Copulas and Their Applications in Risk Management and Finance

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Outline

Motivation

Dependence vs. marginals

- **Fundamental problems** in **finance** & **risk management**: **Solution** is **affected** by **both**
    - Properties of **marginal** distributions (**heavy-tailedness**, **skewness**)
    - **Dependence** (**positive** or **negative** dependence, dependence **asymmetry**)

- Example: Optimal **portfolio choice** & **value at risk** (VaR) theory
  - **Marginal** effects under **independence**: degree of **heavy-tailedness** of risks or returns (not extremely heavy-tailed vs. extremely heavy-tailed) $\Longrightarrow$ **Opposite solutions**
  - **Dependent** $\alpha-$**symmetric** risks: Interplay of **heavy-tailedness** & **common shock dependence** structure; **diversification pays off** even under **extreme heavy-tailedness** in marginals
  - Different solutions: **Positive vs. negative** dependence
- Need reliable **measures** of **risk & dependence**

What is wrong with variances & correlations?

- **Variance** and **correlation**: **most widely** used **measures** of **risk** & **dependence**
  - **Crucial** in many **models** in **economics, finance** & **risk management**
  - Optimal **portfolio selection** in **mean-variance** framework
  - Capital asset pricing theory, arbitrage pricing theory
  - Foundation for **autocorrelation function** (ACF)-based **inference** in **time series** analysis
  - **Linear regression** models

What is wrong with variances & correlations?

- **Problematic** in a number of real-world settings

- Work well under **multivariate normality** and, more generally, **elliptic** distributions (affine transformations of **spherical**)

  Departures from **normality**: **typical** for financial & insurance data

- **Heavy-tailedness** problem
  - **Defined** for risks and returns with **finite second** moments: $Er_t^2 < \infty$
  - A number of time series in finance & insurance: **infinite variances** and even **means**!
  - Reliable **estimation**: **problematic** if **fourth moments** are **infinite**

    Many **risks** & **returns** on financial assets: **tail index** $\alpha \in (2, 4) \Longrightarrow$ Finite variances but **infinite fourth moments**

What is wrong with variances & correlations?

- **Symmetric measures** of **dependence & risk**
  - $\rho(X, Y) = \rho \implies$ **Cannot tell** whether **markets** X & Y are more **likely** to **crash together** or to **boom together**

  Dependence in financial & insurance markets: **typically asymmetric**

  - **Crashes** are more **likely** to **occur together**
  - **Returns & exchange rates** typically exhibit **greater correlation & dependence** during market **downturns** than market **upturns**
  - **Good** story: **individual**; **bad** story: **worldwide**

- **Correlation**: measure of **linear dependence**
  - $\rho(X, Y) = 0 \nRightarrow$ **independence**
  - **Not invariant** under **transformations** of risks: $\rho(X, Y) \neq \rho(f(X), f(Y))$ for nonlinear increasing f
    - **Returns** in financial markets: **uncorrelated** $Corr(r_t, r_{t+h}) \approx 0$
    - **Long-range dependence** in simple **nonlinear functions** of r_t: $Corr(r_t^2, r_{t+h}^2) \approx \frac{C}{h^\beta}$, $\beta \in [0.2, 0.4]$; **decay** to zero **slowly** as $h \to \infty$
    - **Similar** patterns: **Other powers**: $Corr[|r_t|^p, |r_{t+h}|^p]$ & **nonlinear functions**: $Corr[\log(|r_t|), \log(|r_{t+h}|)]$
    - **Volatility clustering**: **large** price **variations** are likely to be **followed** by **large** price **variations**
    - Simple **uncorrelatedness does not account** for nonlinear **dependence**
- **Bivariate measure** of **dependence**
  - **Pairwise** independence $\nRightarrow$ **Joint** independence

Heavy-tailedness paradigm

- Many **economic & financial time series**: **power law** tails: $P(|X| > x) \approx \frac{C}{x^\alpha}$, $\alpha > 0$: **tail index**
- **Moments** of order $p \geq \alpha$: **infinite**; $E|X|^p < \infty$ iff $p < \alpha$
  - $\alpha \leq 1 \implies$ **Infinite first moments**: $E|X| = \infty$
  - $\alpha \leq 2 \implies$ **Infinite variances**: $EX^2 = \infty$

Heavy-tailedness paradigm

- **Several** economic & financial **time series**: **infinite first** & **second moments**
  - **Returns** from **technological innovations**: $\alpha << 1$
  - **Loss** distributions of a number of **operational risks**: $\alpha < 1$
  - Profit in **motion pictures**: $1 < \alpha < 2$
  - **Firm sizes**, **sizes** of largest **mutual funds**, **city sizes**: $\alpha \approx 1$
  - Very **uncertain** demand
- **Returns** on **many stocks** & **stock indices**: $\alpha \in (2, 4)$
- **Variances are finite but fourth moments are infinite**: **not a good news** for an econometrician

Problems in inference on variances & correlations

- $t-$**ratio** of an estimator:

$$t_{\text{estimator}} = \frac{\text{estimator}}{\text{estimated standard error of the estimator}}$$

- Need to estimate **variance** of **sample variance** $\hat{\sigma}_0 = \frac{1}{n}\sum_1^n X_t^2$ and **sample covariances** $\hat{\sigma}_j = \frac{1}{n}\sum_1^n X_t X_{t-j}$

- But $Var(\hat{\sigma}_j) = f(EX_1^4, EX_1^2 EX_2^2, ...) \Rightarrow$ need existence of **fourth moments** for **reliable inference**

- **Infinite fourth** moments (most of data) $\Rightarrow$ **slow convergence**, **wide confidence bands** for **autocorrelations** & **variances**

- Situation: even **worse** for **squares of returns** & risks (analysis of **nonlinear dependence**)

Example: Dependence & marginals in portfolio choice & VaR

Stable distributions & heavy-tailedness

- **Popular** approach to **heavy-tailedness** modeling: **stable** distributions

- $X \sim S_\alpha(\sigma)$: symmetric **stable** distribution, $\alpha \in (0,2]$
  CF: $E(e^{ixX}) = exp\{-\sigma^\alpha |x|^\alpha\}$

  - **Normal** $\mathcal{N}(0,\sigma)$: $\alpha = 2$

  - **Cauchy**: $\alpha = 1$, $f(x) = \frac{\sigma}{\pi(\sigma^2+x^2)}$

  - **Lévy**: $\alpha = 1/2$, support $[0,\infty)$, $f(x) = \frac{\sigma}{\sqrt{2\pi}} x^{-3/2} \exp(-\frac{1}{2x})$

Stable distributions & heavy-tailedness

- **Power law** tails: $P(|X| > x) \approx \frac{C}{x^\alpha}$, $\alpha \in (0, 2)$
  - **Moments** $E|X|^p$: **finite** iff $p < \alpha$
  - **Infinite variances** for $\alpha < 2$ (can be **finite** in **multivariate** generalizations, $\alpha-$**symmetric** distributions)
- **Characteristic property**: i.i.d. $X_1, ..., X_n \sim S_\alpha(\sigma)$ $\sum_{i=1}^n w_i X_i =_d (\sum_{i=1}^n w_i^\alpha)^{1/\alpha} X_1$

Majorization & portfolio diversification

- v **majorized** by w ($\mathbf{v} \prec \mathbf{w}$):

$$\sum_{i=1}^{k} v_{[i]} \leq \sum_{i=1}^{k} w_{[i]}, \quad k = 1, ..., n-1,$$

$$\sum_{i=1}^{n} v_{[i]} = \sum_{i=1}^{n} w_{[i]},$$

$$v_{[1]} \geq \ldots \geq v_{[n]}, \quad w_{[1]} \geq \ldots \geq w_{[n]}$$

- Natural approach to **diversification modeling**

  - $v \prec w \iff$ Components of v **less diverse**

  - Portfolio v: more diversified

  - $\sum_{i=1}^{n} w_i = 1 \Longrightarrow$

$$\underline{w} = \left(\frac{1}{n}, ..., \frac{1}{n}\right) \prec (w_1, ..., w_n) \prec \big(1, 0, ..., 0\big) = \overline{w}$$

    - $\underline{w}$: **equal** weights; **most diversified**

    - $\underline{w}$: **one risk**; **least diversified**

Value at risk (VaR)

- **VaR**
  - **Risk** X; **positive** values = **losses**
  - **Loss probability** q
  - $VaR_q(X) = z : P(X > z) = q$
- **Risks** $X_1, ..., X_n$
- $Z_w = \sum_{i=1}^n w_i X_i$: **return on portfolio** with weights $w = (w_1, ..., w_n)$
- **Problem** of interest:

$$\textbf{Minimize}\, VaR_q(Z_w)$$

s.t. $w_i \geq 0$, $\sum_{i=1}^n w_i = 1$

- When **diversification** $\Rightarrow$ **decrease** in **portfolio riskiness** (VaR)?

Diversification & risk

- **Extreme portfolios**
  - $\underline{w}$: **most diversified**
  - $\underline{w}$: **least diversified**

- $X_1, ..., X_n \sim \mathcal{N}(0, \sigma)$
  - $Z_{\underline{w}} = \frac{1}{n}\sum_{i=1}^{n} X_i =_d \frac{1}{\sqrt{n}} X_1 = \frac{1}{\sqrt{n}} Z_{\overline{w}}$
  - $VaR_q(Z_{\underline{w}}) = \frac{1}{\sqrt{n}} VaR_q(Z_{\overline{w}}) < VaR_q(Z_{\overline{w}})$
  - $VaR_q(Z_{\underline{w}}) : \searrow$ as $n \nearrow$ (**Diversification** $\nearrow$)

Diversification & risk

- $X_1, ..., X_n \sim S_{1/2}(\sigma)$, $\alpha = 1/2$, **Lévy distribution**
  - $Z_{\underline{w}} = \frac{1}{n}\sum_{i=1}^{n} X_i =_d \left[\sum_{i=1}^{n}(\frac{1}{n})^{1/2}\right]^2 X_1 = nX_1 = nZ_{\overline{w}}$
  - $VaR_q(Z_{\underline{w}}) = nVaR_q(Z_{\overline{w}}) > VaR_q(Z_{\overline{w}})$
  - $VaR_q(Z_{\underline{w}}) :\nearrow$ as $n \nearrow$ (**Diversification** $\nearrow$)

- **Heavy-tailedness** (**marginals**) matters: **diversification** $\Longrightarrow$ **opposite** effects on **portfolio riskiness**

- **Skewness**: typically **priced**

Heavy-tailedness & portfolio diversification

- $X_1, ..., X_n \sim S_\alpha(\sigma)$, $\alpha > 1$: **finite first** moments
  - $v \prec w \implies VaR_q(Z_v) < VaR_q(Z_w)$
  - **Diversification** $\implies$ **Decrease** in **riskiness**
  - $VaR_q(Z_{\underline{w}}) < VaR_q(Z_w) < VaR_q(Z_{\overline{w}})$
    - **Optimal** portfolio: $\underline{w}$: **most diversified**; **equal** weights
    - **Worst** portfolio: $\underline{w}$: **least diversified**; **one risk**

Heavy-tailedness & portfolio diversification

- $X_1, ..., X_n \sim S_\alpha(\sigma)$, $\alpha < 1$: **infinite first** moments
  - **Diversification** $\Longrightarrow$ **Increase** in **riskiness!**
  - $VaR_q(Z_{\overline{w}}) < VaR_q(Z_w) < VaR_q(Z_{\underline{w}})$
    - **Optimal** portfolio: $\underline{w}$: **least diversified**; **one risk**
    - **Worst** portfolio: $\underline{w}$: **most diversified**; **equal** weights
- $X_1, ..., X_n \sim S_1(\sigma)$, $\alpha = 1$: **Cauchy**
  - $Z_w =_d X_1 \; \forall w = (w_1, ..., w_n) : w_i \geq 0, \sum_{i=1}^n w_i = 1$
  - **Diversification**: **no effect** at all!

