Appendix A3: Extensions to heterogeneity and skewness

This appendix presents extensions of the results in the paper to the case of skewed and heterogenous risks discussed in the introduction and in Remark 4.1.

Let $\sigma_1, ..., \sigma_n \geq 0$ be some scale parameters and let $X_i \sim S_\alpha(\sigma_i, \beta, 0)$, $\alpha \in (0, 2]$, be independent not necessarily identically distributed stable risks. Further, let $Z_{\tilde{w}} = \sum_{i=1}^n w_{[i]} X_i$ be the return on the portfolio with weights $\tilde{w} = (w_{[1]}, ..., w_{[n]})$, where, as in Section 4, $w_{[1]} \geq ... \geq w_{[n]}$ denote the components of the vector $w = (w_1, ..., w_n) \in \mathbf{R}_+^n$ in decreasing order (a certain ordering in the components of the vector of weights w is necessary for the extensions of the majorization results in this paper to the case of non-identically distributed r.v.'s X_i since property (6) holds for all functions $\phi : \mathbf{R}_+^n \to \mathbf{R}$ that are Schur-convex or Schur-concave).

Observe that, in the case of identically distributed X_i, $i = 1, ..., n$, with $\sigma_1 = ... = \sigma_n$, the risk $Z_{\tilde{w}}$ has the same distribution as the return $Z_w = \sum_{i=1}^n w_i X_i$ on the portfolio of X_i's with weights $w = (w_1, ..., w_n)$. That is, the VaR comparisons for $Z_{\tilde{w}}$ with the ordered weights $\tilde{w}$ under the above distributional assumptions cover the case of portfolio returns Z_w with identically distributed skewed risks.

Theorems A3.3 and A3.4 provide analogues of Theorems 4.3 and 4.4 in the skewed heterogenous case. The proof of the theorems uses Schur-convexity and Schur-concavity properties of the functions

$$\chi(c_1, ..., c_n) = \sum_{i=1}^n \sigma_i^\alpha c_{[i]}^\alpha, \tag{18}$$

$\alpha > 0$, where σ_i, $i = 1, ..., n$, are the scale parameters of the risks in consideration. These properties are similar to the definitions of $p-$majorization for the vectors $(c_1^\alpha, ..., c_n^\alpha)$ with $p = (\sigma_1^\alpha, ..., \sigma_n^\alpha)$ (see Ch. 14 in Marshall & Olkin (1979)). Thus, the results in this section suggest that extensions of the majorization pre-ordering such as $p-$majorization may be useful in formalizations of the concept of diversification for portfolios of heterogenous risks.

Let $Q \sim S_\alpha(1, \beta, 0)$.

Theorem A3.1 *Let $0 < q < P(Q > 0)$ and let $X_1, ..., X_n$ be independent risks such that $X_i \sim S_\alpha(\sigma_i, \beta, 0)$, where $\alpha \in (1, 2]$, $\sigma_1 \geq ... \geq \sigma_n > 0$ and $\beta \in [-1, 1]$. Then*

(i) $VaR_q(Z_{\tilde{v}}) < VaR_q(Z_{\tilde{w}})$ if $v \prec w$ and v is not a permutation of w (in other words, the function $\psi(w, q) = VaR_q(Z_{\tilde{w}})$ is strictly Schur-convex in $w \in \mathbf{R}_+^n$).

(ii) In particular, $VaR_q(Z_{\underline{w}}) < VaR_q(Z_{\tilde{w}}) < VaR_q(Z_{\overline{w}})$ *for all* $q \in (0, 1/2)$ *and all weights* $w \in \mathcal{I}_n$ *such that* $w \neq \underline{w}$ *and* w *is not a permutation of* $\overline{w}$.

