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ON EXTREMAL PROBLEMS AND BEST CONSTANTS IN MOMENT INEQUALITIES

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SUMMARY. In the present paper, we show that the best constant $A^*(t,\gamma)$ in the Rosenthal-type inequality with an arbitrary balancing factor $\gamma > 0$

$$E\left(\sum_{i=1}^{n} X_i\right)^t \le A(t,\gamma)\max\left(\gamma\sum_{i=1}^{n} EX_i^t, \left(\sum_{i=1}^{n} X_i\right)^t\right)$$

for independent nonnegative random variables $X_1, \ldots, X_n$ with finite t-th moment, $1 < t < \infty$, is given by

$$A^*(t,\gamma) = 1 + 1/\gamma, 1 < t \le 2,$$
$$A^*(t,\gamma) = \gamma^{-t/(t-1)} EZ^t(\gamma^{1/(t-1)}), t > 2,$$

where $Z(\gamma^{1/(t-1)})$ is a Poisson random variable with parameter $\gamma^{1/(t-1)}$. In addition to that, we obtain estimates for the best constants in analogues of the above inequality for independent random variables with a set of zero odd moments that generalize and complement the results known for mean-zero random variables and symmetric random variables.

1. **Introduction**

In what follows, $A(.)$, $B(.)$ and $C_i(.), i = 1, 2, 3, 4$, denote constants depending only on parameters in parentheses.

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Let γ be a fixed positive number. Let us consider the following Rosenthal-type inequality:

$$E\left(\sum_{i=1}^{n} X_i\right)^t \leq A(t,\gamma)\max\left(\gamma\sum_{i=1}^{n} EX_i^t, \left(\sum_{i=1}^{n} X_i\right)^t\right) \tag{1}$$

for all independent nonnegative random variables (r.v.'s) $X_1, \ldots, X_n$ with finite t-th moment, $1 < t < \infty$.

Several studies have focused on estimates related to inequality (1) and constants in them (see, e.g., Prokhorov (1962), Dharmadhikari et. al. (1968), Dharmadhikari and Jogdeo (1969), Sazonov (1974), Pinelis (1980), Pinelis and Utev (1984), Utev (1985), Pinelis (1994), Peshkir and Shiryaev (1995), Ostrovskiĭ (1999) and references therein).

Denote by $A^*(t,\gamma)$ the best constant in inequality (1). Johnson et al. (1985) showed that the actual rate of growth of $(A^*(t,1))^{1/t}$ is t/lnt as $t \to \infty$ and obtained the same result for the best constant in the analogue of inequality (1) with $\gamma = 1$ for mean-zero r.v.'s. Figiel et al. (1997) and Ibragimov and Sharakhmetov (1995, 1997) independently obtained the best constant in the analogue of balancing factor free inequality (1) (with $\gamma = 1$) for symmetric r.v.'s. Ibragimov and Sharakhmetov (1998, 2001) obtained the best constant in the analogue of balancing parameter free inequality (1) for mean-zero r.v.'s in the case of even integer t. The results obtained in Ibragimov and Sharakhmetov (1995, 1997, 1998, 2001) and their proofs were presented in Ibragimov (1997).

The "balancing" role that parameter γ plays in inequality (1) is important, because the terms $\sum_{k=1}^{n} EX_k^t$ and $\left(\sum_{k=1}^{n} EX_k\right)^t$ may be, and usually are, of different orders of magnitude (e.g., Pinelis (1994)). In the present paper, we determine the best constant $A^*(t,\gamma)$ in general inequality (1) with an arbitrary balancing factor $\gamma > 0$. We show that the constant $A^*(t,\gamma)$ is given by

$$A^*(t,\gamma) = 1 + 1/\gamma, 1 < t \leq 2,$$
$$A^*(t,\gamma) = \gamma^{-t/(t-1)} EZ^t(\gamma^{1/(t-1)}), t > 2,$$

where $Z(\gamma^{1/(t-1)})$ is a Poisson random variable with parameter $\gamma^{1/(t-1)}$. In addition to the above, we prove estimates for the best constants in the analogues of inequality (1) for r.v.'s with a set of zero odd moments which generalize and complement the results known for mean-zero r.v.'s and symmetric r.v.'s. A part of the results in the paper was announced in Ibragimov and Sharakhmetov (1998) and presented in Ibragimov (1997).

2. An Extremal Problem for Sums of Independent Nonnegative Random Variables with Fixed Sum of Tails of Distributions

Let $\mathbf{R}_+ = [0,\infty), \mathcal{T}$ be the σ-algebra of Borel subsets of $\mathbf{R}_+$ and Λ be the class of finite positive σ-additive measures λ on $\mathcal{T}$ such that $\lambda(\{0\}) = 0$. For a measure $\lambda \in \Lambda$ denote by $T(\lambda)$ the random variable with characteristic function $Ee^{itT(\lambda)} = \exp\left(\int_0^\infty (e^{itx}-1)d\lambda(x)\right)$.