Heavy-tailedness and VaR coherency

- (Artzner et. al., 1999) $\mathcal{R}$: **coherent** measure of risk if
  - A1. (**Monotonicity**) $\mathcal{R}(X) \geq \mathcal{R}(Y)$ if $Y \leq X$
  - A2. (**Translation invariance**) $\mathcal{R}(X + a) = \mathcal{R}(X) + a$
  - A3. (**Positive homogeneity**) $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X) \; \forall \; \lambda \geq 0$
  - A4. (**Subadditivity**) $\mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$
- **Natural conditions**: regulation
- Requirement of **extra capital**, **not coherent** risk measure $\Longrightarrow$ split (**not desirable**)

Heavy-tailedness and VaR coherency

- **Value at risk**:
  - **satisfies** A1-A3; in general, **fails** to satisfy A4 (**subadditivity**)
  - **Coherent** for **not extremely heavy-tailed** risks

    $X, Y \sim S_\alpha(\sigma)$, $\alpha > 1$ (**finite means**) $\Longrightarrow$

    $VaR_q(X+Y) < VaR_q(X) + VaR_q(Y)$
  - **Not coherent** for **extremely heavy-tailed** risks

    $X, Y \sim S_\alpha(\sigma)$, $\alpha < 1$ (**infinite means**) $\Longrightarrow$

    $VaR_q(X+Y) > VaR_q(X) + VaR_q(Y)$

    Superadditivity!

$\alpha-$symmetric distributions and diversification

Interplay of **dependence & marginal** effects

- $(X_1, ..., X_n) \sim \alpha-$**symmetric** if C.f.: $\psi(t) = E\exp(i\sum_{i=1}^n t_i X_i) = \phi[(\sum_{i=1}^n |t_i|^\alpha)^{1/\alpha}]$
  - ϕ: c.f. generator, α : index
- Natural **extension** of **univariate stable** distributions
  - $(X_1, ..., X_n) \sim \alpha-$**symmetric** $\Longrightarrow$ $Z_w = \sum_{i=1}^n w_i X_i =_d (\sum_{i=1}^n w_i^\alpha)^{1/\alpha} X_1$

$\alpha-$symmetric distributions and diversification

- **Spherical** distributions: $\alpha = 2$; $2-$**symmetric**
  - $(X_1, ..., X_n) = (RU_1, ..., RU_n)$
    - $U = (U_1, ..., U_n)$: uniformly distributed on $S_n = \{(x_1, ..., x_n) : \sum_{i=1}^n x_i^2 = 1\}$
    - $R \geq 0$: positive r.v. independent of U; **common shock** (political, macroeconomic)
  - Multivariate **normal**
  - Multivariate t (**moments finite up** to a **certain order**)
  - Multivariate **stable**; C.f.: $\exp[-(\sum_{i=1}^n |t_i|^2)^{\alpha/2}]$

$\alpha-$symmetric distributions and diversification

- $\alpha \in (0, 2]$: Models with **common shocks** $(X_1, ..., X_n) = (RU_1, ..., RU_n)$
  - i.i.d. $U_i \sim S_\alpha(\sigma)$
  - $R \geq 0$: positive r.v. independent of U_i; **common shock** affecting all X_i
  - Exhibit both **dependence** & **heavy-tailedness** or skewness in **marginals**
    - $\alpha < 1$: **Infinite** marginal **first moments**
    - $\alpha > 1$: $E|X_i|^p < \infty$ if $ER^p < \infty$ and $E|U_i|^p < \infty$ $ER = \infty \implies$ **infinite first moments**

$\alpha-$symmetric distributions and diversification

- $\alpha > 1$: **Diversification** $\Longrightarrow$ **decrease** in riskiness; VaR: **coherent**

- $\alpha < 1$: **Diversification** $\Longrightarrow$ **increase** in riskiness; VaR: **not coherent**

- Both **dependence** & **heavy-tailedness** matter: **First moments** can be **infinite** in **both** worlds!

Dependence matters: Extreme examples

- **Minimize** $VaR_q(w_1X_1 + w_2X_2)$ s.t. $w_1, w_2 \geq 0,\ w_1 + w_2 = 1$

- **Independence**:

  - **Optimal** portfolio: $(\tilde{w}_1, \tilde{w}_2) = (\frac{1}{2}, \frac{1}{2})$ (**diversified**) if $\alpha > 1$ (**not extremely heavy-tailed**, **finite** means)

  - $(\tilde{w}_1, \tilde{w}_2) = (1, 0)$ (**not diversified**, **one risk**) if $\alpha < 1$ (**extremely heavy-tailed**, **infinite** means)

Dependence matters: Extreme examples

- Extreme **positive dependence**: $X_1 = X_2$ (a.s.) **comonotonic risks**
  - $VaR_q(w_1 X_1 + w_2 X_2) = VaR_q(X_1) \ \forall w$
  - **Diversification**: **no effect** at all (similar to Cauchy) **regardless** of **heavy-tailedness**
- Extreme **negative dependence** $X_1 = -X_2$ (a.s.) **countermonotonic risks**
  - $VaR_q(w_1 X_1 + w_2 X_2) = (w_1 - w_2) VaR_q(X_1)$
  - Optimal portfolio: $(\tilde{w}_1, \tilde{w}_2) = (1, 0)$ (**not diversified**, **one risk**) **regardless** of **heavy-tailedness**
- Optimal **portfolio choice**: affected by **both dependence** & properties of **marginals**

Copulas and dependence

- **Main idea**: **separate** effects of **dependence** from effects of **marginals**
  - What **matters** more in **portfolio choice**: **heavy-tailedness** & **skewness** or (positive or negative) **dependence**?
- **Copulas**: **functions** that **join together marginal** cdf's to form **multidimensional** cdf

Copulas and dependence

- **Sklar's theorem**
- **Risks** X, Y:
  - **Joint cdf** $H_{XY}(x, y) = P(X \leq x, Y \leq y)$: affected by **dependence** and by **marginal** cdf's $F_X(x) = P(X \leq x)$ and $G_Y(x) = P(Y \leq y)$
  - $C_{XY}(u, v)$: **copula** of X, Y:

$$H_{XY}(x, y) = \underbrace{C_{XY}}_{\textbf{dependence}} \big(\underbrace{F_X(x), G_Y(y)}_{\textbf{marginals}}\big)$$

- C_{XY}: captures **all dependence** between risks X and Y

Copulas in higher dimensions

- **Similar** definition: **arbitrary number** of **risks** $X_1, ..., X_n$
  - **Joint** cdf
    $H_{X_1,...,X_n}(x_1, x_2, ..., x_n) = P(X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n)$
  - **Marginal** cdf's: $F_1(x_1) = P(X_1 \leq x_1)$, $F_2(x_2) = P(X_2 \leq x_2)$, ..., $F_n(x_n) = P(X_n \leq x_n)$
  - $C_{X_1,...,X_n}(u_1, u_2, ..., u_n)$: **copula** if

$$H_{X_1,...,X_n}(x_1, x_2, ..., x_n) = \underbrace{C_{X_1,...,X_n}}_{\textbf{dependence}}\Big(\underbrace{F_1(x_1), F_2(x_2), ..., F_n(x_n)}_{\textbf{marginals}}\Big)$$

- **Separation** of **dependence** from effects of **marginals**:
  - F_i : properties of **univariate** distributions: (unconditional) **heavy-tailedness**, **skewness**, existence of **moments**, **range** of returns
  - C : **dependence** properties: **long memory** vs. **short memory**, **volatility clustering**, **linear** vs. **nonlinear** dependence

Copulas: Main properties

Advantages:

- **Exists** for **any risks** (**correlation**: finiteness of **second moments**)
- Characterizes **all dependence** properties
- **Flexibility** in **dependence modeling**
  - **Asymmetric** dependence: **Crashes** vs. **booms**
  - **Positive** vs. **negative** dependence
  - **Independence**: **Nested** as a particular case: **Product** copula, particular values of **parameter(s)**
  - **Extreme** dependence: $X = Y$ or $X = -Y \Leftrightarrow$ **extreme copulas**; **dependence** in C_{XY} varies in **between**

Copulas: Main properties

- $F_X,\ G_Y$: **continuous** $\Rightarrow$ C_{XY} : **unique**
  - $C_{XY}(u,v) = H_{XY}\big(F_X^{-1}(u), G_Y^{-1}(v)\big)$
  - $F^{-1}(u) = \inf\{t : F(t) \geq u\}$
    (F_X, G_Y : nondecreasing but may be constant on some intervals)
  - $F_X^{-1},\ G_Y^{-1}$: usual **inverses** for strictly increasing F_X & G_Y

Copulas: cdf's with uniform marginals

- $C(u, v)$: **copula** iff $C =$ **cdf** of two (dependent) $\mathcal{U}nif(0,1)$ r.v.'s: $C(u, v) = P(U \leq u, V \leq v)$

