Bubbly Liquidity

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Abstract

This paper analyzes the possibility and the consequences of asset price overvaluation in a dynamic economy where financially constrained firms demand and supply liquidity. Bubbles are more likely to emerge, the scarcer the supply of outside liquidity and the more limited the pledgeability of corporate income; they crowd investment in (out) when liquidity is abundant (scarce). We analyze the economic implications of firm heterogeneity, endogenous corporate governance, and stochastic bubbles. Finally we draw some implications for the way public policy could react to bubbles.

Keywords: liquidity, bubbles, governance, prudential regulation.

JEL numbers: E2, E44.

I Introduction

Economies recurrently generate cycles of asset price overvaluations, credit booms, and, when bubbles burst, recessions. The recent crisis is a case in point; the 2006-2007 burst of the US real estate bubble after 10 years of almost uninterrupted growth had major consequences on levered institutions holding housing assets. In a more historical perspective, Reinhart and Rogoff (2009) document post war crises brought about by an average 35% decline in housing price over the six years after the termination of the bubble.

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Despite some progress in our understanding of asset price bubbles, many challenging questions are left unanswered, including:

- Drivers: What role do macroeconomic conditions and financial institutions play in the emergence of asset price overvaluation?

- Asset-price-driven macro dynamics: is the classic theory of rational bubbles correct in predicting that asset bubbles raise interest rates and crowd out productive investment? While the interest rate response is rather undisputed, some famous episodes seem consistent with a crowding in hypothesis.[1]

- Implications of bubbles for financing patterns: Are the effects of bubbles confined to macroeconomic variables such as interest rates and economic activity, or do they modify the financial structure of firms? Do they benefit/hurt some sectors more than others?

- Public policy: How should capital adequacy requirements react to the presence of a bubble? Can aggregate liquidity policies—aimed at increasing outside liquidity—or growth policies—aimed at fostering financial development—lower the probability of the emergence of bubbles or limit their size?

- Tests for existence of bubbles: Is the classic Abel et al (1989) test appropriate in an economy exhibiting a diversity of rates of interest? If not, can we think of an alternative test for the existence of bubbles?

This paper investigates these questions by adding to the standard growth model an asynchronicity between firms' access to and need for cash. While this asynchronicity is perfectly resolved by capital markets in classic growth theory, capital markets are here imperfect: Factors such as agency costs prevent firms from pledging the entirety of the benefits from investment to uninformed investors, resulting in credit rationing. The anticipation of credit rationing in turn gives rise to a familiar demand for liquidity or capital insurance. The economy then requires a sufficient amount of stores of value to function properly.

Our model has overlapping generations of three-period-lived entrepreneurs. To meet their future needs for cash in the intermediate period of their life, young entrepreneurs know that they will be able to issue securities backed by the pledgeable income they will produce in the last period of their life; they complement this funding liquidity by hoarding stores of

[1] Japan's bubble came with not only high interest rates but also vigorous investment and growth; when it burst, the country went through a prolonged deflation and recession. Similarly, in the US, stores of values do not seem to have hampered productive investment when the public debt rose sharply during the 1980s, or during the Internet bubble; interest rates and investment fell when the latter burst.

value when young: securities issued by previous generations of entrepreneurs, bubbles, and "outside liquidity".

We provide several examples of outside liquidity. In the first illustration, the environment has non-Ricardian features: the state takes advantage of its regalian taxation power, and issues Treasury bonds backed by the consumers' future income. In the second, reverse mortgages allow consumers to borrow against their future income; this securitization of their housing assets increases the number of stores of value that firms can invest in. Finally, we introduce a sector of financially unconstrained firms (i.e. firms such as depicted in standard investment theory); these firms securitize the entirety of the future income associated with their current investment. In all cases, a flow of stores of values is created by the "unconstrained sector," that the constrained sector can build on to meet liquidity needs.

Financially constrained firms both consume and produce stores of value. The impact of outside liquidity on investment and economic activity accordingly hinges on the relative potency of two effects: a liquidity effect and a competition effect. On the consumption side, the firms' hoarding of liquid assets makes them benefit from an increase in the supply, and a reduction in the price of liquid assets. On the production side, their issuing securities on the capital market to finance liquidity needs makes them vulnerable to high interest rate conditions. An increase in outside liquidity raises interest rates and competes with the securities issued by the firms.

Our analysis sheds light on the questions raised above:

Drivers: Bubbles are more likely to emerge, the scarcer the supply of outside liquidity and the more limited the pledgeability of corporate income. That is, when faced with a shortage of stores of value, firms buy overvalued assets to store their wealth. This implies for instance that financial underdevelopment (measured either by the inability of the financial system to mitigate the agency problem or by a low stock of securities issued by the corporate sector) is conducive to the emergence of bubbles, or to larger bubbles.

Developed economies can develop bubbles too if shortages of stores of value develop. The savings glut in countries with a shortage of stores of value may well have fuelled the recent housing bubble in the U.S. by propping up American stores of values.[2]

Asset-price-driven macro dynamics: Outside liquidity is purchased as stores of value by firms, but it also competes with the latter's security offerings. When it is scarce, the

[2] Shiller (2009) makes a strong case that housing was indeed overvalued. To be certain, the rush to acquire real estate assets was also driven by the failure of prudential regulation.

competition effect dominates and outside liquidity crowds investment out. Otherwise the liquidity effect dominates and outside liquidity crowds investment in.

A bubble, except at its inception, always crowds investment in as long as outside liquidity exceeds some threshold and crowds investment out otherwise. Bubbles increase the interest rate and induce a transfer of net worth from suppliers of liquidity to purchasers of liquidity.

Bubble bursts are accompanied by low interest rates and high leverage. Because stochastic bubbles do not pay off in states of the world where internal funds can be levered the most, they command a liquidity discount—they have higher expected returns. We show that bad shocks hitting firms' balance sheets reduce the demand for liquidity and lead endogenously to bubble bursts. Bad shocks to corporate balance sheets can have an amplified effect on investment over and above that described in the literature emphasizing the importance of corporate net worth—for example Kiyotaki-Moore (1997)—by triggering liquidity dry-ups in the form of bubble bursts.

Implications of bubbles for financing patterns: First, outside liquidity impacts firms differently. Firms with limited ability to pledge future cash-flows (family and private equity firms, start ups) are little hit by competing claims as they issue no or few securities. Accordingly, they benefit more from a bubble and are hurt more by a bubble crash. Second, we predict that the emergence of a bubble should lead firms to sacrifice collateral for value, for example by adopting corporate policies that are less investor friendly, such as a relaxed governance.