Theorem A3.2 *Let* $0 < q < P(Q > 0)$ *and let* $X_1, ..., X_n$ *be independent risks such that* $X_i \sim S_\alpha(\sigma_i, \beta, 0)$, *where* $\alpha \in (0, 1)$, $\sigma_n \geq ... \geq \sigma_1 > 0$ *and* $\beta \in [-1, 1]$. *Then*

(i) $VaR_q(Z_{\tilde{v}}) > VaR_q(Z_{\tilde{w}})$ *if* $v \prec w$ *and* v *is not a permutation of* w *(in other words, the function* $\psi(w, q) = VaR_q(Z_w)$, *is strictly Schur-concave in* $w \in \mathbf{R}^n_+$*).*

(ii) In particular, $VaR_q(Z_{\overline{w}}) < VaR_q(Z_{\tilde{w}}) < VaR_q(Z_{\underline{w}})$ *for all* $q \in (0, 1/2)$ *and all weights* $w \in \mathcal{I}_n$ *such that* $w \neq \underline{w}$ *and* w *is not a permutation of* $\overline{w}$.

Theorem A3.3 *Let* $r \in (0, 2]$, $0 < q < P(Q > 0)$ *and let* $X_1, ..., X_n$ *be independent risks such that* $X_i \sim S_\alpha(\sigma_i, \beta, 0)$, $\alpha \in (0, 2]$, $\sigma_i > 0$, $\beta \in [-1, 1]$, $\beta = 0$ *for* $\alpha = 1$. *If* $\alpha > r$ *and* $\sigma_1 \geq ... \geq \sigma_n > 0$, *then*

(i) $VaR_q(Z_{\tilde{v}}) < VaR_q(Z_{\tilde{w}})$ *if* $(v_1^r, ..., v_n^r) \prec (w_1^r, ..., w_n^r)$ *and* $(v_1^r, ..., v_n^r)$ *is not a permutation of* $(w_1^r, ..., w_n^r)$ *(that is, the function* $\psi(w, q) = VaR_q(Z_{\tilde{w}})$, $w \in \mathbf{R}^n_+$, *is strictly Schur-convex in* $(w_1^r, ..., w_n^r)$*).*

(ii) The following sharp bounds hold:

$$n^{1-1/r}(\sum_{i=1}^{n} w_i^r)^{1/r} VaR_q(Z_{\underline{w}}) < VaR_q(Z_{\tilde{w}}) < (\sum_{i=1}^{n} w_i^r)^{1/r} VaR_q(Z_{\overline{w}})$$

for all $q \in (0, 1/2)$ *and all weights* $w \in \mathcal{I}_n$ *such that* $w \neq \underline{w}$ *and* w *is not a permutation of* $\overline{w}$.

Theorem A3.4 *Let* $r \in (0, 2]$, $0 < q < P(Q > 0)$ *and let* $X_1, ..., X_n$ *be independent risks such that* $X_i \sim S_\alpha(\sigma_i, \beta, 0)$, $\alpha \in (0, 2]$, $\sigma_i > 0$, $\beta \in [-1, 1]$, $\beta = 0$ *for* $\alpha = 1$. *If* $\alpha < r$ *and* $\sigma_n \geq ... \geq \sigma_1 > 0$, *then*

(i) $VaR_q(Z_{\underline{v}}) > VaR_q(Z_{\underline{w}})$ *if* $(v_1^r, ..., v_n^r) \prec (w_1^r, ..., w_n^r)$ *and* $(v_1^r, ..., v_n^r)$ *is not a permutation of* $(w_1^r, ..., w_n^r)$ *(that is, the function* $\psi(w, q) = VaR_q(Z_{\underline{w}})$, $w \in \mathbf{R}^n_+$ *is strictly Schur-concave in* $(w_1^r, ..., w_n^r)$*).*

(ii) The following sharp bounds hold :

$$(\sum_{i=1}^{n} w_i^r)^{1/r} VaR_q(Z_{\overline{w}}) < VaR_q(Z_{\underline{w}}) < n^{1-1/r}(\sum_{i=1}^{n} w_i^r)^{1/r} VaR_q(Z_{\underline{w}})$$

for all $q \in (0, 1/2)$ *and all weights* $w \in \mathcal{I}_n$ *such that* $w \neq \underline{w}$ *and* w *is not a permutation of* $\overline{w}$.

Remark A3.1 *Using conditioning arguments, one gets that the extensions provided by Theorems A3.3- A3.2 also hold in the case of random scale parameters* σ_i. *Similar to the proof of Theorems 5.1-5.4 and A3.3- A3.2, one can also show that analogues of the theorems hold for dependent heterogenous skewed risks, including convolutions (10) of common shock models (8) with skewed non-identically distributed risks* Y_{ij}.