Let $\lambda \in \Lambda$. Set (here and in what follows (X,n) denotes the set of independent nonnegative r.v.'s $(X_1,\dots,X_n)$) $W(\lambda) = \{(X,n) : n \geq 1, \sum\limits_{i=1}^{n} P(X_i \in B\backslash\{0\}) = \lambda(B), B \in \mathcal{T}\}$. Denote by $W^{(i.d.)}(\lambda)$ the subset of $W(\lambda)$ consisting of identically distributed r.v.'s. Let $Z(d)$ be a r.v. with Poisson distribution with parameter $d > 0 : P(Z(d) = k) = e^{-d}d^k/k!, k = 0,1,2,\dots$

The following theorem complements the results obtained by Utev (1985).

Theorem 1. *If* $t > 1, \lambda \in \Lambda$, and $\int_0^\infty x^t d\lambda(x) < \infty$, then

$$\sup_{(X,n)\in W(\lambda)} E\left(\sum_{i=1}^{n} X_i\right)^t = \sup_{(X,n)\in W^{(i.d.)}(\lambda)} E\left(\sum_{i=1}^{n} X_i\right)^t = ET^t(\lambda) < \infty. \quad (2)$$

Proof of Theorem 1. Suppose that X and Y are nonnegative r.v.'s, X has a distribution $\lambda \in \Lambda$, the r.v.'s X, Y and $T(\lambda)$ are independent, $\int_0^\infty x^t d\lambda(x) < \infty$ and $EY^t < \infty$. The distribution of the r.v. $T(\lambda)$ is the same as the distribution of the r.v. $\sum\limits_{i=1}^{Z(1)} X_i$, where $X_1, X_2, \dots,$ is a sequence of independent r.v.'s with distribution λ. We have

$$\sum_{k=1}^{\infty} E\left(\sum_{i=1}^{k} X_i + Y\right)^t /k! \leq \sum_{k=1}^{\infty} (k+1)^{t-1}(kEX_1^t + EY^t)/k! < \infty.$$

From Proposition 3.I.1.b in Marshall and Olkin (1979) it follows that if $t > 1$, then the function $\sum\limits_{i=1}^{n} x_i^t$ is Schur-convex on $\mathbf{R}_+^n$, that is, $\sum_{i=1}^n x_i^t \leq \sum_{i=1}^n y_i^t$ if $x_i, y_i \geq 0$, $i = 1,...,n$, $\sum\limits_{i=1}^{k} x_{[i]} \leq \sum\limits_{i=1}^{k} y_{[i]}, k = 1,\dots,n-1$, $\sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i$, where $x_{[1]} \geq \dots \geq x_{[n]}$ and $y_{[1]} \geq \dots \geq y_{[n]}$ denote the components of the vectors $(x_1,\dots,x_n)$ and $(y_1,\dots,y_n)$ arranged in descending order. Taking here $(x_1,\dots,x_{n-1},x_n) = (a_1+x,\dots,a_{n-1}+x,a_n+x)$ and $(y_1,\dots,y_{n-1},y_n) = (x,\dots,x,\sum\limits_{i=1}^{n} a_i + x)$, we obtain that

$$(n-1)x^t + \left(\sum_{i=1}^{n} a_i + x\right)^t \geq \sum_{i=1}^{n} (a_i + x)^t$$

for all $t > 1$, $a_1, \ldots, a_n, x \in \mathbf{R}_+$. Therefore, similarly to the proof of Lemmas 3.6 and 6.2 in Utev (1985), we get

$$E(X+Y)^t - e^{-1}EY^t = e^{-1} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{k} E(X_i + Y)^t - (k-1)EY^t \right) /k! \leq$$

$$e^{-1} \sum_{k=1}^{\infty} E \left(\sum_{i=1}^{k} X_i + Y \right)^t /k!,$$

that is, $E(X+Y)^t \leq e^{-1} \sum_{k=0}^{\infty} E \left(\sum_{i=1}^{k} X_i + Y \right)^t /k! = E(T(\lambda) + Y)^t < \infty$. From the latter inequality by induction it follows that

$$E \left(\sum_{i=1}^{n} X_i \right)^t \leq ET^t(\lambda) < \infty \tag{3}$$

for all $(X, n) \in W(\lambda)$. Taking a triangular array of independent nonnegative r.v.'s $X_{1n} \ldots X_{nn}$ such that $P(X_{in} \in B \setminus \{0\}) = n^{-1}\lambda(B)$ for $B \in \mathcal{T}, i = 1, \ldots, n, n \geq \lambda(\mathbf{R}_+)$, and using the Fatou lemma for convergence in distribution similarly to Pinelis and Utev (1984) and Utev (1985), we get that bounds (3) are exact and relations (2) hold. The proof is complete.