Capital adequacy ratios (CARs): In our model, leverage increases in the aftermath of a bubble crash. Thus, if we follow the "representation hypothesis", according to which prudential regulators act on behalf of small depositors (insurees, future pensioners) and monitor the financial structure of retail institutions[3], the CARs should be higher during asset-price driven booms and smaller during busts. We thus provide some support to the view that regulation should react to asset prices, and be somewhat countercyclical. Countercyclical CARs has long been frowned upon within the regulators' community, but has gained more visibility with the Spanish "dynamic provisioning" policy during the housing bubble and with the recent statements by the Committee of European Banking Supervisors, the US Treasury, and the G20 countries.

Aggregate liquidity policies**:** Authorities can to some extent regulate the possibility and the size of bubbles by controlling the supply of outside liquidity—for example by adjusting

[3] See Dewatripont-Tirole (1994) for arguments in favor of this view.

the level of public debt, or by tightening or relaxing securitization standards (how much a consumer is allowed to borrow against his house).

Financial development: Financial development in our model is captured in two related ways, governance and the existing stock of traded securities. In practice, a country's quality of governance relates to its legal system's investor protection or to the existence of intermediaries with financial and human capital that can monitor industry and thereby enhance its access to capital. In our theory, governance increases funding liquidity and reduces the demand for stores of value, making bubbles smaller and less likely ceteris paribus. Similarly, a larger inherited stock of traded securities increases the amount of stores of value and therefore also makes bubbles smaller or less likely.

Rehabilitation of interest-rate tests to gauge the possibility of bubbles: Standard tests for the possibility of bubbles (Abel et al 1989) are ill suited for our environment. With imperfect capital markets, the economy can be dynamically efficient while the interest rate is lower than the growth rate of the economy. This is because the rate of return on internal funds exceeds that on borrowed ones; therefore the social rate of return on investments is higher than the market interest rate when returns can be only imperfectly collateralized.

Interest-rate tests have been abandoned because of the actual diversity of interest rates in the economy. Our agency-based approach provides a rationale for such a diversity and argues in favor of the use of (relatively low) interest rates received by uninformed investors in the comparison with the economy's rate of growth.

Outline of the paper. The paper proceeds as follows. Section II sets up the model and describes the solution when there are no bubbles. It characterizes its unique steady state and derives some key comparative statics results. Section III introduces the possibility of rational asset price bubbles. It derives the dynamics with bubbles, describes the properties of the unique bubbly steady state. Section IV first analyzes how bubbles affect the financial structure of firms; it then introduces stochastic bubbles and derives the mechanics of a bubbly boom-bust episode. Section V checks the robustness of the results in several variants of the model. Section VI offers some speculations on policy, and attempts to use the model to highlight some forces behind the ongoing subprime crisis. Finally, Section VII summarizes the main insights and discusses alleys for research.

Relation to the literature. The paper builds on a number of contributions. Most obviously, it brings together the literature on (rational) bubbles and that on aggregate liquidity. The competition effect, however, differs from that featured in Diamond (1965)'s celebrated analysis of national debt, and prominent in the theory of rational bubbles (Tirole 1985). The standard competition effect captures the idea that unconstrained firms want to invest less when interest rates are high. Our competition effect has it that high interest rates aggravate credit rationing and so firms cannot invest as much. In particular, Diamond's version of the competition effect is inconsistent with the existence of a liquidity effect.

The role of stores of values in supporting investment when income is not fully pledgeable has been stressed for example by Woodford (1990), Holmström-Tirole (1998) and a large recent literature, including independent contributions by Kiyotaki-Moore (2008) and Kocherlakota (2009). In Woodford's and Kocherlakota's contributions, which are most closely related to ours, firms are net demanders of liquidity and there is always a potential shortage of stores of value. These two papers assume that firms do not enjoy any funding liquidity by positing that none of the future cash flow is pledgeable to investors and so firms do not issue securities. As will be discussed in Appendix A3, though, our model, even in the absence of bubbles, does not converge to Woodford's when pledgeability converges to zero but remains positive: the dynamic system representing the economy features bifurcation at the point of zero pledgeability, making this case special. The existence of funding liquidity is central to many of our insights (existence of a liqudity effect, impact of financial development on the feasibility of bubbles, impact of bubbles on prudential regulation or on cross-sectoral financing patterns).

Saint-Paul (2005) shows that government debt (a store of value), while deterring capital accumulation, can increase the efficiency of the financial sector. Entrepreneurs can buy public debt and use it as collateral. The existence of collateral reduces agency costs (Saint-Paul uses the costly-state-verification model as an illustration). Accordingly, public debt boosts growth over a range of parameters.

The paper shares with Kiyotaki-Moore (1997) the idea that investment decisions are intertemporal complements. In Kiyotaki-Moore, tomorrow's investment will raise the price of the store of value, which is used as an input in the production process; this future increase in the price of the store of value raises the firms' wealth and thereby today's investment. In our paper, it is yesterday's investment that supports today's investment, by creating securities that firms can hoard to meet their liquidity needs. Thus, Kiyotaki and Moore's dynamics are forward looking (in the absence of bubbles) while ours are essentially backward looking. Also, Kiyotaki-Moore's focus is rather different as it has no bubbles.

The rational bubble literature has addressed the crowding-out critique in alternative ways. Bubbles are attached to investment in Olivier (2000) and to entrepreneurship in Ventura (2003), generating an incentive and a wealth effect respectively; in both papers, bubbles can crowd investment in. Saint-Paul (1992) addresses the dynamic-efficiency critique by studying an endogenous growth model with bubbles, in which the social return on investment exceeds the private return due to spillovers. The long-term rate of interest can then be smaller than the rate of growth of the economy, and yet the economy be dynamically efficient. Caballero and Krishnamurthy (2006) develop a theory of bubbles in emerging markets. They introduce, as we do, an investment driven demand for liquidity and show in the presence of fragile (stochastic) bubbles, the economy overinvests in the bubbly asset and is overexposed to bubble crashes due to a pecuniary externality.