- **Equivalent** definition: $C(u, v)$: **copula** if

  - $C(u, v)$: nondecreasing in u & v

  - $C(1, v) = v$, $C(u, 1) = u$

  - $u_1 \leq u_2$, $v_1 \leq v_2 \Longrightarrow$

    $$C(u_2, v_2) + C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) \geq 0$$

    $C-$**volume** of any rectangle: **nonnegative**
    C induces a **probability measure** on $[0, 1]^2$

Multivariate case

- A **copula** $C(u_1, ..., u_n)$ exists for any n dependent risks $X_1, ..., X_n$
  - For any **joint cdf** $H(x_1, ..., x_n)$ with **one-dimensional marginal** cdf's $F_1(x_1), ..., F_n(x_n)$
- C is **unique** if **univariate cdf's** $F_1(x_1), ..., F_n(x_n)$ are **continuous**
  - $F_1^{-1}(x_1)$, ..., $F_n^{-1}(x_n)$: **inverses** of **marginal cdf's** $F_1(x_1), ..., F_n(x_n)$
  - $C(u_1, ..., u_n) = H\Big(F_1^{-1}(x_1), ..., F_n^{-1}(x_n)\Big)$
  - **Inversion** method for **constructing** copulas
- **Equivalent** definition: $C(u_1, ..., u_n)$: **copula** iff it is a **joint cdf** of a **random vector** $U_1, ..., U_n$ with **uniform** $(0, 1)$ **marginals**: $C(u_1, ..., u_n) = P(U_1 \leq u_1, ..., U_n \leq u_n)$

Copulas: Main properties

- X, Y: **independent** $\Leftrightarrow H_{XY}(x,y) = F_X(x)G_Y(y) \Leftrightarrow C_{UV}(u,v) = uv$
    - **Minimal** dependence
- **Invariant** under **increasing transformations**:
    - $f(X), g(Y)$, **increasing** f, g: **same copula** C_{XY}
    - **Not true** for **correlation**: $\rho\big[f(X), g(Y)\big] \neq \rho(X,Y)$
    - Take $f(x) = F_X^{-1}(x)$, $g(x) = G_Y^{-1}(x)$
        - $f(X), g(Y) \sim \mathcal{U}nif(0,1)$ & **same copula** $C_{XY}(u,v)$
        - But **all moments** of $f(X), g(Y)$ **exist**: **solution** to **heavy-tailedness** problem!
        - Contrast to **correlation**: can deal with dependence under **infinite variances**

Copulas: Examples

- **Product (independence)** copula: $C(u, v) = uv$
- **Clayton:** $C^{Clayton}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, \;\; \theta > 0$
- **Gumbel:** $C^{Gumbel}(u, v) = \exp\big(- \big[(-ln\ u)^{\theta} + (-ln\ v)^{\theta}\big]\big)^{-1/\theta}$
- **Frank:** $C^{Frank}(u, v) = -\frac{1}{\theta} ln\Big(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\Big)$
- **Eyraud-Farlie-Gumbel-Morgenstern (EFGM)** copulas: $C(u, v) = uv(1 + \theta(1 - u)(1 - v)), -1 \leq \theta \leq 1$
  - θ : **dependence parameter**

Copulas: construction

Inversion method
$C(u, v) = H\left[F_X^{-1}(u), G_Y^{-1}(v)\right]$

- **Gaussian** copula
  - $H(x, y) = \Phi_\rho(x, y)$: **bivariate normal**, correlation ρ
  - $F_X, G_Y \sim F_{\mathcal{N}}$: standard **univariate normal** cdf
  - $C^{Gaussian}(u, v, \rho) = \Phi_\rho\left[F_{\mathcal{N}}^{-1}(u), F_{\mathcal{N}}^{-1}(u)\right]$
- **Student** t copula
  - $F = T_{\rho,\nu}(x, y)$: **bivariate Student** t, correlation ρ, ν degree of freedom
  - $F_X, G_Y \sim t_\nu$: **univariate Student** cdf with ν d.f.'s
  - $C^{Student}(u, v, \rho, \nu) = T_{\rho,\nu}\left[t_\nu^{-1}(u), t_\nu^{-1}(v)\right]$

Copulas: construction

- **Multivariate** analogues (Σ : **correlation** matrix, ν : d.f.)
  - $C_{\Sigma}^{Gaussian}(u_1, ..., u_n) = \Phi_{\Sigma}\left[F_{\mathcal{N}}^{-1}(u_1), ..., F_{\mathcal{N}}^{-1}(u_n)\right]$
  - $C_{\Sigma}^{Student}(u_1, ..., u_n) = T_{\Sigma,\nu}\left[t_{\nu}^{-1}(u_1), ..., t_{\nu}^{-1}(u_n)\right]$

Copulas: construction

Archimedean copulas: $C(u, v, \phi) = \phi^{-1}\Big[\phi(u) + \phi(v)\Big]$

ϕ: continuous strictly decreasing convex function (**generator**)

- $C^{Clayton}: \phi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$
- $C^{Gumbel}: \phi_\theta(t) = (-\ln t)^\theta$
- $C^{Frank}: \phi_\theta(t) = -\ln(\frac{e^{-\theta t}-1}{e^{-\theta}-1})$

First application: Probability of "perfect storm"

- **Important**: Are **crashes** (**booms**) in markets X & Y **likely** to **occur together**?
  - $x = VaR_q(X), y = VaR_q(Y)$: **disaster** levels in markets X & Y
  - **How likely** is the **event** $\{X > x\} \bigcap \{Y > y\}$ (**"perfect storm"**)
  - Need to **characterize** $P(X > x, Y > y)$: **Joint survival** function
  - Note $P(X > x, Y > y) = P(U > q, V > q) = \overline{C}_{XY}(q, q)$
    - $U, V \sim \mathcal{U}(0, 1)$, **joint cdf** $C_{XY}(u, v)$
    - $\overline{C}_{XY}(u, v) = P(U > u, V > v)$: **joint survival** function of U, V
    - **Easy to find** if **copula is known**: $\overline{C}_{XY}(u, v) = 1 - u - v + C(u, v)$

Hoeffding-Fréchet bounds: best & worst dependence

When the **probability** of "**perfect storm**"
$P(X > x, Y > y) = \overline{C}_{XY}(q, q)$ **maximal** or **minimal**?

- Answer: **Hoeffding-Fréchet** bounds
  - **Maximal** under **perfect positive** dependence
  - **Minimal** under **perfect negative** dependence
- $C_L(u, v) = \max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v) = C_U(u, v)$
- Same ordering: **Survival functions**; **joint crash** probabilities $(u, v = q)$
  - $\overline{C}_L(u, v) = \max(1 - u - v, 0) \leq \overline{C}(u, v) \leq \min(1 - u, 1 - v) = \overline{C}_U(u, v)$
  - $\overline{C}_L(q, q) = \max(1 - 2q, 0) \leq \overline{C}(q, q) \leq 1 - q = \overline{C}_U(q, q)$

Hoeffding-Fréchet bounds: best & worst dependence

- C_L and C_U: **perfect dependence** copulas
- $(X, Y) \sim C_U(u, v) \Leftrightarrow (X, Y) = (f(Z), g(Z))$, $f \nearrow$, $g \nearrow$, X, Y: **comonotonic**
- $(X, Y) \sim C_L(u, v) \Leftrightarrow (X, Y) = (f(Z), g(Z))$, $f \nearrow$, $g \searrow$, X, Y: **countermonotonic**

Bounds in multivariate case

- $C_L(u_1, u_2, ..., u_n) = \max(u_1 + ... + u_n + 1 - n, 0) \leq C(u_1, ..., u_n) \leq \min(u_1, u_2, ..., u_n) = C_U(u_1, u_2, ..., u_n)$
  - C_U : **copula** $\forall n \geq 2$
  - C_L : **no longer** copula for $n > 2$
  - However, the **lower bound** is **sharp**

Bounds in multivariate case

- Similar bounds: **Survival function** (**SF**)
  $\overline{C}(u_1, ..., u_n) = P(U_1 > u_1, ..., U_n > u_n)$
  - $x_i = VaR_q(X_i)$: **disaster levels** in markets $i = 1, ..., n$
  - $P_{Crash} = P(U_1 > q, ..., U_n > q) = \overline{C}(q, ..., q)$: **Probability** that **crash occurs** at the same **in all** n **markets**
  - Probability of a "**perfect storm**"
    $$\overline{C}_L(u_1, ..., u_n) = \max(1 - u_1 - ... - u_n, 0) \leq P_{Crash} \leq$$
    $$\overline{C}_U(u_1, ..., u_n) = \min(1 - u_1, ..., 1 - u_n)$$
  - $\overline{C}_L$: **SF** of $U_1, ..., U_n \sim C_L$
  - $\overline{C}_U$: **SF** of $U_1, ..., U_n \sim C_U$

Tail dependence

- **Will crash** (**boom**) in market X lead to a **crash** (**boom**) in market Y?
- Are **extreme values** of X & Y (**very large** losses) **likely** to **occur together**?
- **How likely** is $X \nearrow (X \searrow) \Longrightarrow Y \nearrow (Y \searrow)$?
- **Loss probability** $q \to 1- \Longrightarrow VaR_q(X), VaR_q(Y) \to \infty$: **extremely large** values of X & Y
  - Coefficient of **upper tail** dependence: $\lambda_U = \lim_{q \to 1-} P[Y > VaR_q(Y) | X > VaR_q(X)]$
- **Loss probability** $q \to 1- \Longrightarrow VaR_q(X), VaR_q(Y) \to -\infty$: extremely small values of X & Y
  - Coefficient of **lower tail** dependence: $\lambda_L = \lim_{q \to 0+} P[Y < VaR_q(Y) | X < VaR_q(X)]$

- **Copula** representations $(U, V) \sim C_{XY}(u, v)$
  $\overline{C}_{XY}(u, v) = P(U > u, V > v) = C_{XY}(1 - u, 1 - v) = 1 - u - v + C_{XY}(u, v)$

  - $\lambda_U = \lim_{q \to 1-} \frac{P(U > q, V > q)}{1 - q} = \lim_{q \to 1-} \frac{\overline{C}(q, q)}{1 - q}$
  - $\lambda_L = \lim_{q \to 0} \frac{P(U \leq q, V \leq q)}{q} = \lim_{q \to 0} \frac{C(q, q)}{q}$

- λ_U, λ_L : **Measures** of **asymptotic** dependence

Different copulas $\Longrightarrow$ different tail dependence

- $C^{Gaussian}(u, v, \rho) : \lambda_U = \lambda_L = 0$ iff $\rho < 1$; $= 1$ iff $\rho = 1$;
  Asymptotic independence $\forall \rho < 1$