3. Extrema of Moments of Sums of Independent Nonnegative Random Variables

Let $\gamma > 0, t > 1, a_i \geq 0, b_i \geq 0, a_i^t \leq b_i, i = 1, \ldots, n$; $a_i > 0$ if $b_i > 0$, $i = 1, ..., n$; $F_t, G, H_t > 0$, and let $X_1, ..., X_n, \ldots$ denote independent nonnegative r.v.'s with finite t-th moment. In what follows, write $(y, n) = (y_1, \ldots, y_n)$. Similarly to Utev (1985) and Ibragimov and Sharakhmetov (1997), set

$$M_1(n, t, a, b) = \{(X, n) : EX_i = a_i, EX_i^t = b_i, i = 1, \ldots, n\},$$

$$M_2(n, t, a, b) = \{(X, n) : EX_i \leq a_i, EX_i^t \leq b_i, i = 1, \ldots, n\},$$

$$\begin{aligned} U_1(F_t, G) &= \bigcup_{\substack{n \geq 1, (a,n), (b,n): \\ \Sigma a_i = G, \Sigma b_i = F_t}} M_1(n, t, a, b) \\ &= \{(X, n) : n \geq 1, \sum_{i=1}^{n} EX_i = G, \sum_{i=1}^{n} EX_i^t = F_t\}, \end{aligned}$$

$$\begin{aligned} U_2(F_t, G) &= \bigcup_{\substack{n\ge 1,(a,n),(b,n):\\ \Sigma a_i \le G, \Sigma b_i \le F_t}} M_2(n,t,a,b) \\ &= \{(X,n) : n \ge 1, \sum_{i=1}^n EX_i \le G, \sum_{i=1}^n EX_i^t \le F_t\}, \end{aligned}$$

$$\overline{U}(\gamma, H_t) = \{(X,n) : n \ge 1, \max\left(\left(\sum_{i=1}^n EX_i\right)^t, \gamma \sum_{i=1}^n EX_i^t\right) = H_t\}.$$

Denote $\bar{Z}(F_t, G) = Z((G^t/F_t)^{1/(t-1)})$. Let $V_1(t, a_1, b_1), \dots, V_n(t, a_n, b_n)$ be independent r.v.'s with distributions $P(V(t,a,b) = 0) = 1 - (a^t/b)^{1/(t-1)}$, $P(V(t,a,b) = (b/a)^{1/(t-1)}) = (a^t/b)^{1/(t-1)}$, if $a, b > 0$; and $P(V(t,a,b) = 0) = 1$, if $a = b = 0$.

Theorem 2. *If* $1 < t \le 2$, *then*

$$\sup_{(X,n)\in M_k(n,t,a,b)} E\left(\sum_{i=1}^n X_i\right)^t = \sum_{i=1}^n (b_i - a_i^t) + \left(\sum_{i=1}^n a_i\right)^t, \tag{4}$$

$$\sup_{(X,n)\in U_k(F_t,G)} E\left(\sum_{i=1}^n X_i\right)^t = F_t + G^t, k = 1, 2. \tag{5}$$

If $t > 2$, *then*

$$\sup_{(X,n)\in M_k(n,t,a,b)} E\left(\sum_{i=1}^n X_i\right)^t = E\left(\sum_{i=1}^n V_i(t, a_i, b_i)\right)^t, \tag{6}$$

$$\sup_{(X,n)\in U_k(F_t,G)} E\left(\sum_{i=1}^n X_i\right)^t = (F_t/G)^{t/(t-1)} E\bar{Z}^t(F_t, G), k = 1, 2. \tag{7}$$

Remark 1. From relation (5) it follows that for $1 < t \le 2$ the problem of determining the best constants $C_1^*(t)$ and $C_2^*(t)$ in the inequality $E\left(\sum_{i=1}^n X_i\right)^t \le C_1(t)\left(\sum_{i=1}^n EX_i^t\right) + C_2(t)\left(\sum_{i=1}^n EX_i\right)^t$ for independent nonnegative r.v.'s $X_1, \dots, X_n$ with finite t-th moment, $1 < t < \infty$, and a more general problem of determining the exact upper bounds for $E\left(\sum_{i=1}^n X_i\right)^t$ in terms of $\sum_{i=1}^n EX_i^t$ and $\sum_{i=1}^n EX_i$ are equivalent. (5) also implies that $C_1^*(t) = C_2^*(t) =$