Our paper sheds some light on the debate as to whether monetary authorities should try to lean against bubbles (or, in a more extreme form, try to make them pop) by raising interest rates or denying access to the discount window to banks that extend too many loans. Some scholars (e.g., Bernanke 2002, Bernanke-Gertler 2000, 2001, Gilchrist-Leahy 2002) argue that the central bank should not pay attention to asset prices unless these signal future inflation; others (e.g., Bordo-Jeanne 2002) are in favor of a moderate reaction.[4] All concur that a restrictive policy leads to a lower output and a significant risk of collateral-induced credit crunch. Our model is consistent with this premise, as the pricking of the bubble leads to a collateral shortage and reduced investment and production.

II The model

II.1 Description

Demographics, preferences and technology. Our model has overlapping generations of risk-neutral entrepreneurs. The population is constant (all our results generalize to the case of positive population growth). Entrepreneurs live for three periods: young, middle aged and old. For simplicity, we assume that entrepreneurs consume only when old. We relax this assumption in Section V.1. They are risk-neutral and seek to maximize expected consumption.

There is a single good in the economy. When young, entrepreneurs of generation t are endowed with A units of good (wealth). When middle aged, they invest i_t to produce $\rho_1 i_t$

[4]This is only a partial list of references on the topic. See Adrian-Shin (2008) for a more complete list.

when old. However, only a fraction $\rho_0 i_t < \rho_1 i_t$ of the return on investment is pledgeable.

Market for liquidity. In every period, a market for *liquidity* allows entrepreneurs to lend and borrow, subject to the borrowing constraints imposed by the limited pledgeability of their future income. The interest rate prevailing between date $t+1$ and date $t+2$ is $1 + r_{t+1}$. In equilibrium, it will always be the case that the pledgeability parameter ρ_0 is strictly less than $1 + r_{t+1}$. Because pledgeability is limited, firms can only partially rely on outside financing at the investment stage. We will only analyze equilibria where $\rho_1 > 1 + r_{t+1}$ so that the investment opportunities of entrepreneurs are strictly positive net-present-value projects.

The ingredients that determine supply and demand on the market for liquidity are as follows. The asynchronicity between the availability of cash and investment opportunities, together with the imperfect pledgeability of cash flows from investment, lead to a demand for liquidity from young entrepreneurs: they purchase stores of value in their youth when their wealth is high, and sell them in their middle age when they have an attractive investment opportunity that can only be partially financed by the market. In turn, middle-aged entrepreneurs are also suppliers of liquidity: they supply assets which capitalize the pledgeable cash flows from their investment project.

At the heart of this paper is the interplay between different forms of liquidity. Specifically, we investigate the interaction of *inside liquidity*—stores of values produced by middle-aged entrepreneurs of generation t when they pledge a fraction of the return on their investment project—and *outside liquidity*—stores of values that come from a different sector in the economy.

We model outside liquidity as follows. At each point of time, there is a net supply of l units of "rents". Rents at date t are short-term real bonds, paying one unit of good at date $t+1$. These stores of value will be purchased in equilibrium by young entrepreneurs so as to be able to invest when middle aged. We will not restrict l to be positive. Indeed, as will be discussed below, it may be the case that the corporate sector supplies stores of value to other sectors of the economy: consumers, foreigners, etc. At this stage, l is just an exogenous supply and the focus is entirely on entrepreneurs. In addition to rents, outside liquidity also comes in the form of a rational bubble. We denote by b_t the size of the bubble at date $t+1$.

The problem of entrepreneurs. Entrepreneurs invest all their wealth in their youth in stores of values—rents, bubble, and investment projects of the previous generation—and use these savings when middle-aged to produce internal funds for their investment project.

In their youth, entrepreneurs of generation t must decide how many bonds $\hat{l}_t$ to purchase, how much $\hat{b}_t$ of the bubble to acquire and how much x_t to invest in projects of entrepreneurs of generation $t-1$ realized in period t and delivering output in period $t+1$. We denote by $\hat{x}_t = x_t(1+r_t)$ the period-$t+1$ cash-flows of the securities issued in period t by entrepreneurs of generation $t-1$ to entrepreneurs of generation t in exchange of x_t. We have:

$$A = \frac{\hat{l}_t + \hat{b}_t + \hat{x}_t}{1+r_t}.$$

At date $t+1$, the date-t entrepreneurs' borrowing capacity is the sum of the value of claims on the future cash flows from their investment $\rho_0 i_t/(1+r_{t+1})$, hoarded rents $\hat{l}_t$, the bubble $\hat{b}_t$, and securities $\hat{x}_t$ purchased from the previous generation of entrepreneurs:

$$i_t = \frac{\rho_0 i_t}{1+r_{t+1}} + \hat{l}_t + \hat{b}_t + \hat{x}_t \qquad \text{or} \qquad i_t = \frac{\hat{l}_t + \hat{b}_t + \hat{x}_t}{1 - \frac{\rho_0}{1+r_{t+1}}}.$$

As is standard from the corporate finance literature, investment i_t increases with the entrepreneurs' *net worth* $\hat{l}_t + \hat{b}_t + \hat{x}_t$ at the time when the investment is made. The *investment multiplier* $1/[1-\rho_0/(1+r_{t+1})]$ is a measure of *leverage*. Investment increases with the fraction of income that is pledgeable to investors ρ_0, and decreases with the interest rate $1+r_{t+1}$ through the decrease in the value of the collateral generated by the project.

Discussion. *Overlapping generations.* We have adopted a framework with overlapping generations of entrepreneurs. The concept of generation should not be interpreted too literally—a period in our model need not last for 25 years. Rather, overlapping generations are the simplest modelling device that allows us to capture two features that are essential to our analysis. First, at any point in time, some entrepreneurs are net suppliers of liquidity while others are net demanders of liquidity. Second, interest rates can be lower than the rate of growth of the economy (here, zero), which makes room for rational bubbles.

Other modelling options would have delivered these same features. For example we could have analyzed a model à la Woodford (1990) where entrepreneurs are segmented into groups with alternating investment opportunities and borrowing constraints. Or we could have opted for a model à la Aiyagari (1994) and Bewley (1986) where the investment opportunities of entrepreneurs are stochastic, with idiosyncratic risk (and possibly aggregate risk as well). Under both types of models, occasionally binding borrowing constraints segment the horizons of agents with essentially the same effects as overlapping generations.