- $C^{Gumbel}(u, v, \theta)$
  $\lambda_L = 0$, $\lambda_U = 2 - 2^{1/\theta}$: **upper tail dependence** provided $\theta \neq 1$

- $C^{Clayton}(u, v, \theta)$
  $\lambda_L = 2^{-1/\theta}$ for $\theta > 0$, $\lambda_U = 0$ for $\theta > 0$: **lower tail dependence** provided $\theta \neq 1$

  - C^{Gumbel}, $C^{Clayton}$: models for **asymmetric** dependence

  - **More dependence** in market **upturns or downturns**

Different copulas $\Longrightarrow$ different tail dependence

- $C^{Student}(u, v, n, \rho)$
  $\lambda_U = \lambda_L 2P(t_{n+1} > \frac{\sqrt{n+1}\sqrt{1-\rho}}{\sqrt{1+\rho}})$:
  Tail dependence for all $\rho > -1$
  Asymptotic dependence in tails even for zero correlations

- **Archimedean copulas**: $\phi'(0) \neq 0 \Longrightarrow \lambda_U = 0$
  $\phi'(0) = 0 \Longrightarrow \lambda_U = 2 - 2\lim_{s\to 0+} \frac{\phi'(s)}{\phi'(2s)}$
  $\lambda_L = 2\lim_{s\to +\infty} \frac{\phi'(s)}{\phi'(2s)}$

Copula-based dependence measures

- **Copula**-based **dependence measures**: **functionals** of **copulas**; inherit **nice properties** of **copula** functions
  - **Invariance** under increasing **transformations**
  - **Exist regardless of** properties of **marginal** distributions (**heavy-tailedness**, existence of moments)
  - **Independence** & **extreme** positive or negative **dependence** (**Hoeffding-Fréchet** bounds): typically **nested within**

Copula-based dependence measures

- $X \sim F(x)$, $Y \sim G(y)$, $X, Y \sim C(u, v)$
- **Spearman's rank correlation ρ_S**
  - **Main idea**: $F(X), G(Y) \sim \mathcal{U}nif(0,1) \Rightarrow$ all moments are finite $\Rightarrow$ **Rank correlation**: **always defined**
  - $\rho_S(X, Y) = Corr(F(X), G(Y))$
  - **Functional** of copula: $C(u, v) \to \rho_S(X, Y) = \rho_S(C) = 12 \int_0^1 \int_0^1 \Big(C(u, v) - uv \Big) dudv$
  - ρ_S : **"signed" deviation** from **independence** (**product**) copula uv

Copula-based dependence measures

- **Kendall's rank correlation**
  - $\rho_\tau(X,Y) = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]$
    - $\{(X_1 - X_2)(Y_1 - Y_2) > 0\}$: **Boom** in $i \approx$ **Boom** in j; **Crash** in $i \approx$ **Crash** in j
    - $\{(X_1 - X_2)(Y_1 - Y_2) < 0\}$: **Boom** in $i \approx$ **Crash** in j; **Crash** in $i \approx$ **Boom** in j

    $(X_1, Y_1), (X_2, Y_2)$: **independent copies** of (X, Y)
    $(X_i, Y_i) \sim C(u,v)$, $X_i \sim F(x)$, $Y_i \sim G(y)$
  - Probability of **concordance** minus probability of **discordance**
  - **Functional** of copula:
    $C(u,v) \to \rho_\tau(X,Y) = \rho_S(C) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1$

Rank correlations: main properties

- **Properties** $k(X, Y) = \rho_S(X, Y)$ or $\rho_T(X, Y)$
  - X, Y : **independent** (product copula $C(u, v) = uv$) $\Rightarrow$ $k(X, Y) = 0$
    - **Converse** is **not true** in general (counterexample: spherical distributions)
  - $-1 \leq k(X, Y) \leq 1$: any **rank correlation** in $-1, 1]$ can be **obtained** (**not true** for **linear correlation**)
  - $k(X, Y) = -1 \Leftrightarrow C(u, v) = C_L(u, v) = \max(u + v - 1, 0) \Leftrightarrow$ $(X, Y) = (f(Z), g(Z))$, $f \nearrow$, $g \nearrow$, X, Y: **comonotonic**
  - $k(X, Y) = 1 \Leftrightarrow C(u, v) = C_U(u, v) = \min(u, v) \Leftrightarrow$ $(X, Y) = (f(Z), g(Z))$, $f \nearrow$, $g \searrow$, X, Y: **countermonotonic**
- $C^{Gaussian}(u, v, \rho)$: $\rho_S(X, Y) = \frac{6}{\pi} \arcsin \frac{\rho}{2}$, $\rho_\tau(X, Y) = \frac{2}{\pi} \arcsin \rho$ (**useful** in estimation)
- **Copula estimates** $\Rightarrow$ **inference** on ρ_S & ρ_τ

Related measures of dependence

- **Natural conditions** to be imposed on a **dependence measure** (Renyi, 1959):
  - C1. $\delta(X, Y) = \delta(Y, X)$ (**symmetry**)
  - C2. $-1 \leq \delta(X, Y) \leq 1$ (**normalization**)
  - C3. $\delta(X, Y) \Leftrightarrow X, Y$ **comonotonic** $\delta(X, Y) = -1 \Leftrightarrow X, Y$ **countermonotonic**
  - C4. f, g : strictly **monotone**: $\delta(f(X), g(Y)) = \delta(X, Y)$ (**invariance**)
    - δ : **marginally free**; the same as corresponding **measure on copula**; **functional** of copula
  - C5: $\delta(X, Y) = 0 \Leftrightarrow X, Y$: **independent**
- **Linear correlation**: satisfies C1 & C2 only
- **Kendall's tau** ρ_τ & **Spearman's** ρ_S : satisfy C1-C4 and $(\Leftarrow)$ in C5

- Different **functionals of copulas**: other **measures of dependence**
- Schweiser & Wolff (1981)
- $\delta_1(X,Y) = 12\int_0^1\int_0^1 |C(u,v) - uv|dudv$
- $\delta_2(X,Y) = \left[90\int_0^1\int_0^1 \left(C(u,v) - uv\right)^2 dudv\right]^{1/2}$
- $\delta_3(X,Y) = 4\sup_{u,v\in[0,1]^2}|C(u,v) - uv|$
    - **Distances to independence**
    - Satisfy C1-C5
    - **Estimates** of **copulas** $\Rightarrow$ **estimates** for **functionals** & **dependence measures**
    - **Drawback**: Constrained to be **nonnegative** $\Rightarrow$ **cannot differentiate** between **positive & negative dependence**

Further applications: Bounds on portfolio VaR & expected payoffs & fair prices of contingent claims

- Fix weights $w_1, w_2 \geq 0$, $w_1 + w_2 = 1$
- $X \sim F_X(x)$, $Y \sim G_Y(y)$
- Determine **extrema** $VaR_q(Z_w) = VaR_q(w_1 X + w_2 Y)$ over all possible **dependence** structures for X, Y
- What is the **best** (**worst**) VaR portfolio scenario?

Further applications: Bounds on portfolio VaR & expected payoffs & fair prices of contingent claims

- Key: **comonotonic** & **countermonotonic** copulas C_U & C_L
  - $VaR_q(M) \leq VaR_q(Z_w) \leq VaR_q(L)$
  - Risk L: cdf

$$P(L \leq z) = \sup_{w_1 x + w_2 y = z} C_L\Big[F_X(x), G_Y(y)\Big] =$$

$$\sup_{w_1 x + w_2 y = s} \max\Big[F_X(x) + G_Y(y) - 1, 0\Big]$$

  - Risk M: cdf

$$P(M \leq z) = \inf_{w_1 x + w_2 y = z} \max\Big[F_X(x) + G_Y(y), 1\Big]$$

Contrasting VaR & correlation

- Note: **Worst case** portfolio VaR $\not\Leftrightarrow$ **comonotonic** $X, Y \sim C_U$ (**maximal correlation** $\rho(X, Y)$)!
- Crucial **difference** from **variance** as measure of risk:
  - $\sigma^2(w_1 X + w_2 Y) = w_1^2\sigma^2(X) + w_2^2\sigma^2(Y) + 2w_1 w_2 \rho(X, Y)\sigma^2(X)\sigma^2(Y)$
  - **Maximal** when $\rho(X, Y)$: **maximal**
  - **Breaks down** when **VaR** is used as **measure of riskiness** of portfolio return Z_w!
  - Corollary: **option & contingent claim bounds**
    - $EU(M) \leq EU(Z_w) \leq EU(L)\ \forall$ increasing U
    - $U(x) = \max(x - K, 0)$: **bounds** on **expected payoffs** & fair **prices** of European call **option**
    - Arbitrary **contingent claims**; **utility** bounds!

Natural restrictions on dependence: positive vs. negative quadrant dependence

Positive quadrant dependence

- **Similar** conclusions: **multivariate** case $n > 2$
- **Sharper restrictions** on copulas (on dependence)
  - $\tilde{C}_L(u,v) \leq C(u,v) \leq \tilde{C}_U(u,v)$
  - **Idea**: take **independence** as **one of bounds**
  - $\tilde{C}_L(u,v) = uv = C_{\text{independence}}(u,v) \Rightarrow$
    $uv \leq C_{XY}(u,v) \leq C_U(u,v) = \min(u,v) \Leftrightarrow$

    $\overline{C}_{\text{independence}}(u,v) = (1-u)(1-v) \leq \overline{C}(u,v) \leq \min(1-u, 1-v) = \overline{C}_U(u,v)$
  - **Positive quadrant** dependence (**PQD**): **Crashes** are **more likely** to **occur together than** under **independence**

Negative quadrant dependence

- $\tilde{C}_U(u,v) = uv \Rightarrow$

  $C_L(u,v) = \max(u+v-1,0) \leq C_{XY}(u,v) \leq uv \Leftrightarrow$

  $\overline{C}_L(u,v) = \max(1-u-v,0) \leq \overline{C}(u,v) \leq (1-u)(1-v) =$

  $\overline{C}_{\text{independence}}(u,v)$

- **Negative quadrant** dependence: **Crashes** are **less likely** to **occur together than** under **independence**

  Crash in market X **is likely** to occur **together** with **boom** in Y

Multivariate quadrant dependence

- **Similar definition**: $n > 2$
  - **Positive upper quadrant** dependence (**PUQD**):

$$\overline{C}(u_1, ..., u_n) = P(U_1 > u_1, ..., U_n > u_n) \geq$$
$$(1 - u_1)...(1 - u_n) = \overline{C}_{independence}(u_1, ..., u_n)$$