$1, 1 < t \leq 2$. Similarly, from Proposition in Ibragimov and Sharakhmetov (1997) it follows that in the case $2 < t \leq 4$ the problem of determining the best constants $C_3^*(t)$ and $C_4^*(t)$ in the inequality $E\left|\sum_{i=1}^n X_i\right|^t \leq C_3(t)\left(\sum_{i=1}^n E|X_i|^t\right) + C_4(t)\left(\sum_{i=1}^n EX_i^2\right)^{t/2}$ for independent symmetric r.v.'s $X_1, \ldots, X_n$ with finite t-th moment, $2 < t < \infty$, is equivalent to the problem of determining the exact upper estimates for $E\left|\sum_{i=1}^n X_i\right|^t$ in terms of $\sum_{i=1}^n E|X_i|^t$ and $\sum_{i=1}^n EX_i^2$. Moreover, $C_3^*(t) = 1$ and, $C_4^*(t) = 2^{t/2}\Gamma((t+1)/2)/\sqrt{\pi}, 2 < t \leq 4$.

Let us formulate some auxiliary results needed for the proof of Theorem 2.

Let $\mathcal{T}$ and Λ be the same as in Section 2, and let

$$\bar{\Lambda}_1(F_t, G) = \{\lambda \in \Lambda : \int_0^\infty x d\lambda(x) = G, \int_0^\infty x^t d\lambda(x) = F_t\}.$$

$$\bar{\Lambda}_2(F_1, G) = \{\lambda \in \Lambda : \int_0^\infty x d\lambda(x) \leq G, \int_0^\infty x^t d\lambda(x) \leq F_t\}.$$

Denote by $U_j^{(i.d.)}(F_t, G)$, $j = 1, 2$, the subsets of $U_j(F_t, G), j = 1, 2$, respectively, consisting of identically distributed r.v.'s.

It is evident that

$$\sup_{(X,n)\in U_k(F_t,G)} E\left(\sum_{i=1}^n X_i\right)^t = \sup_{\lambda\in\bar{\Lambda}_k(F_t,G)} \sup_{(X,n)\in W(\lambda)} E\left(\sum_{i=1}^n X_i\right)^t,$$

$$\sup_{(X,n)\in U_k^{(i.d.)}(F_t,G)} E\left(\sum_{i=1}^n X_i\right)^t = \sup_{\lambda\in\bar{\Lambda}_k(F_t,G)} \sup_{(X,n)\in W^{(i.d.)}(\lambda)} E\left(\sum_{i=1}^n X_i\right)^t,$$

$k = 1, 2$.

Using the latter relations and Theorem 1, we obtain the following lemma.

Lemma 1. *If $t \geq 1$, then*

$$\begin{aligned}\sup_{(X,n)\in U_k(F_t,G)} E\left(\sum_{i=1}^n X_i\right)^t &= \sup_{(X,n)\in U_k^{(i.d.)}(F_t,G)} E\left(\sum_{i=1}^n X_i\right)^t \\ &= \sup_{\lambda\in\bar{\Lambda}_k(F_t,G)} E(T(\lambda))^t, k = 1, 2.\end{aligned}$$

Let $f(t,z,v) = v^{-1/(t-1)}((v^{1/(t-1)}+z)^t - z^t), g(t,z,v) = (v+z)^t - v^t - z^t, z, v > 0$.

LEMMA 2. *If* $1 < t \le 2$, *then the function* $g(t,z,v)$ *is concave in* $v > 0$. *If* $t > 2$, *then the function* $f(t,z,v)$ *is concave in* $v > 0$.

PROOF. It is not difficult to see that

$$\partial^2 f(t,z,v)/\partial v^2 = (t/(t-1)^2)v^{-1/(t-1)-2}z^t(((1+u_1)^{t-1}-1)-$$

$$(t-1)u_1(1+u_1)^{t-2}),$$

$$\partial^2 g(t,z,v)/\partial v^2 = t(t-1)v^{t-2}((1+u_2)^{t-2}-1),$$

where $u_1 = v^{1/(t-1)}/z, u_2 = z/v$. Since $1 \le (1+u)^{2-t}$ for $1 < t \le 2, u > 0$, and $1 + (2-t)u \le (1+u)^{2-1}$ for $t > 2, u > 0$, we have $\partial^2 g(t,z,v)/\partial v^2 \le 0$, if $1 < t \le 2$, and $\partial^2 f(t,z,v)/\partial v^2 \le 0$, if $t > 2$. The proof is complete.