The potential benefit of Aiyagari-Bewley models over ours is that they are in principle more suitable for realistic quantitative explorations. However, the parameters for a realistic calibration in the context of our model (i.e. a precautionary savings model for firms instead of the more customary income fluctuation problem for consumers) are currently largely unknown. Moreover, this benefit has to be weighted against the cost in terms of loss of tractability. Indeed, the dynamics of such models can be hard to characterize theoretically because of the need to keep track of the evolving cross-sectional distribution of wealth. By contrast, we are basically able to derive the solution of our model in closed form. Since our objective is mostly theoretical, we view our model as preferable.

Outside liquidity. Although we will provide more microfoundations for outside liquidity in Section V.2, it is useful to give a simple example. Imagine the following non-Ricardian environment: there are consumers who live for one period, receive income w at home or abroad. They incur a cost $\tilde{l} < w$ if they move abroad. So the state can tax them $\tilde{l}$. The state issues bonds one period ahead. Let π be the number of newly-born consumers per newly-born entrepreneur. The state receives $\tilde{l}\pi/(1+r_t)$ from the bond issuance and distributes it to consumers. Then $l = \tilde{l}\pi$. Note that individual consumers live for a single period. Individually, they are neither lenders nor borrowers. Collectively, though, they are net borrowers as the state issues "on their behalf" pledges on their future income.[5] We will encounter in Section V.2 other microfoundations for outside liquidity in which l responds to the interest rate; as we will see, the theory extends to such situations.

II.2 Competitive equilibrium

A competitive equilibrium imposes market clearing: $\hat{l}_t = l$, $\hat{b}_t = b_t$ and $\hat{x}_t = \rho_0 i_{t-1}$. We will use a version of recursive equilibrium as our running definition. The economy is amenable to a recursive representation with two state variables: past investment i_{t-1} and the bubble b_t. The laws of motion for these variables can be derived from three simple equations: a bubble dynamics equation, an asset supply equation and an asset demand equation.

Bubble dynamics. The bubble must grow at the rate of interest:

$$b_{t+1} = (1 + r_{t+1})b_t \tag{1}$$

[5]A private-sector variant of this would have private lenders, who subsidize consumption when young of two-period-lived consumers up to a reimbursement limit of $\tilde{l}$ as consumers can move abroad in the second period of their life. This model is isomorphic to the one with public supply just outlined.

Asset supply. The supply equation describes how generation t's investment at date $t+1$ is constrained by the available liquidity, $l + b_t + \rho_0 i_{t-1}$, and by the investment-related pledgeable income, $\frac{\rho_0 i_t}{1+r_{t+1}}$:

$$i_t = \frac{\rho_0 i_t}{1 + r_{t+1}} + [l + b_t + \rho_0 i_{t-1}] \tag{2}$$

and can be expressed as

$$i_t = \frac{l + b_t + \rho_0 i_{t-1}}{1 - \frac{\rho_0}{1+r_{t+1}}}.$$

Asset demand. The demand equation says that generation $t+1$'s wealth goes into buying outside liquidity (l), the bubble and the stores of value generated by the previous generation's investment ($\rho_0 i_t$):

$$A = \frac{l}{1 + r_{t+1}} + b_t + \frac{\rho_0 i_t}{1 + r_{t+1}}. \tag{3}$$

It can be expressed as

$$i_t = \frac{A\,(1 + r_{t+1}) - l - b_t\,(1 + r_{t+1})}{\rho_0}.$$

We define a competitive equilibrium as a sequence of investment levels, bubble and interest rates $\{i_t, b_t, r_t\}$ such that at every date t, the asset market clears:

Definition 1 *A competitive equilibrium is a sequence* $\{i_t, b_t, r_t\}_{t\geq 0}$ *together with an initial investment level* $i_{-1} \geq 0$ *and an initial bubble* b_0 *such that the bubble condition (1) and the asset supply and asset demand equations (2) and (3) hold, and for all* $t \geq 0$, $i_t \geq 0$, $b_{t-1} \geq 0$ *and* $1 + r_t > 0$. *In addition, we require that* $\rho_1 > 1 + r_t$ *for all* t.

As mentioned above, we will focus exclusively on the case where $\rho_1 > 1 + r_t$ for all t. Rather than stating this condition in each proposition below, we impose it as a global assumption here and do not mention it again. Our characterizations apply only the portions of the state space where it holds.

Phase diagram analysis. The economy is a two-dimensional dynamic system with state variables i_{t-1} and b_t, that can be described conveniently with a phase diagram. This requires characterizing the $i_t = i_{t-1}$ schedule and the $b_{t+1} = b_t$ schedule.[6] The characteristics

[6] Note that $b_{t+1} = b_t = 0$ whenever $b_t = 0$. Literally speaking the $b_{t+1} = b_t$ schedule consists of two parts:

of the dynamics of the economy, and the nature of the phase diagram changes depending on whether $l > 0$ or $l < 0$. We first treat the bubbleless case in Section II.3. We then turn to the bubbly case in Section III, when $l > 0$ in Section III.2, and when $l < 0$ in Section III.3.

The expressions for the $i_t = i_{t-1}$ and $b_{t+1} = b_t$ schedules do not depend on the sign of l. The $i_t = i_{t-1}$ schedule is given by

$$b_t = i_{t-1}^2 \frac{\rho_0}{l} (1 - \rho_0) - \frac{\rho_0 i_{t-1}}{l} \left[A + \left(2 - \frac{1}{\rho_0} \right) l \right] - l,$$

which defines a function $b_t^i(i_{t-1})$. The $b_{t+1} = b_t$ schedule is given by

$$b_t = -\rho_0^2 i_{t-1} + (1 - \rho_0)(A - l)$$

which defines a function $b_t^b(i_{t-1})$.

II.3 The bubbleless case

Let us first assume that $b_t = 0$ for all t. The asset market clears at date $t+1$ when the demand and the supply curves intersect, determining i_t and r_{t+1} as a function of the state variable i_{t-1}. This involves solving a quadratic equation $[1 - \rho_0 A / (l + \rho_0 i_t)] i_t = l + \rho_0 i_{t-1}$. We derive the exact solution for the dynamics of investment (i_t as a function of i_{t-1}) in appendix A1.

Inside and outside liquidity. The productive sector provides its own liquidity in a dynamic fashion: an increase in i_{t-1} leads to an increase in i_t. The asset supply and asset demand equations (2) and (3) can also be used to determine the impact of outside liquidity (l) on investment. This impact can be decomposed into two effects. On the one hand, increasing outside liquidity l available at date t shifts the asset supply curve (2) upwards, raising investment i_t for all interest rate levels r_{t+1}—a *liquidity effect*. On the other hand, increasing outside liquidity l available at date $t+1$ increases the interest rate r_{t+1}. Leverage $1/[1 - \rho_0/(1 + r_{t+1})]$ decreases, which just expresses the fact that financing is harder when interest rates are high. This we call the *leverage or competition effect*. The resulting effect on investment i_t at date $t+1$ is ambiguous.

