) ... f^2(u_{k+1}) f(u_{k+2}) du_2 du_3 = 0,$$

that is,

$$\alpha^2 \Big[\int_0^1 f^2(u_2) du_2 \Big] \Big[\int_0^1 f^2(u_3) du_3 \Big] f(u_1) f^2(u_3) ... f^2(u_{k+1}) f(u_{k+2}) = 0.$$

This evidently implies that $g(u_1, u_2, ..., u_{k+1}) = \alpha f(u_1) ... f(u_{k+1}) = 0$ and, thus, $\{X_t\}$ is a sequence of i.i.d. r.v.'s. ■

Proof of Corollary 2. The corollary is a consequence of Theorem 5 applied to the function $f(u) = 1 - 2u$. ■

Proof of Corollary 3. The corollary is a consequence of Theorem 5 applied to the function $f(u) = (l+1)u^l - (l+2)u^{l+1}$. ■

Proof of Theorem 6. Using relation (17) in Theorem 3, we obtain that, if, under the conditions of Theorem 6, $\{X_t\}_{t=1}^{\infty}$ is a stationary $m-$dependent $C-$based Markov process, then

$$\int_0^1 ... \int_0^1 \alpha^{m+1} f(u_1) f^2(u_2) ... f^2(u_{m+1}) f(u_{m+2}) du_2 ... du_{m+1} = 0,$$

that is,

$$\alpha^{m+1} f(u_1) f(u_{m+2}) \Big[\int_0^1 f^2(u_2) du_2 \Big]^m = 0.$$

This evidently implies that $\alpha = 0$ or $f(u) = 0$ and, thus, $\{X_t\}_{t=1}^{\infty}$ is a sequence of i.i.d. r.v.'s. ■

Proof of Corollary 4. The corollary is a consequence of Theorem 6 applied to the function $f(u) = 1 - 2u$. ■

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