  - **Positive lower quadrant** dependence (**PLQD**):

$$C(u_1, ..., u_n) = P(U_1 \leq u_1, ..., U_n \leq u_n) \geq$$
$$u_1...u_n = C_{independence}(u_1, ..., u_n)$$

  - **Negative upper & lower quadrant** dependence (**NUPD & NLQD**): reversals of inequalities
  - Note that $PUQD \not\Leftrightarrow PLQD$ ($NUQD \not\Leftrightarrow NLQD$) for $n > 2$

Multivariate quadrant dependence

- **PLQD: Partial sums & portfolio returns**
  - $X_1, ..., X_n$: independent risks
  - $S_k = \sum_{i=1}^{k} w_i X_i$: **PLQD**

$$P(S_1 \leq x_1, S_2 \leq x_2, ..., S_n \leq x_n) \geq$$

$$P(S_1 \leq x_1)P(S_2 \leq x_2)...P(S_n \leq x_n)$$

(Robbins, 1954)

Association

- **Positive & negative association**
  - X, Y: **positively** (**negatively**) **associated** if

    $Cov[f(X, Y), g(X, Y)] \geq 0$ (≤ 0)

    $\forall$ **increasing** (in each argument) $f \& g$
  - Random vector $X = (X_1, ..., X_n)$: **positively** (**negatively**)

    associated if $Cov[f(X_1, ..., X_n), g(X_1, ..., X_n)] \geq 0$

    $\forall$ **increasing** $f \& g$
- Stronger than **PUOD** (**NUOD**)
  - **Positive association** $\Rightarrow$ **PUOD**
  - **Negative association** $\Rightarrow$ **NUOD**

Association

- **Negative association**:
  - Negatively correlated normal, multinomial, multivariate hypergeometric, Dirichlet, permutation distribution

- **Positive association**:
  - **Linear processes** with nonnegative coefficients
    $x_t = \sum_{j=0}^{\infty} c_j \epsilon_{t-j}$,
    ϵ_t : i.i.d.

Bounds for **expected payoffs** & **fair prices** of **contingent claims**

- **Risks** $X_1, ..., X_n$, portfolio **weights** $w_1, ..., w_n$
- $Ef(Z_w)$: **Measure of riskiness** of Z_w
  - $f(x) = [\max(x-\mu, 0)]^2$: **semivariance**
- $Ef(Z_w)$: **Expected payoff** of a **contingent claim** (CC) written **on portfolio** with return $Z_w = \sum_{i=1}^n w_i X_i$
  - $f(x) = \max(x-K, 0)$: payoff of **European call option**

  Fair prices: **Discounted** expected payoffs: $e^{-rT} Ef\left(Z_w\right)$
- Problem: what is **riskiest** dependence? **Most expensive** dependence?
- **Maximize** $Ef(Z_w)$ over all $X_1, ..., X_n$ with prescribed **dependence** structure

Complete solution: negatively associated risks

- $X_1, ..., X_n$: **negatively associated** with univariate marginals $F_{X_1}(x_1), ..., F_{X_n}(x_n)$
- $\xi_1, ..., \xi_n$: independent copies of X_i
  - $Z_w = \sum_{i=1}^n w_i X_i$: return on portfolio of **dependent** risks
  - $\tilde{Z}_w = \sum_{i=1}^n w_i \xi_i$: return on portfolio of **independent** risks
- **Independence**: **worst** case scenario for contingent claims with convex payoff function (**most expensive!**):

  Expected payoffs: $Ef(Z_w) \leq Ef(\tilde{Z}_w)$ $\forall$ **convex** f

  (Shao, 2000); $E(\cdot)$: expectation w/r to true probability measure

  Fair prices: $e^{-rT}Ef(Z_w) \leq e^{-rT}Ef(\tilde{Z}_w)$ $\forall$ **convex** f

  $E(\cdot)$: expectation w/r to equivalent probability measure

Bounds for **expected payoffs** & **fair prices** of **contingent claims**

- Important particular cases: $f(x) = \max(x - K, 0)$: **bounds** for **European call** option on Z_w

- **Risk bounds**

    - E.g., **semivariance** $f(x) = [\max(x - \mu, 0)]^2$, $\mu = EX_i = E\xi_i$

    - **Independence** = **most risky** portfolio

    - **Expected shortfall**

- **Positive association**: typically, **constants change** (Doukhan & Louhichi, 1999; Nze & Doukhan, 2004)

Copulas & time series

Important **application** of **copulas**: **characterization** of **Markov processes** of arbitrary order

- **Markov processes**: completely **characterized** by bivariate **copulas** and **marginals** (Darsow, Nguyen and Olsen, 1992)

- $\{X_t\}_{t \in T}$: **Markov process** if $\forall\ t_1 < \ldots < t_{n-1} < t_n < t$,

$$P\left(X_t < x_t \middle| X_{t_1}, \ldots, X_{t_{n-1}}, X_{t_n}\right) = P\left(X_t < x_t \middle| X_{t_n}\right)$$

  - **Today's** stock **price** is **affected** only by the **price** in **previous period**

- $*-$**product** of copulas
  $A, B : [0,1]^2 \to [0,1]$:

$$(A * B)(x,y) = \int_0^1 \frac{\partial A(x,t)}{\partial t} \cdot \frac{\partial B(t,y)}{\partial t} dt$$

- $\star-$**product** $A : [0,1]^m \to [0,1]$, $B : [0,1] \to [0,1]$:

$$A \star B(x_1, ..., x_{m+n-1}) =$$

$$\int_0^{x_m} \frac{\partial A(x_1, ..., x_{m-1}, \xi)}{\partial \xi} \cdot \frac{\partial B(\xi, x_{m+1}, ..., x_{m+n-1})}{\partial \xi} d\xi$$

- X_t: **Markov** iff

$$C_{t_1,...,t_n} = C_{t_1 t_2} \star C_{t_2 t_3} \star ... \star C_{t_{n-1} t_n}$$

- Similar **characterizations**: hold for **Markov processes** of **arbitrary order** (Ibragimov, 2005)

Alternative approach to modeling Markov processes

- Instead of **initial distribution** & **transition** probabilities:
  - Prescribe **marginals** & $(k+1)-$**copulas**
  - Generate **copulas of higher order** & finite-dimensional **cdf's**
- **Advantage: separation** of properties of **marginals** (fat-tailedness) & **dependence** properties (conditional symmetry, $m-$dependence, $r-$independence, mixing)

Inversion method: **New** $k-$**Markov** with **dependence similar** to a given Markov process **Different marginals**

- X_t: **stationary** $k-$**Markov**
  $(k+1)-$cdf $\tilde{F}(x_1, ..., x_{k+1})$, $1-$cdf F
  $\Rightarrow (k+1)-$copula:

$$C(u_1, ..., u_{k+1}) = \tilde{F}\Big(F^{-1}(u_1), ..., F^{-1}(u_{k+1})\Big)$$

- Another $1-$cdf G: **Stationary** $k-$**Markov**, **same** dependence as $\{X_t\}$, **different** 1-marginal G:
  $(k+1)-$copula:

$$C(u_1, ..., u_{k+1}) = \tilde{F}\Big(G^{-1}(u_1), ..., G^{-1}(u_{k+1})\Big)$$

- Representation $\Rightarrow$ **Higher-order copulas & cdf's**
- Example: **Processes** based on **Gaussian copulas** C & **non-normal** marginals G

  Dependence: similar to **Brownian motion**

  Non-normal univariate distributions

Copulas: simulation

- **Goal**: **Simulate** a **sample** $(U_i, V_i) \sim C(u, v)$, $i = 1, ..., N$
- **Key**: $(U, V) \sim C(u, v) \Rightarrow$ **Conditional cdf** of V **given** $U = u$ is $C_u(v) = P(V \le v | U = u) = \frac{\partial C(u,v)}{\partial u}$
  - Step 1: Generate $U, Z \sim$ i.i.d. $\mathcal{U}nif(0, 1)$
  - Step 2: Calculate $V = C_U^{-1}(Z)$
  - $(U, V) \sim C(u, v)$
  - Repeat N times to get a sample (U_i, V_i), $i = 1, ..., N$

Copulas: simulation

- One can use the sample to simulate **copula functionals** $G(C) = Eg(U, V)$ via $\hat{G} = \frac{1}{N}\sum_{i=1}^{N} g(U_i, V_i)$
    - **Kendall's tau**: $C(u, v) \to \rho_\tau(X, Y) = \rho_S(C) = 4\int_0^1\int_0^1 C(u, v)dC(u, v) - 1$
    - $G(C) = EC(U, V) - 1$, $g(u, v) = C(u, v)$
    - Simulate via $\hat{G} = \frac{1}{n}\sum_{i=1}^{n} C(U_i, V_i) - 1$

Copulas: simulation

- **Similar approach**: **functionals of copula** & **marginals** (say, option prices)

- Simulate, in addition to copula $X_i \sim F_X(x)$, $Y_i \sim G_Y(y)$:

  - **Functionals** of random vector (X, Y) with **prescribed copulas** (**option prices**)

  - Can **vary** both **copula** C & **marginals** $\Rightarrow$ analysis of **robustness** of **pricing formulae** to **dependence** & **marginal** properties

Copulas: simulation

- **Easily implementable** provided **analytical expression** for $C_u^{-1}(z)$ ($\Leftrightarrow$ **solution** v to $C_u(v) = z$) is found
  Easy to determine for **commonly used copulas** on, e.g., Mathematica

- Example: **Simulating Clayton** copulas
  $C^{Clayton}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$

  - Calculate $C_u(v) = \frac{\partial C(u,v)}{\partial u} = \left[1 + u^{\theta}(v^{-\theta} - 1)\right]^{-(1+\theta)/\theta}$

  - Solve $C_u(v) = z$ for v:
    $C_u^{-1}(z) = v = \left((z^{-\frac{\theta}{\theta+1}} - 1)u^{-\theta} + 1\right)^{-1/\theta}$

  - Generate $U, Z \sim$ i.i.d. $\mathcal{U}nif(0, 1)$

  - Set $V = \left((Z^{-\frac{\theta}{\theta+1}} - 1)U^{-\theta} + 1\right)^{-1/\theta}$

  - $(U, V) \sim C^{Clayton}(u, v)$

Copulas: simulation
Multivariate case: similar approach

- **Simulate** N **random vectors** $(U_1^i, ..., U_n^i) \sim C(u_1, ..., u_n)$, $i = 1, ..., N$

- **Key**: $(U_1, ..., U_n) \sim C(u_1, ..., u_n) \Rightarrow$ **Conditional cdf** of U_k **given** $U_1, ..., U_{k-1}$ is

$$C_k(u_k | u_1, ..., u_{k-1}) =$$

$$P(U_k \leq u_k | U_1 = u_1, ..., U_{k-1} = u_{k-1}) =$$

$$\frac{\partial^{k-1} C_k(u_1, ..., u_k)}{\partial u_1 ... \partial u_{k-1}} \Big/ \frac{\partial^{k-1} C_{k-1}(u_1, ..., u_{k-1})}{\partial u_1 ... \partial u_{k-1}}$$

$C_k(u_1, ..., u_k) = C(u_1, ..., u_k, 1, ..., 1)$: $k-$**variate** marginals, $k = 2, ..., n-1$

Multivariate case: similar approach

- Simulate $U_1 \sim \mathcal{U}nif(0,1)$
- Simulate $U_2 \sim C_2(u_2|u_1)$
  - Simulate $Z_2 \sim \mathcal{U}nif(0,1)$
  - Set $U_2 = C_2^{-1}(Z_2|U_1)$

...