Intuitively, firms demand liquidity which is akin to an input in production. This tends

the one characterized by $b_t^b(i_{t-1})$ which applies whenever $b_t > 0$, and the line $b_t = 0$; abusing terminology, we refer to the former as the $b_{t+1} = b_t$ schedule.

to make investment and outside liquidity complements. But investments made by the private sector also play the role of inside liquidity. Inside liquidity is in direct competition with outside liquidity. This tends to make investment and outside liquidity substitutes. This distinction between the liquidity effect and the competition effect also has a temporal dimension. Existing liquidity—inside liquidity i_{t-1} or outside liquidity—and contemporaneous investment i_t are complements. Future liquidity and contemporaneous investment i_t are substitutes.

Steady state. The bubbleless economy has a unique steady state determined by the unique intersection with $i^* > 0$ of the steady state asset supply and demand curves:

$$i^* = \frac{l}{1 - \frac{\rho_0}{1+r^*} - \rho_0} \qquad \text{and} \qquad i^* = \frac{A\left(1 + r^*\right) - l}{\rho_0}.$$

This steady state is stable.

At a bubbleless steady state, we can clarify the circumstances under which rents and investment are complements or substitutes. We can rearrange these two equations to get:

$$i^* = \frac{A(1 + r^*)}{1 - \frac{\rho_0}{1+r^*}}. \tag{4}$$

Since r^* increases with l, i^* increases with l if and only if i^* increases with r^* in (4).[7] It can be verified that i^* increases with r^* if and only if the following condition holds:

$$\frac{1}{2} \geq \frac{\rho_0}{1 + r^*}. \tag{5}$$

Proposition 1 *In the bubbleless economy, steady state investment i^* increases with outside liquidity l when the interest rate is high enough so that (5) is verified. More precisely, there exists l_0 such that $\frac{\partial i^*}{\partial l} > 0$ if and only if $l > l_0$. Moreover, $l_0 > 0$ if and only if $\rho_0 < 1/2$.*

This proposition characterizes the situations where inside liquidity (investment) and outside liquidity (rents) are complements or substitutes. When liquidity is abundant – because

[7] When $l > 0$, the steady state asset supply curve for i^* is downward slopping in r^* and the steady state asset demand curve for i^* is upward slopping in r^*. They have a unique intersection. Since an increase in l shifts the asset supply curve up and the asset demand curve down, it unambiguously increases the steady state interest rate r^*. When $l < 0$, the asset supply curve becomes upward slopping. However, the formula

$$\frac{dr^*}{dl} = \frac{1 - \rho_0/\left(1 + r^*\right)}{A\left(1 - \rho_0\right) - \rho_0 l/\left(1 + r^*\right)^2}$$

shows that the steady state interest rate r^* once again increases with l.

pledgeability ρ_0 is high or because there are a lot of rents l—the price of liquidity is low—the interest rate r^* is high—and the liquidity effect outweighs the competition effect so that investment i^* increases with rents l. An intuition for this result is that an increase in the interest rate r^* has a constant positive marginal effect on net worth at the time of investment $A(1 + r^*)$ but a decreasing negative marginal effect on leverage $1/[1 - \rho_0/(1 + r^*)]$.

III Bubbles

In this section, we consider the possibility of rational bubbles. We first characterize the conditions of existence of a bubble steady state. We then turn to the dynamics of the economy, which turn out to hinge on the sign of l.

III.1 Bubbly steady state and investment dynamics

Bubbly steady state. There exists either zero or a unique saddle-path-stable bubbly steady state. When $l \geq 0$, bubbly steady states are always saddle path stable when they exist. When $l < 0$, there might also exist an unstable bubbly steady state, which we ignore. In what follows, we refer to the unique saddle-path-stable bubbly steady state as the bubbly steady state. When it exists the bubbly steady state is given by

$$i^{**} = \frac{A}{1 - \rho_0}, \; b^{**} = A\frac{1 - 2\rho_0}{1 - \rho_0} - l \text{ and } r^{**} = 0.$$

The condition of existence of a bubbly steady state is

$$\frac{1 - 2\rho_0}{1 - \rho_0} > \frac{l}{A}. \tag{B}$$

Condition (B) shows that bubbles can emerge when inside (ρ_0) and outside (l) liquidity is scarce, creating a high demand for stores of value. Note also that in a bubbly steady state, the interest rate is pinned down at 0. In Section (III.4) below, we show that condition (B) is equivalent to the standard condition that the interest rate in the bubbleless steady state (r^*) be less than the rate of growth of the economy (0). There, we also analyze the connection between dynamic efficiency and the condition for the existence of bubbles.

Note that variations in l are compensated one for one by variations in the size of the bubble: the number of stores of value is invariant to outside liquidity l. As a result, investment i^{**} at the bubbly steady state does not depend on l.

Investment dynamics. Returning to (2) and (3), one can eliminate the rate of interest and rewrite generation-t's investment as a function of the previous generation's investment and the bubble:

$$i_t = A + \rho_0 i_{t-1} + l\left(1 - \frac{1}{\rho_0}\right) + \frac{l}{\rho_0 i_t}[b_t + l + \rho_0 i_{t-1}].$$

Note that for $l = 0$, investment dynamics are unaffected by the existence of a bubble:

$$i_t = A + \rho_0 i_{t-1}.$$

The reason is that the sum of the value of the securities issued by the middle-aged generation, $\rho_0 i_t/(1+r_{t+1})$, and the bubble b_t in the end is equal to the savings A of the young generation. Put differently, the bubble fully crowds out the securities offering; it impacts only the rate of interest, as

$$b_t + \frac{\rho_0 i_t}{1 + r_{t+1}} = A.$$

This invariance disappears when $l \neq 0$. This can be understood as follows. The presence of the bubble lowers the price of outside liquidity—the inverse of the interest rate. As we will see, when $l > 0$, this boosts corporate net worth and investment to the detriment of the originators of this outside liquidity (the consumers/state in the example in Section II). The opposite occurs when $l < 0$.