For $a, b > 0$, $a^t \le b$, let $V(t,a,b)$ denote a r.v. with distribution

$$P(V(t,a,b) = 0) = 1 - (a^t/b)^{1/(t-1)},$$

$$P(V(t,a,b) = (b/a)^{1/(t-1)}) = (a^t/b)^{1/(t-1)}.$$

LEMMA 3. *If* $t > 2, z \ge 0$ *and* X *is a nonnegative r.v. such that* $EX = a > 0, EX^t = b > 0$, *then*

$$E(X+z)^t \le E(V(t,a,b)+z)^t. \tag{8}$$

PROOF. According to Hoeffding (1955), it suffices it prove (8) for discrete r.v.'s X. Let $P(X = x_i) = p_i$, $x_i > 0$, $i = 1, \ldots, n$, $\sum_{i=1}^n p_i \le 1$, $\sum_{i=1}^n x_i p_i = a$, $\sum_{i=1}^n x_i^t p_i = b$. Since $EX^t = EV^t(t,a,b) = b$, one can assume that $z > 0$. Denote $v_i = x_i^{t-1}$, $q_i = x_i p_i/a$, $i = 1, \ldots, n$. Since $\sum_{i=1}^n q_i = 1$, $\sum_{i=1}^n v_i q_i = b/a$, from Lemma 2 it follows that $\sum_{i=1}^n f(t,z,v_i)q_i \le f(t,z,b/a)$. It is easy to see that the latter inequality is equivalent to relation (8). The proof is complete.

LEMMA 4. *If* $t > 2, 0 < a_1 \le a_2, 0 < b_1 \le b_2, a_i^t \le b_i, i = 1,2, z \ge 0$, *then*

$$E(V(t,a_1,b_1)+z)^t \le E(V(t,a_2,b_2)+z)^t \tag{9}$$

Proof. It is not difficult to see that (9) is equivalent to the inequality

$$f(t,z,y_1)r \le f(t,z,y_2), \tag{10}$$

where $r = a_1/a_2, y_j = b_j/a_j, j = 1,2$. It is evident that $0 < r \le 1, y_1 r \le y_2, y_1, y_2 > 0$. Let $r < 1$. Set $x = (y_2 - r y_1)/(1-r)$. Under the assumptions of the lemma, the function f is nonnegative and concave in $v > 0$. Consequently, $f(t,z,y_1)r \le f(t,z,y_1)r + f(t,z,x)(1-r) \le f(t,z,y_1 r + x(1-r)) = f(t,z,y_2)$. It is easy to see that the function $f(t,z,v)$ is nondecreasing in $v > 0$. This implies inequality (10) for $r = 1$. The proof is complete.

Lemmas 3 and 4 imply the following

Lemma 5. If $t > 2, a, b > 0, a^t \le b, X, Y$ and $V(t,a,b)$ are independent nonnegative r.v.'s such that $EX \le a, EX^t \le b, EY^t < \infty$, then $E(X+Y)^t \le E(V(t,a,b)+Y)^t$.

Lemma 6. If $1 < t \le 2, X, Y$ are independent nonnegative r.v.'s such that $EX = a > 0, EX^t = b > 0, EY^t < \infty$, then $E(X+Y)^t - EX^t \le E(a+Y)^t - a^t$.

Proof. It is evident that it suffices to consider the case $Y = z > 0$. Let us show that

$$E(X+z)^t - EX^t \le (a+z)^t - a^t. \tag{11}$$

It suffices to consider discrete r.v.'s X. Let $P(X = x_i) = p_i$, $x_i > 0$, $i = 1,\dots,n$, $\sum_{i=1}^n p_i \le 1$, $\sum_{i=1}^n x_i p_i = a$. From Lemma 2 it follows that

$$\begin{aligned}
\sum_{i=1}^n g(t,z,x_i)p_i &= \left(\sum_{i=1}^n g(t,z,x_i)p_i \Big/ \left(\sum_{i=1}^n p_i\right)\right)\left(\sum_{i=1}^n p_i\right) \\
&\le g\left(t,z,a\Big/\left(\sum_{i=1}^n p_i\right)\right)\left(\sum_{i=1}^n p_i\right) \\
&= g\left(t,z,a\Big/\left(\sum_{i=1}^n p_i\right)\right)\left(\sum_{i=1}^n p_i\right) + g(t,z,0)\left(1-\left(\sum_{i=1}^n p_i\right)\right) \\
&\le g(t,z,a).
\end{aligned}$$

It is easy to see that the latter inequality is equivalent to (11).