- Simulate $U_n \sim C_n(u_n|u_1, ..., u_{n-1})$
  - Simulate Z_n
  - Set $U_n \sim C_n^{-1}(Z_n|U_1, ..., U_{n-1})$
- $U_1, ..., U_n \sim C(u_1, ..., u_n)$

Copulas: simulation
Archimedean copulas

- $C(u,v) = \phi^{-1}\Big[\phi(u) + \phi(v)\Big] \Rightarrow \phi(C) = \phi(u) + \phi(v) \Rightarrow \phi'(C)C_u = \phi'(u)$

  - Step 1: Generate $U, Z \sim$ i.i.d. $\mathcal{U}nif(0,1)$
  - Step 2: Calculate $W = C(U,Z) = (\phi')^{-1}\left(\frac{\phi'(U)}{\partial C(U,Z)/\partial u}\right)$
  - Step 3: Calculate $V = \phi^{-1}[\phi(W) - \phi(U)]$
  - $(U,V) \sim C(u,v) = \phi^{-1}\Big[\phi(u) + \phi(v)\Big]$

Copulas: simulation
Inverse copulas: simple solution

$$C(u_1, ..., u_n) = F\left[F_1^{-1}(u_1), ..., F_n^{-1}(u_n)\right]$$

- Simulate $X_1, ..., X_n \sim F$
- Set $U_i = F_i^{-1}(X_i)$, $i = 1, ..., n$

Copulas: simulation

- Example: **Gaussian** copula

- $C_{\Sigma}^{Gaussian}(u_1, ..., u_n) = \Phi_{\Sigma}(F_{\mathcal{N}}^{-1}(u_1), ..., F_{\mathcal{N}}^{-1}(u_n))$

  $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq n}$: **covariance** matrix

  $R = (\rho_{ij})_{1 \leq i < j \leq n}$: corresponding **correlation** matrix

  $\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} = \frac{Corr(Y_i, Y_j)}{\sqrt{Var(Y_i)Var(Y_j)}}$

  $Y = (Y_1, ..., Y_n) \sim \mathcal{N}(0, \Sigma)$

  $R = AA^T \Rightarrow RZ \sim \mathcal{N}(0, R)$

  $Z = (Z_1, ..., Z_n) \sim$ i.i.d. $\mathcal{N}(0, 1)$

Gaussian copula: simulation

- Find the **correlation** matrix R corresponding to Σ
- Find the **Cholesky decomposition** of R: $R = AA^T$ (implemented in standard packages)
- Generate i.i.d. $Z = (Z_1, ..., Z_n) \sim$ i.i.d. $\mathcal{N}(0,1)$
- Set $X = AZ^T$
- Set $U_i = \Phi(X_i)$, $i = 1, ..., n$
- $(U_1, ..., U_n) \sim C_{\Sigma}^{Gaussian}$

Copulas: simulation

- Example: **Student** copula
  $C_{\Sigma}^{Student}(u_1, ..., u_n) = T_{\Sigma,\nu}\left[t_{\nu}^{-1}(u_1), ..., t_{\nu}^{-1}(u_n)\right]$

  - Find the **correlation** matrix R corresponding to Σ
  - Find the **Cholesky decomposition** of R: $R = AA^T$
  - Simulate i.i.d. $Z = (Z_1, ..., Z_n) \sim$ i.i.d. $\mathcal{N}(0, 1)$
  - Set $Y = AZ^T$
  - Generate a r.v. $s \sim \chi^2(\nu)$ independent of Z_i
  - Set $X = \frac{\sqrt{\nu}}{\sqrt{s}} Y$
  - Set $U_i = t_{\nu}(X_i)$, $i = 1, ..., n$
  - $(U_1, ..., U_n) \sim C_{\Sigma}^{Student}$

Copulas: estimation

- Problem: Given **observations** (X_i, Y_i), $i = 1, ..., n$, of a **random vector** $(X, Y) \sim C(u, v)$, **estimate** C

  - (X_i, Y_i): **i.i.d. observations** on (X, Y) = two different **exchange rates**

  - $Y_i = X_{i+1}$: **Dependent** stationary **time series** $X_1, X_2, ...,$ generated by a copula C (**Markov** process) C : copula of X_t, X_{t+1}
    The same exchange rate = dependence **over time** (asymptotics: more complicated)

- **Parametric, semiparametric**: Know that
  $C = C_{\theta_0} \in \{C_\theta : \theta \in \Theta\}$
  Goal: estimate θ_0

- **Nonparametric** procedures

Parametric estimation for copulas

Exact maximum likelihood

- θ: **copula** parameter C_θ
  β, γ : parameters of **marginals** $X \sim F_\beta, f_\beta$, $Y \sim G_\gamma, g_\gamma$
  Joint cdf: $H_{\theta,\beta,\gamma}(x, y) = C_\theta(F_\beta(x), G_\gamma(y))$

- Form **log-likelihood** for a **sample** based on $H_{\theta,\beta,\gamma}(x, y)$

- **Maximize** it w/r to θ, β, γ

Exact maximum likelihood

- **Idea**: choose $\hat{\theta}, \hat{\beta}, \hat{\gamma}$ for which the **likelihood** of observing the given **sample** (X_i, Y_i), $i = 1, ..., n$, is **maximal**

- **Cdf** $H_{\theta,\beta,\gamma}(x, y) = C_\theta\left[F_\beta(x), G_\gamma(y)\right] \Rightarrow$

  Density $f_{\theta,\beta,\gamma}(x, y) = \frac{\partial^2 C_\theta\left[F_\beta(x), G_\gamma(y)\right]}{\partial u \partial v} f_\beta(x) g_\gamma(y) = c_\theta\left[F_\beta(x), G_\gamma(y)\right] f_\beta(x) g_\gamma(y)$

  - $c_\theta(u, v) = \frac{\partial^2 C_\theta(u,v)}{\partial u \partial v}$: **copula density**

  - f_β, g_γ : **marginal densities**

Exact maximum likelihood

- **Likelihood** for **sample** (X_i, Y_i), $i = 1, ..., n$ (**joint density**)

$$L_{\theta,\beta,\gamma}(X_1, Y_1, ..., X_n, Y_n) = f_{\theta,\beta,\gamma}(X_1, Y_1)...f_{\theta,\beta,\gamma}(X_n, Y_n) =$$

$$c_\theta\left[F_\beta(X_1), G_\gamma(Y_1)\right]...c_\theta\left[F_\beta(X_n), G_\gamma(Y_n)\right]\times$$

$$f_\beta(X_1)...f_\beta(X_n)g_\gamma(Y_1)...g_\gamma(Y_n)$$

  - Joint **copula density** $\times$ joint **marginal density**

Exact maximum likelihood

- **Log-likelihood**

$$l_{\theta,\beta,\gamma}(X_1, Y_1, ..., X_n, Y_n) = \log L_{\theta,\beta,\gamma}(X_1, Y_1, ..., X_n, Y_n) =$$

$$\underbrace{\sum_{t=1}^{n} c_\theta\left[F_\beta(X_t), G_\gamma(Y_t)\right]}_{\text{Loglikelihood}_{copula}(\theta,\beta,\gamma)} + \underbrace{\sum_{t=1}^{n} f_\beta(X_t) + \sum_{t=1}^{n} g_\gamma(Y_t)}_{\text{Loglikelihood}_{marginals}(\beta,\gamma)} =$$

$$\text{Loglikelihood}_{copula}(\theta,\beta,\gamma) \quad + \quad \text{Loglikelihood}_{marginals}(\beta,\gamma)$$

  - Key: **Separation** of **dependence** from **marginals** by **copula** approach

Exact maximum likelihood for copulas

- **Maximum likelihood** estimates:
  - Choose as an estimate of (θ, β, γ) their values that **maximize log-likelihood** $l_{\theta,\beta,\gamma}(X_1, Y_1, ..., X_n, Y_n)$
  - $(\hat{\theta}_{MLE}, \hat{\beta}_{MLE}, \hat{\gamma}_{MLE}) = Argmax_{\,\theta,\beta,\gamma} \;\; l_{\theta,\beta,\gamma}(X_1, Y_1, ..., X_n, Y_n)$

Exact maximum likelihood for copulas

- **Consistent**
  $\delta_n^{MLE} = (\hat{\theta}_{MLE}, \hat{\beta}_{MLE}, \hat{\gamma}_{MLE}) \to_P \delta = (\theta_{MLE}, \beta_{MLE}, \gamma_{MLE})$
  as **sample size** $n \to \infty$

  - **Distribution** of **estimate** δ_n^{MLE}: **concentrates** around **true** δ

- **Asymptotic normality**: $\sqrt{n}(\delta_n^{MLE} - \delta) \to \mathcal{N}(0, \Im^{-1}(\delta))$
  $\Im(\delta)$: Fisher's information matrix

  - Distributional **shape**: **normal**

  - **Key** to constructing **confidence** intervals for $\delta = (\theta, \beta, \gamma)$ and **testing**

- **Efficient** (smallest asymptotic variance)

Two-stage parametric estimation for copulas: IFM

- **Drawback** of **exact MLE**: can be very **computationally expensive**, especially in **higher dimensions**
    - Needs to **estimate jointly** parameters of **marginals** (β, γ) and those of **copula** (θ)