III.2 Positive outside liquidity

In this section, we assume that there is a positive net supply of stores of value $l \geq 0$.

Lemma 1 *Investment i_t increases with the state variables i_{t-1} and b_t.*

The presence of the bubble lowers the price of outside liquidity, or in other words, increases the interest rate. This increases corporate net worth and investment, to the detriment of the suppliers of outside liquidity.

Figure 1 is a phase diagram representing the dynamics of the economy.[8] The bubbly steady state always features more investment than the bubbleless steady state. The latter is stable while the former features a downward-sloping saddle path. If the economy starts on the saddle path, it will eventually converge to the bubbly steady state. If it starts below the saddle path, it will eventually converge to the bubbleless steady state. The economy cannot start above the saddle path without eventually violating one of the constraints.

We are now in position to describe the dynamics when a bubble bursts. Suppose for example that we are in the bubbly steady state. As the bubble crashes, the economy jump downwards to the $b_t = 0$ line. Investment collapses, the interest rate decreases and the economy gradually converges to the bubbleless steady state. More generally, the following proposition summarizes the effects of a bubble.

Proposition 2 *Assume that (B) holds. For any i_{t_0-1}, there exists a maximum feasible bubble $\bar{b}(i_{t_0-1})$. The path of productions/investments $\{i_t\}_{t\geq t_0}$ and interest rates $\{r_t\}_{t\geq t_0}$ are increasing in the size of the original bubble b_{t_0}. For $b_{t_0} < \bar{b}(i_{t_0-1})$, the economy is asymptotically bubbleless: it converges to the bubbleless steady state. For $b_{t_0} = \bar{b}(i_{t_0-1})$, the economy is asymptotically bubbly: it converges to the bubbly steady state.*

Proposition 3 *Assume that (B) holds. On an asymptotically bubbly path, the bubble*
(i) decreases with the fraction of income that is pledgeable (ρ_0); indeed a bubble can exist if and only if the pledgeable income ρ_0 is smaller than $(A - l) / (2A - l)$;
(ii) decreases with the supply of existing stores of value (l).

Remark 1 *Woodford (1990) shows that the introduction of bubbles always crowds investment in, starting in a situation where there are neither outside stores of value ($l = 0$) nor inside stores of value ($\rho_0 = 0$). Introducing even a minimal amount of inside (ρ_0) and outside (l) liquidity produces a bifurcation: a different (and stable) growth path appears, that does not converge to the steady state considered by Woodford when $\rho_0 = 0$ and $l = 0$. A bubble need not crowd investment in. Appendix A3 expands on this remark and provides a detailed discussion.*

[8] We have $b_t^i(0) = -l < 0$ and $b_t^b(0) = (1 - \rho_0)(A - l)$ which is strictly positive as long as (B) holds. It is easy to verify that b_t^i is increasing when it intersects b_t^b. The sign of $\frac{db_t^i}{di_{t-1}}|_{i_{t-1}=0}$ on the other hand, is unclear a priori. Note that the bubbleless steady state is always stable.

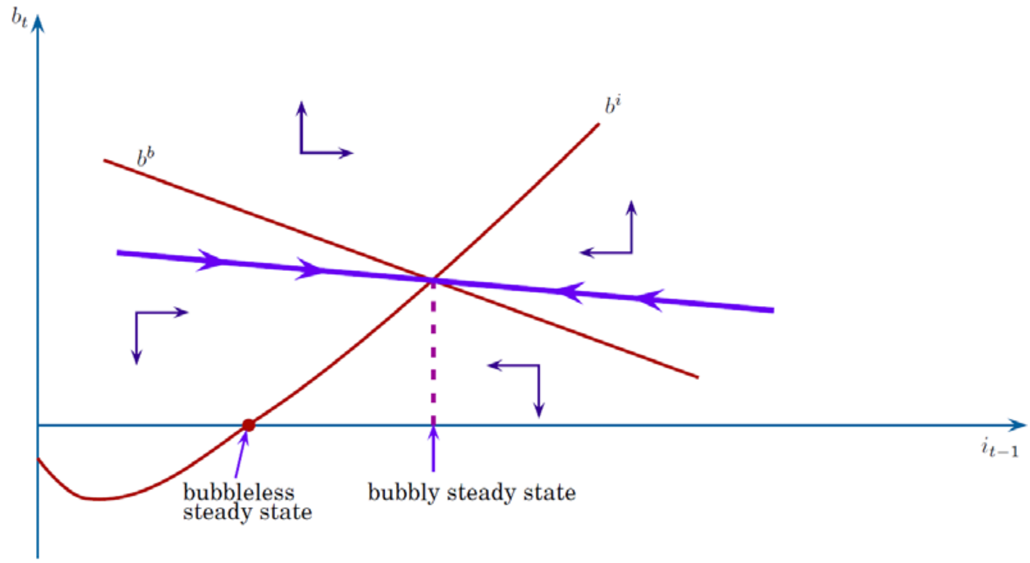

Figure 1: Figure 1: Phase diagram with positive outside liquidity.

III.3 Negative outside liquidity

We now assume that there is a negative net supply of stores of value: $l < 0$.

Lemma 2 *Investment i_t decreases with the state variable b_t.*

The presence of the bubble lowers the price of outside liquidity, or in other words, increases the interest rate. Since entrepreneurs are now net suppliers of liquidity, this decreases corporate net worth and investment.

There is a unique bubbleless steady state which corresponds to the intersection of the $b_t = 0$ locus with $i_t = i_{t-1} \geq 0$; this steady state is always stable.[9] If (B) holds, there are two intersections between the $i_t = i_{t-1}$ and the $b_{t+1} = b_t$ schedules such that $b_t > 0$. The corresponding investment levels are given by $-l\,(1-\rho_0)\,/\rho_0$ and $A/\,(1-\rho_0)$. As long as the following condition is satisfied, the lowest solution corresponds to an unstable bubbly steady state and the highest solution to a saddle-path-stable bubble steady state:

$$-\frac{l}{A} < \frac{\rho_0}{(1-\rho_0)^2}. \tag{6}$$

Condition (6) is more likely to be verified, the lower the net demand $-l$ for stores of value, the higher the level of pledgeable income ρ_0 and the higher the net worth of entrepreneurs

[9] We have $b_t^i(0) = -l > 0$ and $b_t^b(0) = (1-\rho_0)\,(A-l)$ which is strictly positive as long as (B) holds.