Lemma 7. *Let* $1 < t \le 2, a, b > 0, a^t \le b, Y$ *be a nonnegative r.v. with* $EY^t < \infty$*,* J *be the set of nonnegative r.v.'s* X *which are independent of* Y *and satisfy the conditions* $EX = a, EX^t = b$*. Then*

$$\sup_{X \in J} E(X+Y)^t = b + E(a+Y)^t - a^t. \tag{12}$$

Proof. From Lemma 6 it follows that it suffices to find a sequence of nonnegative r.v.'s X_n which are independent of Y and satisfy the conditions $EX_n = a, EX_n^t = b$ and $\lim_{n\to\infty} E(X_n+Y)^t = b+E(a+Y)^t - a^t$. If $b = a^t$, then it suffices to take $X_n = a$. Let $a^t < b$. Similarly to the proof of Lemma 7.6 in Utev (1985) set $\delta_n = 1/n, P(X_n = a) = 1 - \delta_n$, $P(X_n = b_n) = \delta_n^*, P(X_n = 0) = \delta_n - \delta_n^*$, where $\delta_n^* = a\delta_n/b_n, b_n = ((b - a^t(1-\delta_n))/(a\delta_n))^{1/(t-1)}$. It is evident that $b_n \ge a, 0 \le \delta_n^* \le \delta_n, \delta_n \to 0, b_n \to \infty, b_n^t\delta_n^* \to b - a^t$. We have

$$E(X_n+Y)^t = E(a+Y)^t(1-\delta_n) + EY^t(\delta_n - \delta_n^*) + (E(b_n+Y)^t - b_n^t)\delta_n^* + b_n^t\delta_n^*.$$

It suffices to check that $(E(b_n+Y)^t - b_n^t)\delta_n^* \to 0$. Since (see Lemma 7.5 in Utev (1985))

$$\Big||1+x|^t - 1\Big| \le 2^t t(|x| + |x|^t)$$

for $t \ge 1, x \in \mathbf{R}$, we obtain $(E(b_n+Y)^t - b_n^t)\delta_n^* \le b_n^t\delta_n^* 2^t t(EY/b_n + EY^t/b_n^t) \to 0$. The proof is complete.

Since the function $(a+z)^t - a^t, t > 1$, is nondecreasing in $a \ge 0$ for $z \ge 0$, we obtain the following

Lemma 8. *If* $t > 1, 0 \le a_1 \le a_2, 0 \le b_1 \le b_2, Y$ *is a nonnegative r.v. such that* $EY^t < \infty$*, then*

$$b_1 + E(a_1+Y)^t - a_1^t \le b_2 + E(a_2+Y)^t - a_2^t.$$

Proof of Theorem 2. Relations (4) and (6) easily follow from Lemmas 5, 7 and 8 by induction and conditioning argument. If $1 < t \le 2$, then, applying Lemma 1 and relation (4), we obtain

$$\begin{aligned}\sup_{(X,n)\in U_1(F_t,G)} E\left(\sum_{i=1}^n X_i\right)^t &= \sup_{(X,n)\in U_1^{(i.d.)}(F_t,G)} E\left(\sum_{i=1}^n X_i\right)^t \\ &= \sup_n (F_t + G^t - G^t/n^{t-1}) = F_t + G^t\end{aligned} \tag{13}$$

$$\begin{aligned}\sup_{(X,n)\in U_2(F_t,G)} E\left(\sum_{i=1}^{n} X_i\right)^t &= \sup_{(X,n)\in U_2^{(i.d.)}(F_t,G)} E\left(\sum_{i=1}^{n} X_i\right)^t \\ &= \sup_{\substack{0<F_t'\le F_t\\ 0<G'\le G}} (F_t' + G'^t) = F_t + G^t.\end{aligned} \tag{14}$$

(13) and (14) imply relation (5). If $t > 2$, then from Lemma 1 and relation (6) it follows that

$$\begin{aligned}\sup_{(X,n)\in U_1(F_t,G)} E\left(\sum_{i=1}^{n} X_i\right)^t &= \sup_{(X,n)\in U_1^{(i.d.)}(F_t,G)} E\left(\sum_{i=1}^{n} X_i\right)^t \\ &= \sup_n E\left(\sum_{i=1}^{n} V_i(t, Gn^{-1}, F_t n^{-1})\right)^t,\end{aligned} \tag{15}$$

and, in addition to that,

$$\begin{aligned}\sup_{(X,n)\in U_2(F_t,G)} E\left(\sum_{i=1}^{n} X_i\right)^t &= \sup_{(X,n)\in U_2^{(i.d.)}(F_t,G)} E\left(\sum_{i=1}^{n} X_i\right)^t \\ &= \sup_{\substack{0<F_t'\le F_t\\ 0<G'\le G}} \; \sup_{(X,n)\in U_1^{(i.d.)}(F_t,G)} E\left(\sum_{i=1}^{n} X_i\right)^t \\ &= \sup_n \sup_{\substack{0<F_t'\le F_t\\ 0<G'\le G}} E\left(\sum_{i=1}^{n} V_i(t, G'n^{-1}, F_t' n^{-1})\right)^t \\ &= \sup_n E\left(\sum_{i=1}^{n} V_i(t, Gn^{-1}, F_t n^{-1})\right)^t.\end{aligned} \tag{16}$$