Proof of Theorems A3.3-A3.2.

Let $r, \alpha \in (0, 2]$, $\sigma_1, ..., \sigma_n > 0$, and let $v = (v_1, ..., v_n) \in \mathbf{R}_+^n$ and $w = (w_1, ..., w_n) \in \mathbf{R}_+^n$ be two vectors of portfolio weights such that $(v_1^r, ..., v_n^r) \prec (w_1^r, ..., w_n^r)$ and $(v_1^r, ..., v_n^r)$ is not a permutation of $(w_1^r, ..., w_n^r)$ (as in the proof of Theorems 4.3 and 4.4, these assumptions evidently imply $\sum_{i=1}^n v_i \neq 0$ and $\sum_{i=1}^n w_i \neq 0$). Let $X_1, ..., X_n$ be independent risks such that $X_i \sim S_\alpha(\sigma, 0, 0)$, $i = 1, ..., n$. Similar to the proof of Theorems 4.3 and 4.4, we note that from (3) it follows that if $c = (c_1, ..., c_n) \in \mathbf{R}_+^n$, $\sum_{i=1}^n c_i \neq 0$, then $\sum_{i=1}^n c_{[i]} X_i / \Big(\sum_{i=1}^n c_{[i]}^\alpha \sigma_i\Big)^{1/\alpha} \sim S_\alpha(1, \beta, 0)$. Using positive homogeneity of the value at risk (see property a3 in Section 3), we thus obtain that, for all $0 < q < P(Q > 0)$,

$$VaR_q(\sum_{i=1}^n c_{[i]} X_i) = VaR_q(Q)\Big(\sum_{i=1}^n c_{[i]}^\alpha \sigma_i\Big)^{1/\alpha}. \tag{19}$$

By Theorem 3.A.4 in Marshall & Olkin (1979), the function $\chi(c_1, ..., c_n)$ defined in (18) is strictly Schur-convex in $(c_1, ..., c_n) \in \mathbf{R}_+^n$ if $\alpha > 1$ and $\sigma_1 \geq ... \geq \sigma_n \geq 0$ and is strictly Schur-concave in $(c_1, ..., c_n) \in \mathbf{R}_+^n$ if $\alpha < 1$ and $\sigma_n \geq ... \geq \sigma_1 \geq 0$ (see also Propositions 3.H.2.b and 4.B.7 in Marshall & Olkin (1979)). Therefore, we have $\sum_{i=1}^n v_{[i]}^\alpha \sigma_i^\alpha = \sum_{i=1}^n (v_{[i]}^r)^{\alpha/r} \sigma_i^\alpha < \sum_{i=1}^n (w_{[i]}^r)^{\alpha/r} \sigma_i^\alpha = \sum_{i=1}^n w_{[i]}^\alpha \sigma_i^\alpha$, if $\alpha/r > 1$ and $\sigma_1 \geq ... \geq \sigma_n > 0$. Similarly, $\sum_{i=1}^n w_{[i]}^\alpha \sigma_i^\alpha = \sum_{i=1}^n (w_{[i]}^r)^{\alpha/r} \sigma_i^\alpha < \sum_{i=1}^n (v_{[i]}^r)^{\alpha/r} \sigma_i^\alpha = \sum_{i=1}^n v_{[i]}^\alpha \sigma_i^\alpha$, if $\alpha/r < 1$ and $\sigma_n \geq ... \geq \sigma_1 > 0$. This, together with (19), implies that, for all $q \in (0, 1/2)$, $VaR_q(Z_{\tilde{v}}) < VaR_q(Z_{\tilde{w}}$ if $\alpha > r$ and $\sigma_1 \geq ... \geq \sigma_n \geq 0$ and $VaR_q(Z_{\tilde{v}}) > VaR_q(Z_{\tilde{w}})$ if $\alpha < r$ and $\sigma_n \geq ... \geq \sigma_1 \geq 0$ This completes the proof of parts (i) of Theorems A3.3 and A3.4.

The bounds in parts (ii) of Theorems A3.3 and A3.4 follow from their parts (i) and majorization comparisons (15). Sharpness of the bounds is already established in Theorems 4.3 and 4.4.

Theorems A3.1 and A3.2 are consequences of Theorems A3.3 and A3.4 with $r = 1$.

■