A. We maintain it throughout this section. When there is no risk of confusion, we refer to the saddle-path-stable bubbly steady state, with a slight abuse of terminology, as the bubbly steady state.

The saddle path of the saddle-path-stable bubble steady state spirals around the unstable bubble steady state. Following this saddle path backwards (toward the left) from the saddle-path-stable bubble steady state in the direction of $i_{t-1} < i^{**}$ and $b_t > b^{**}$, we eventually cross the $i_t = i_{t-1}$ schedule; we denote by $\underline{i}$ the level of investment at this intersection. The following proposition characterizes the dynamics of the economy when investment i_{t-1} is higher than investment $\underline{i}$. We do not seek to characterize further the dynamics of the economy around the unstable bubbly steady state.

Proposition 4 *Assume that (B) holds, that $l < 0$, and that (6) holds. Then $r^* < 0$ and $i^{**} < i^*$. There exists $\underline{i} < i^{**}$ such that for any $i_{t_0-1} \geq \underline{i}$, there exists a maximum feasible bubble $\bar{b}(i_{t_0-1})$. Investment decreases with the size of the bubble. For $b_{t_0} < \bar{b}(i_{t_0-1})$, the economy is asymptotically bubbleless: it converges to the bubbleless steady state. For $b_{t_0} = \bar{b}(i_{t_0-1})$, the economy is asymptotically bubbly: it converges to the bubbly steady state.*

When $l > 0$, the non-corporate sector is a net seller of stores of value. The bubble then operates a transfer from the non-corporate sector to the corporate sector, which increases investment. *Bubbles and investment are complements.* When $l < 0$, the opposite happens and bubbles crowd investment out. *Bubbles and investment are then substitutes.*

III.4 Tests for bubbles and dynamic efficiency

In this section, we explain how dynamic efficiency, interest rates and the possibility of bubbles are related. Most of the results below are not new in substance. Indeed, similar results are discussed in Woodford (1990). Nevertheless, we find it useful to extend these insights to our setup; this exercise further allows us to revisit the question of the measurement of dynamic efficiency and the possibility of bubbles.

Dynamic efficiency. Abel *et al*'s (1989) test of dynamic efficiency involves comparing the value of resources used for investment every period to the value of resources produced. It is believed to be superior to an interest rate test involving a comparison of the interest rate and the growth rate (here 0) since it is hard in practice to determine which interest rate to use in this comparison.

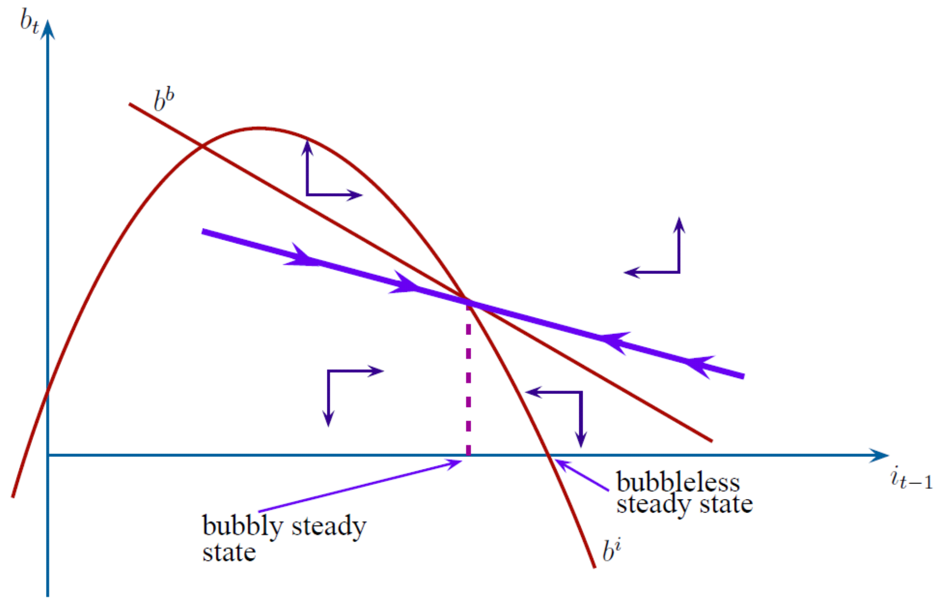

Figure 2: Figure 2: Phase diagram with negative outside liquidity.

We will see that in our model, an interest rate below the rate of growth does not necessarily imply that the economy is dynamically inefficient, and so interest rate tests are inadequate to gauge dynamic efficiency, but appropriate to detect if bubbles are possible.

Indeed, consider a steady state, with or without bubbles, where investment and interest rates are given by $\tilde{\imath}$ and $\tilde{r}$. In steady state, resources being used for investment in period t, or equivalently the total wealth of generation t at birth are A. In steady state, resources being produced from investment in period t are $\rho_1 \tilde{\imath} + l$. Hence Abel *et al*'s criterion tests whether or not

$$\rho_1 \tilde{\imath} + l - A \geq 0.$$

The bubbleless steady state is dynamically efficient if and only if

$$(\rho_1 - \rho_0)\, i^* + Ar^* \geq 0. \tag{DE}$$

Hence we can have $r^* < 0$ and still Abel *et al*'s test accepting dynamic efficiency (DE). Note, further, that if all output were pledgeable and therefore the rate of return were equal for internal and external funds ($\rho_0 = \rho_1$), then (DE) would boil down to the standard comparison between the rate of interest and the rate of growth (here 0).

The bubbly steady state is dynamically efficient if and only if

$$\frac{l}{A} \geq \frac{1-(\rho_1+\rho_0)}{1-\rho_0}. \tag{DE'}$$

This condition is consistent with (B) since $\rho_1 > \rho_0$: depending on parameter values, the bubbly steady state may or may not be dynamically efficient. The bubbly steady state is always dynamically efficient when $l \geq 0$: the left hand side of (DE') is then positive, and the right hand side is always negative (since $\rho_1 > 1$). Note also that $r^{**} = 0$ so that Abel *et al*'s test always rejects dynamic inefficiency if the economy is at the bubbly steady state.

Dynamic efficiency and bubbles. Let us now discuss the predictive content of dynamic efficiency tests such as Abel et al's test for the presence or the possibility of bubbles. In the model of Tirole (1985), bubbles can arise only if the bubbleless steady state is dynamically inefficient. In that case, the bubbly steady state is dynamically efficient and all asymptotically non bubbly paths are dynamically inefficient. Therefore, if the actual economy is found to be dynamically inefficient, bubbles are possible. If on the other hand the economy is dynamically efficient, then either bubbles are impossible or we are on an asymptotically bubbly path.