We have that for all $B \in \mathcal{T}$ $\sum_{i=1}^{n} P(V_i(t, Gn^{-1}, F_t n^{-1}) \in B\setminus\{0\}) = \lambda(B)$, where $\lambda \in \Lambda, \lambda(\{F_t/G)^{1/(t-1)}\}) = \lambda(\mathbf{R}_+) = (G^t/F_t)^{1/(t-1)}$. The distribution of the r.v. $T(\lambda)$ is the same as the distribution of the r.v. $\bar{Z}(F_t, G)$. From Theorem 1 it follows that

$$\sup_n E\left(\sum_{i=1}^{n} V_i(t, Gn^{-1}, F_t n^{-1})\right)^t = E\bar{Z}^t(F_t, G). \tag{17}$$

(relation (17) also follows from the results obtained by Hoeffding (1956)). Using (15)-(17), we get (7).

4. **The Best Constants in Rosenthal's Inequalities**

The following theorem gives the best constant in general inequality (1) with an arbitrary balancing factor $\gamma > 0$.

THEOREM 3. The best constant in inequality (1) is given by

$$A^*(t,\gamma) = 1 + 1/\gamma, 1 < t \leq 2, \tag{18}$$

$$A^*(t,\gamma) = \gamma^{-t/(t-1)} EZ^t(\gamma^{1/(t-1)}), t > 2. \tag{19}$$

Proof. Set

$$K(\gamma, H_t) = \sup_{(X,n)\in \bar{U}(\gamma,H_t)} E\left(\sum_{i=1}^{n} X_i\right)^t.$$

Using obvious inequalities

$$\sup_{(X,n)\in U_1(H_t/\gamma, H_t^{1/t})} E\left(\sum_{i=1}^{n} X_i\right)^t \leq K(\gamma, H_t) \leq \sup_{(X,n)\in U_2(H_t/\gamma, H_t^{1/t})} E\left(\sum_{i=1}^{n} X_i\right)^t$$

and relations (5) and (7), we obtain $K(\gamma, H_t) = (1 + 1/\gamma)H_t$ for $1 < t \leq 2$ and $F(H_t) = \gamma^{-t(t-1)} EZ^t(\gamma^{1/(t-1)})H_t$ for $t > 2$. Since $A^*(t,\gamma) = \sup_{H_t > 0} K(\gamma, H_t)/H_t$, we get (18) and (19). The proof is complete.

REMARK 3. Using the results obtained in Ibragimov and Sharakhmetov (1997) similarly to the proof of Theorem 3, we get that the best constant $B^*_{sym}(t,\gamma)$ in the analogue of inequality (1) for symmetric r.v.'s

$$E\left|\sum_{i=1}^{n} X_i\right|^t \leq B(t,\gamma) \max\left(\gamma \sum_{i=1}^{n} E|X_i|^t, \left(\sum_{i=1}^{n} EX_i^2\right)^{t/2}\right) \tag{20}$$

is given by

$$\begin{aligned} B^*_{sym}(t,\gamma) &= 1 + E|N|^t/\gamma, 2 < t \leq 4 \\ B^*_{sym}(t,\gamma) &= \gamma^{-t/(t-1)} E|Z_1(0.5\gamma^{1/(t-1)}) - Z_2(0.5\gamma^{1/(t-1)})|^t, t > 4, \end{aligned}$$

where N is the standard normal r.v. and $Z_1(0.5\gamma^{1/(t-1)})$ and $Z_2(0.5\gamma^{1/(t-1)})$ are independent Poisson r.v.'s with parameter $0.5\gamma^{1/(t-1)}$.

In conclusion, let us mention a combinatorial interpretation of the best constants in balancing factor free inequality (1) and its analogues (20) for symmetric and mean-zero r.v.'s and provide estimates with a similar interpretation for the best constants in inequality (20) for r.v.'s with a set of zero odd moments.

From Dobinski's formula (see, e.g., Sachkov (1996), p. 262) it follows that the best constant $A^*(m,1) = EZ^m(1)$ in inequality (1) with $\gamma = 1$ for $t = m$ equals the $m-$the Bell number, that is, the total number of partitions of an $m-$element set into blocks;

By Theorem 2 in Ibragimov and Sharakhmetov (1998), the best constant $B^*_{sym}(2m,1)$ in inequality (20) with $\gamma = 1$ and $t = 2m$ for symmetric r.v.'s equals the number of partitions of a $2m-$element set into blocks consisting of an even number of elements.

According to Ibragimov and Sharakhmetov (2001), the best constant $B^*_{mean-zero}(2m,1)$ in estimate (20) with $\gamma = 1$ and $t = 2m$ for mean-zero r.v.'s equals the number of partitions of a $2m-$element set into blocks each of which contains more than one element.