- **Computationally easier** procedure: **two-stage** estimation of parameters = **Inference for the Margins** (**IFM**) estimation (Joe and Xu, 1996)

Two-stage parametric estimation for copulas: IFM

- **Log-likelihood**

  $$l_{\theta,\beta,\gamma}(X_1, Y_1, ..., X_n, Y_n) = \log L_{\theta,\beta,\gamma}(X_1, Y_1, ..., X_n, Y_n) =$$

  $$\underbrace{\sum_{t=1}^{n} c_\theta\big[F_\beta(X_t), G_\gamma(Y_t)\big]}_{\text{Loglikelihood}_{copula}(\theta,\beta,\gamma)} + \underbrace{\sum_{t=1}^{n} f_\beta(X_t) + \sum_{t=1}^{n} g_\gamma(Y_t)}_{\text{Loglikelihood}_{marginals}(\beta,\gamma)} =$$

  - Key: **Separation** of **dependence** from **marginals** by **copula** approach

Two-stage parametric estimation for copulas: IFM

- Idea: First, **maximize** Loglikelihood$_{marginals}(\beta, \gamma)$ w/r to **marginal parameters** (β, γ): Obtain $(\hat{\beta}, \hat{\gamma})$

- **Maximize** Loglikelihood$_{copula}(\gamma, \hat{\beta}, \hat{\gamma})$ w/r to γ: Obtain **copula parameter** estimate $\hat{\gamma}$

- Note: **Separation** of log-likelihood (dependence from marginals) is the **key**

- **Stage 1. Estimate**
  $(\hat{\beta}, \hat{\gamma}) = argmax_{\beta,\gamma}$ Loglikelihood$_{marginals}(\beta, \gamma) =$
  $argmax_{\beta,\gamma} \sum_{t=1}^{n} f_\beta(X_t) + \sum_{t=1}^{n} g_\gamma(Y_t)$

- **Stage 2. Replace** β & γ in Loglikelihood$_{copula}(\theta, \beta, \gamma)$ by the **estimates** in **first stage** and **estimate**

  - $\hat{\gamma} = argmax$ Loglikelihood$_{copula}(\theta, \hat{\beta}, \hat{\gamma}) =$
    $argmax \sum_{t=1}^{n} c_\theta \left[F_{\hat{\beta}}(X_t), G_{\hat{\gamma}}(Y_t) \right]$

- $(\hat{\theta}, \hat{\beta}, \hat{\gamma}) = (\hat{\theta}_{IFM}, \hat{\beta}_{IFM}, \hat{\gamma}_{IFM}) = \delta_{IFM}$

- Under **regularity** conditions: **asymptotically normal** (Joe, 1997): $\sqrt{n}(\delta_{IFM} - \delta) \to \mathcal{N}(0, G^{-1}(\delta))$
  $G(\delta)$: Godambe **information matrix**

Semiparametric estimation

- Genest, Ghoudi & Rivest (1995)
- **Two-stage** estimation
  - Estimate **marginals nonparametrically**: $\hat{F}_n(x)$ & $\hat{G}_n(x)$
  - Estimate **copula parameter** θ by **maximizing** $\text{Loglikelihood}_{copula}(\theta)$ with marginals **replaced** by their **estimates**
- **Empirical cdf's** $\hat{F}_n(x) = \frac{1}{n}\sum_{t=1}^{n} I(X_t \leq x)$,
  $\hat{G}_n(y) = \frac{1}{n} I(Y_t \leq y)$, $I(\cdot)$: indicator function
  $\hat{F}_n(X_i)$: **fraction** of X_t's that are **not greater** than X_i
  $\hat{F}_n(Y_i)$: **fraction** of Y_t's that are **not greater** than Y_i
- **Estimate** $\hat{\gamma} = argmax \sum_{t=1}^{n} c_\theta\left[\hat{F}(X_t), \hat{G}_n(Y_t)\right]$
- $\hat{\theta}$: **Consistent** & **asymptotically normal** under regularity conditions

Nonparametric estimation

- **Empirical copula** process
  Deheveuls (1979), Fermanian, Radulovic & Wegkamp (2004)

  - Estimate **both univariate** & **multivariate** cdf's **nonparametrically**: **empirical cdf's**

  - Use **inversion method** to **estimate copula**

  - $C(u, v) = H(F^{-1}(u), G^{-1}(v))$

  - Replace H by $\hat{H}_n$ and F^{-1} and G^{-1} by inverses of $\hat{F}_n$ and $\hat{G}_n$

  - Obtain $\hat{C}(u, v)$: **empirical copula**

Nonparametric estimation

- Estimate **marginals nonparametrically**: $\hat{F}_n(x)$ & $\hat{G}_n(x)$

  Empirical cdf's $\hat{F}_n(x) = \frac{1}{n}\sum_{t=1}^{n} I(X_t \leq x)$,
  $\hat{G}_n(y) = \frac{1}{n}\sum_{t=1}^{n} I(Y_t \leq y)$

- Estimate **joint cdf nonparametrically**

  $\hat{H}(x,y) = \frac{1}{n}\sum_{t=1}^{n} I(X_t \leq x, Y_t \leq y) =$
  $\frac{1}{n}\sum_{t=1}^{n} I(X_t \leq x) I(Y_t \leq y)$

- Estimate **copula** via **inversion** method
  $\hat{C}(u,v) = \hat{H}(\hat{F}_n^{-1}(u), \hat{G}_n^{-1}(v))$

  $\hat{F}_n^{-1} = \inf\{t \in \mathbf{R} : \hat{F}_n(t) \geq u\}$
  $\hat{G}_n^{-1} = \inf\{t \in \mathbf{R} : \hat{G}_n(t) \geq u\}$: Inverses of $\hat{F}_n$, $\hat{G}_n$

- **Empirical copula** process: asymptotically **Gaussian** (normal)

Copula-based time series estimation
Semiparametric estimation for time series
Chen & Fan (2004)

- **Estimation** procedures: **similar** to the case of i.i.d. vectors
- **Additional regularity** conditions
- $\{X_t\}$: **copula**-based stationary **dependent time series** (**Markov** chain); C : **copula** of (X_t, X_{t+1})

Copula-based time series estimation

- **Two-stage** semiparametric estimation

  - Estimate **univariate** marginals **nonparametrically**: $\hat{F}_n(x)$

  - Estimate **copula parameter** θ by **maximizing** Loglikelihood$_{copula}(\theta)$ for time series with **marginals replaced** by their **estimates**

  - Empirical **univariate cdf** $\hat{F}_n(x) = \frac{1}{n}\sum_{t=1}^{n} I(X_t \leq x)$
    $\hat{F}_n(X_i)$: **fraction** of X_t's that are **not greater** than X_i

  - **Estimate** $\hat{\theta} = argmax \sum_{t=1}^{n} c_\theta\left[\hat{F}(X_t), \hat{F}(X_{t+1})\right]$

  - $\hat{\theta}$: **Consistent** & **asymptotically normal** under regularity conditions

Nonparametric copula estimation for time series

- **Nonparametric** estimation: **empirical copula** process for **dependent time series**

- $\{X_t\}$: **copula-based** stationary **dependent** time series (**Markov** chain); C : **copula** of (X_t, X_{t+1})

  - Estimate **univariate** marginals **nonparametrically**: $\hat{F}_n(x) = \frac{1}{n}\sum_{t=1}^{n} I(X_t \leq x)$

  - Estimate **joint cdf** for subsequent observations **nonparametrically** $\hat{F}_n(x,y) = \frac{1}{n}\sum_{t=1}^{n} I(X_t \leq x) I(X_{t+1} \leq y)$

  - Use **inversion** method to **estimate copula** $\hat{C}_n(u,v) = \hat{F}_n\left[\hat{F}_n^{-1}(u), \hat{F}_n^{-1}(v)\right]$

- Process: **asymptotically Gaussian** under $\beta-$mixing (**weak dependence**) assumptions (Ibragimov, 2005)

Empirical evidence on copula models

- Patton (2004): Parametric estimation of **conditional copula** models for **exchange rates** (Deutsche mark & yen)
- Significant evidence that **dependence** between DM-USD & Yen-USD exchange rates is **asymmetric**
- Strong evidence of a **structural break** in conditional **copula** following the **introduction of euro** in Jan 1999
    - **Tail dependence**: **decreases**
    - Significant **upper tail** dependence **changes** to weak **lower tail** dependence

Empirical evidence on copula models
Patton (2004)

- **Symmetrized Joe-Clayton** copula

$$C^{SJC}(u,v) = \frac{1}{2}\Big(C^{JC}(u,v) + C^{JC}(1-u,1-v) + u + v - 1\Big)$$

$$C^{JC}(u,v) = 1 - \Big[1 - \big([1-(1-u)^{\kappa}]^{-\gamma} +$$

$$[1-(1-v)^{\kappa}]^{-\gamma} - 1\big)^{-1/\gamma}\Big]^{1/\kappa}$$

$$\kappa = 1/\log_2(2-\tau^{U}), \quad \gamma = -1/\log_2(\tau^{L})$$

  - **Extension** of $C^{Clayton}(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$
  - **Tail dependence** parameters λ_U and λ_L **determine copula**
  - **Symmetric iff** $\lambda_U = \lambda_L$

Empirical evidence on copula models
Patton (2004)

- Parameter estimates (**MLE**):

  - **Pre-Euro**: $\tau^U = 0.359$ (s.e: 0.025), $\tau_L = 0.294$ (s.e.: 0.027)

  - **Post-Euro**: $\tau_U = 0$ (imposed), $\tau_L = 0.093$ (s.e.: 0.038)

- **Structural break**; **reversal** of the **dependence direction** following the **introduction of Euro**

Empirical evidence on copula models

- Hu (2004): **modeling & estimation** of **copula models** for international financial markets using **mixed copulas**
- **Semiparametric** estimation
  - **First stage**: **Nonparametric** estimation of **marginals**
  - **Second stage**: Estimation of **mixed copulas**

Empirical evidence on copula models Hu (2004):

- **Mixed copula** model

  - Idea: account for **asymmetry** in **tail dependence** using a **mixture** of $C^{Gaussian}$ (**no tail dependence** $\lambda_U = \lambda_L = 0$), C^{Gumbel} (**positive** λ^U) and Gumbel survival copula (**positive** λ_L)

  - $C(u,v) = w_1 C^{Gaussian}(u,v,\rho) + w_2 C^{Gumbel}(u,v,\theta_1) + w_3 C^{SGumbel}(u,v,\theta_2)$