Let $1 \le l \le m$, $k_1 = 1 < k_2 < ... < k_l$ be arbitrary elements of the set $\{2s-1, s = 1,2,...,m\}$ and let $B^*_{k_1,k_2,...,k_l}(2m)$ denote the best constant in inequality (20) with $\gamma = 1$, $t = 2m$ for independent r.v.'s $X_1, X_2, ..., X_n$ with $EX_i^{2m} < \infty$, $EX_i^{k_1} = EX_i^{k_2} = ... = EX_i^{k_l} = 0$, $i = 1,2,...,n$. Also, let $D_{k_1,k_2,...,k_l}(2m)$ be the number of partitions of a $2m-$element set into blocks, the number of elements in which does not equal k_s, $s = 1,2,...,l$. According to Ibragimov and Sharakhmetov (2001), if $X_1, X_2, ..., X_n$ are independent r.v.'s with $EX_i = 0$, $EX_i^{2m} < \infty$, $i = 1,2,...,n$, then

$$E(\sum_{i=1}^n X_i)^{2m} \le (2m)! \sum_{r=0}^{2m} \sum \prod_{k=1}^r \frac{A^{j_k}_{m_k,n}(m_k!)^{-j_k}}{j_k!},$$

where $A_{m_k,n} = \sum_{i=1}^n EX_i^{m_k}$, and the inner sum is taken over all natural $m_1 > m_2 > ... > m_r > 1$ and $j_1, j_2, ..., j_r$ such that $m_1 j_1 + m_2 j_2 + ... + m_r j_r = 2m$. From the latter inequality it follows that if $X_1, X_2, ..., X_n$ are independent r.v.'s such that $EX_i^{2m} < \infty$, $EX_i^{k_1} = EX_i^{k_2} = ... = EX_i^{k_l} = 0$, $i = 1,2,...,n$, then

$$E(\sum_{i=1}^n X_i)^{2m} \le (2m)! \sum_{r=0}^{2m} \sum \prod_{k=1}^r \frac{A^{j_k}_{m_k,n}(m_k!)^{-j_k}}{j_k!},$$

where the inner sum is taken over all natural $m_1 > m_2 > ... > m_r$ and $j_1, j_2, ..., j_r$ such that $m_1 j_1 + m_2 j_2 + ... + m_r j_r = 2m$, $m_i \ne k_s$, $s = 1,2,...,l$,

$i = 1, 2, ..., r$. From the inequality $|A_{s,n}| \le \left(A_{2m,n}^{s-2}A_{2,n}^{2m-s}\right)^{1/(2m-2)}$ for all integer $s \in (2, 2m)$ (see, e.g., Lemma 2 in Ibragimov and Sharakhmetov (2001)) and Sachkov (1996), pp. 257-259, we get that for all independent r.v.'s $X_1, X_2, ..., X_n$ such that $EX_i^{2m} < \infty$, $EX_i^{k_1} = EX_i^{k_2} = ... = EX_i^{k_l} = 0$, $i = 1, 2, ..., n$,

$$E(\sum_{i=1}^{n} X_i)^{2m} \le (2m)! \sum_{j=1}^{2m} \left(\sum_{r=1}^{j} \sum \prod_{k=1}^{r} \frac{(m_k!)^{-j_k}}{j_k!} \right) (A_{2m,n}^{m-j} A_{2,n}^{m(j-1)})^{1/(m-1)} \le$$

$$(2m)! \sum_{j=1}^{2m} \left(\sum_{r=1}^{j} \sum \prod_{k=1}^{r} \frac{(m_k!)^{-j_k}}{j_k!} \right) max(A_{2m,n}, A_{2,n}^{m}) =$$

$$D_{k_1,k_2,...,k_l}(2m) max(A_{2m,n}, A_{2,n}^{m}),$$

where the inner sums are taken over all natural $m_1 > m_2 > ... > m_r$ and $j_1, j_2, ..., j_r$ satisfying the conditions $m_1 j_1 + m_2 j_2 + ... + m_r j_r = 2m$, $j_1 + j_2 + ... + j_r = j$, $m_i \neq k_s$, $i = 1, 2, ..., r$, $s = 1, 2, ..., l$. Consequently,

$$B_{k_1,k_2,...,k_l}^{*}(2m) \le D_{k_1,k_2,...,k_l}(2m).$$

Therefore, the best constants in inequality (20) for r.v.'s with zero odd moments of order $k_1, k_2, \ldots .k_l$ are majorized by the numbers of partitions of a $2m-$element set into blocks, the number of elements in which does not equal k_s, $s = 1, 2, ..., l$. Moreover, from the above it follows that the latter bounds are sharp in the extremal cases $l = 1$ and $l = m$, $k_s = 2s - 1$, $s = 1, 2, ..., m$, that is, in the cases of mean-zero random variables and random variables with m zero first odd moments.

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