    - $C^{Gumbel}(u,v,\theta_1) = \exp\left(-\left[(-ln\ u)^{1/\theta_1} + (-ln\ v)^{1/\theta_1}\right]\right)^{-\theta_1}$

    - $C^{SGumbel}(u,v,\theta_2) = u + v - 1 + \exp\left(-\left[(-ln\ (1-u))^{1/\theta_2} + (-ln\ v)^{1/\theta_2}\right]\right)^{-\theta_2}$

Empirical evidence on copula models
Hu (2004)

- Four **stock market indices**: S&P 500 (US), FTSE 100 (UK), Nikkei 225 (Japan), Hang Seng (HK)
  - Need to **estimate**: one-dimensional **marginals**, individual copula **parameters** (ρ, θ_1, θ_2) and mixture **weights** w_i
  - **Parameters** of C: $\rho, \theta_1, \theta_2, w_1, w_2, w_3$
  - Problem: financial data is **not i.i.d.** over time
    GARCH filter is applied to data on stock indices before empirical distributions are computed

    Alternative: apply **asymptotic theory** for **empirical processes** for **dependent time series** or **copula-based time series**

Hu (2004)

- Estimation results:

  For **all pairs** of stock indices: $w_2 = 0$: **no upper tail** dependence

  - (S& P, FTSE): $w_1 = 0.16$, $w_3 = 0.84$ (some **weight** on **symmetric** $C^{Gaussian}$)
  - (FTSE, Nikkei): $w_1 = 1$, $w_2 = w_3 = 0$ (**symmetric dependence**)
  - **All remaining pairs** (S& P, Nikkei), (S& P, Hang Seng), (FTSE, Hang Seng), (Nikkei, Hang Seng): $w_3 = 1$, $w_1 = w_2 = 0$ (**positive lower tail** dependence!)

    Good story is **individual**, while **bad story** is **worldwide**

Characterizations of copulas & dependence

de la Peña & Ibragimov (2003), de la Peña, Ibragimov & Sharakhmetov (2003)

Copulas: characterization by $U-$statistics

- $V_1, ..., V_n$: i.i.d. $\mathcal{U}\big([0,1]\big)$
- C: $n-$**copula** iff $\exists\ \tilde{g}_{i_1,...,i_c}$ s.t.
  A1 (integrability):
  $$\int_0^1 \ldots \int_0^1 |\tilde{g}_{i_1,...,i_c}(t_{i_1}, ..., t_{i_c})| dt_{i_1} ... dt_{i_c} < \infty$$
  A2 (degeneracy):
  $$E_{V_{i_k}}\left[\tilde{g}_{i_1,...,i_c}(V_{i_1}, ..., V_{i_{k-1}}, V_{i_k}, V_{i_{k+1}}, ..., V_{i_c})\right] = 0$$
  A3 (positive definiteness):
  $$\tilde{U}_n(V_1, ..., V_n) \equiv \sum_{c=2}^{n} \sum_{1 \le i_1 < ... < i_c \le n} \tilde{g}_{i_1,...,i_c}(V_{i_1}, ..., V_{i_c}) \ge -1$$

- **Representation for** C :

$$C(u_1, ..., u_n) = \int_0^{u_1} ... \int_0^{u_n} \left(1 + \tilde{U}_n(t_1, ..., t_n)\right) \prod_{i=1}^{n} dt_i$$

- $\tilde{U}_n$: sum of **degenerate** $U-$**statistics**

Device for **constructing $n-$copulas and cdf's**

- **Bivariate Eyraud-Farlie-Gumbel-Morgenstern copulas & cdf's:**

$$C_\theta(u,v) = uv\left(1+\theta(1-u)(1-v)\right)$$

$$H_\theta(x,y) = F(x)G(y)\Big(1+\theta(1-F(x))(1-G(y)\Big)$$

$n=2$; $\tilde{g}_{1,2}(t_1,t_2) = \theta(1-2t_1)(1-2t_2)$, $\theta \in [-1,1]$

- **Multivariate EFGM copulas & cdf's:**

$$C_\theta(u_1,u_2,...,u_n) = \prod_{i=1}^{n} u_i \left(1+\theta\prod_{i=1}^{n}(1-u_i)\right)$$

$$\tilde{g}_{i_1,...,i_c}(t_{i_1},...,t_{i_c}) = \theta_{i_1,...,i_c}(1-2t_{i_1})(1-2t_{i_2})...(1-2t_{i_c})$$

- **Generalized multivariate EFGM copulas** (Johnson and Kotz, 1975, Cambanis, 1977)

$$C(u_1, \dots, u_n) = \prod_{k=1}^{n} u_k \Big(1 + \sum_{c=2}^{n} \sum_{1 \le i_1 < \dots < i_c \le n} \theta_{i_1,\dots,i_c}(1 - u_{i_k})\Big)$$

$\tilde{g}_{i_1,\dots,i_c}(t_{i_1}, \dots, t_{i_c}) = 0,\ c < n - 1$
$\tilde{g}_{1,2,\dots,n}(t_1, t_2, \dots, t_n) = \theta(1 - 2t_1)(1 - 2t_2)\dots(1 - 2t_n)$

- **Generalized EFGM copulas:** complete **characterization** of joint **cdf's** of **two-valued r.v.'s** (Sharakhmetov & Ibragimov, 2002)

From dependence to independence through $U-$statistics
$\mathcal{G}_n$: **sums of $U-$statistics**

$$U_n(\xi_1, ..., \xi_n) = \sum_{c=2}^{n} \sum_{1 \leq i_1 < ... < i_c \leq n} g_{i_1,...,i_c}(\xi_{i_1}, ..., \xi_{i_c})$$

$g_{i_1,...,i_c}$: satisfy A1-A3

- Arbitrarily **dependent r.v.'s**:
  sum of $U-$statistics in **independent** r.v.'s
  with canonical kernels

- **Reduction** of problems for **dependence** to **well-studied** objects

- Transfer of results for $U-$**statistics** under **independence**

Characterizations of dependence
Canonical $\tilde{g}'$s: complete **characterizations** of **dependence** properties

- $X_1, ..., X_n$: **independent** iff

$$\tilde{g}_{i_1,...,i_c}(V_{i_1}, ..., V_{i_c}) = 0 \ \ \forall i_1, ..., i_c$$

- $X_1, ..., X_n$: $r-$**independent** if $\forall$ r jointly independent

  $r = 2$: pairwise independence
  $\Leftrightarrow \tilde{g}_{i_1,...,i_c}(V_{i_1}, ..., V_{i_c}) = 0$ (a.s.) $1 \leq i_1 < ... < i_c \leq n$,
  $c = 2, ..., r$

Measures of dependence & sharp bounds for contingent claims & VaR

- **Measures of dependence:**

$$\phi^2_{X_1,\ldots,X_n} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{(dF(x_1,\ldots,x_n))^2}{dF_1(x_1)\ldots dF_n(x_n)} - 1$$

- (**multivariate Pearson's** ϕ^2)

$$\delta_{X_1,\ldots,X_n} =$$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} log\left(\frac{dF(x_1,\ldots,x_n)}{dF_1\ldots dF_n}\right) dF(x_1,\ldots,x_n)$$

(**relative entropy**)

Measures of dependence & sharp inequalities

- **Normal** distribution: **explicit** expressions $(X_1, ..., X_n)' \sim N(\mu, \Sigma)$:
    - $\phi^2_{X_1,...,X_n} = |R(2I_n - R)|^{-1/2} - 1$, $\lambda(R) < 2$

- $U-$**statistic** representations $\Rightarrow$ **Generalized EFGM:**

$$\phi^2_{X_1,...,X_n} = \sum_{c=2}^{n} \sum_{1 \leq i_1 < ... < i_c \leq n} \theta^2_{i_1,...,i_c}$$

Reduction of expectations under dependence to independence

- **Bounds** for **expected payoffs** & **fair prices** of **contingent claims** under **dependence**

- $f : \mathbf{R}^n \to \mathbf{R}$: **arbitrary** nonnegative

  $Ef(X_1, ..., X_n)$: **expected payoff** of a **contingent claim** on **risks** $X_1, ..., X_n$

  $\xi_1, ..., \xi_n$: **independent** copies

Reduction of expectations under dependence to independence

Sharp inequalities: $Ef(X_1, ..., X_n) \leq$

- $(1 + \phi^2_{X_1,...,X_n})^{1/q}(Ef^q(\xi_1, ..., \xi_n))^{1/q}, \ \ q \geq 2$
- $\underbrace{Ef(\xi_1, ..., \xi_n)}_{\text{Expected payoffs under independence}} + \underbrace{\phi_{X_1,...,X_n}\Big(Ef^2(\xi_1, ..., \xi_n)\Big)^{1/2}}_{\text{price for dependence}}$

Expected payoffs
under **independence**

price for
dependence

Bounds for tail probabilities: value at risk

- **Sharp inequalities:** $P\Big[h(X_1, ..., X_n) > x\Big] \leq$

  - $\Big[1 + \phi^2_{X_1,\dots,X_n}\Big]^{1/2}\Big[P(h(\xi_1, ..., \xi_n) > x)\Big]^{1/2}$

  - $(e-1)P\Big[h(\xi_1, ..., \xi_n) > x\Big] + \delta_{X_1,\dots,X_n}$

  - $\underbrace{P\Big[h(\xi_1, ..., \xi_n) > x\Big]}_{\substack{\textbf{Tail probability} \\ \text{under } \textbf{independence}}} + \underbrace{\phi_{X_1,\dots,X_n}\Big[P(h(\xi_1, ..., \xi_n) > x)\Big]^{1/2}}_{\textbf{price} \text{ for } \textbf{dependence}}$

- **Bounds** for **portfolio VaR** in the case $h(X_1, ..., X_n) = \sum_{i=1}^n w_i X_i$

- **Reduction** to the case of **i.i.d. risks**

Conclusion

- **Fundamental problems** in **economics, finance** & **risk management**:
  - **Properties** of **marginal** distributions
  - **Dependence**
- **Copulas**: **convenient tool** to **account** for **both effects** and to **separate** one from the other
- **Separation** of **marginal effects** from **dependence**: **Key** to
  - **Reduction** of problems under **dependence** to **independent case**
  - **Sharp bounds** on **value at risk** for financial portfolios & contingent claim (option) **prices**
  - **Estimation** of **co-movements** in financial & insurance markets
- Theory & applications of **copulas**: **main results** lie **ahead**