In our model, the link between bubbles and dynamic efficiency is considerably weakened. Condition (B) is consistent with the bubbleless steady state being either dynamically efficient or inefficient. Moreover, when bubbles are possible the bubbly steady state may or may not be dynamically efficient.

Bubbles. Whereas dynamic inefficiency is not determined by $\tilde{r} \leq 0$, the possibility of bubbles is still determined however, by the interest rate test $\tilde{r} \leq 0$.

Proposition 5 *The higher rate of return on internal funds than on borrowed ones implies that dynamic efficiency (in the sense of Abel et al) is consistent with bubbleless rates of interest below the rate of growth of the economy, and with the existence of asymptotically bubbly paths. The possibility of bubbles is exactly determined by an (uninformed investor) interest rate test of the form $\tilde{r} \leq 0$.*

Proof. Only the last claim still needs to be proven. Suppose that the steady state is the bubbleless steady state. In the appendix, we derive the following expression for r^* :

$$1+r^* = \frac{A\rho_0 + l + \sqrt{[A\rho_0+l]^2 - 4Al(1-\rho_0)\rho_0}}{2A(1-\rho_0)}.$$

It is straightforward to verify that (B) is equivalent to $r^* < 0$. If the steady state is the bubbly steady state, then by construction, (B) holds and $\tilde{r} = 0$. ■

The considerations brought about by our analysis go part of the way towards rehabilitating interest rate tests as an indication for the possibility of bubbles. They shed light on which interest rate to use in these tests: this rate corresponds to an *"uninformed" interest rate—a relatively low interest rate.*

IV Economic implications

IV.1 Collateral heterogeneity

We have assumed so far for simplicity that firms are homogenous (perhaps up to a scaling factor). When firms differ, say, in the pledgeability of their income, those with limited access to unsophisticated investors, i.e., low ρ_0 firms (family firms, private equity, startups), benefit relatively more from the presence of a bubble. They enjoy the liquidity effect without being much impacted by the competition effect as they do not resort much to small investors' money. Conversely, they also suffer more from a bubble crash.

Let k be an index for firms and let ρ_0^k be an increasing function of k. We can assume without loss of generality that k is distributed uniformly on $[0, 1]$. We then have the following aggregation result. The economy is described by two state variables: the value of the bubble b_t and the integral $\int \rho_0^k i_{t-1}^k dk$ describing the total value of the existing stock of securities. The law of motion for b_t is still $b_{t+1} = b_t(1 + r_{t+1})$, while $\int \rho_0^k i_{t-1}^k dk$ and r_{t+1} are jointly determined as the intersection of the aggregate supply and the demand curves for assets:

$$\int \rho_0^k i_t^k dk = \left(\int \frac{\rho_0^k}{1 - \frac{\rho_0^k}{1+r_{t+1}}} dk \right) \left(b_t + l + \int \rho_0^k i_{t-1}^k dk \right)$$

and

$$\int \rho_0^k i_t^k dk = [A\,(1 + r_{t+1}) - l - (1 + r_{t+1}) b_t].$$

Investment by firm k can be computed as

$$i_t^k = \frac{b_t + l + \int \rho_0^k i_{t-1}^k dk}{1 - \frac{\rho_0^k}{1+r_{t+1}}}.$$

There exists either zero or a unique bubbly steady state. When a bubbly steady state exists, it is given by

$$i^{**k} = \frac{A}{1-\rho_0^k},\ b^{**} = \left(\int \frac{1-2\rho_0^k}{1-\rho_0^k} dk\right) A - l, \text{ and } r^{**} = 0.$$

The condition for a bubble to exist is now given by

$$\left(\int \frac{1-2\rho_0^k}{1-\rho_0^k} dk\right) > \frac{l}{A}. \tag{B'}$$

The analysis of the dynamics of the economy are exactly as in Sections III.2 and III.3. Replacing the representative firm's pledged income by the industry-average pledged income, we see that the previous analysis generalizes to heterogenous firms.

The relative size of firms with low pledgeable income increases when bubbles arise and decreases when bubbles crash:

$$\begin{aligned} \frac{di_t^k}{i_t^k db_t} &= \frac{\partial i_t^k}{i_t^k \partial b_t} + \frac{\partial i_t^k}{i_t^k \partial r_{t+1}} \frac{dr_{t+1}}{db_t} \\ &= \frac{1}{b_t + l + \int \rho_0^k i_{t-1}^k dk} - \frac{\rho_0^k}{(1 + r_{t+1} - \rho_0^k)^2} \frac{dr_{t+1}}{db_t} \end{aligned}$$

For example, when a bubble crashes, there are two immediate opposing effects on investment: (a) firms which used the bubble to hoard liquidity have less net worth (b) the interest rate decreases, which allows for more leverage. The percentage decrease in investment resulting from a lower net worth is independent of ρ_0^k. The percentage increase in investment from a higher leverage is higher, the higher is pledgeable income ρ_0^k. As a result, in relative terms, the size of firms with low pledgeable income compared to the size of firms with high pledgeable income.decreases when a bubble crashes.

Proposition 6 *Assume that (B') holds. Then:*
(i) for any value of $\int \rho_0^k i_{t_0-1}^k dk$, there exists a maximum feasible bubble $\bar{b}\left(\int \rho_0^k i_{t_0-1}^k dk\right)$. The path of interest rates $\{r_t\}_{t\geq t_0}$ are increasing in the size of the initial bubble b_{t_0}; the path of productions/investments $\{i_t\}_{t\geq t_0}$ is increasing in the size of the initial bubble $b_{t_0}$when outside liquidity l is positive, and decreasing when outside liquidity l is negative. For $b_{t_0} < \bar{b}\left(\int \rho_0^k i_{t_0-1}^k dk\right)$, the economy is asymptotically bubbleless: it converges monotonically to the bubbleless steady state. For $b_{t_0} = \bar{b}\left(\int \rho_0^k i_{t_0-1}^k dk\right)$, the economy is asymptotically bubbly: it converges monotonically to the bubbly steady state;

(ii) the relative size of firms with low pledgeable income (low ρ_0^k) compared to the size of firms with high pledgeable income (high ρ_0^k) increases when a bubble pops up and decreases when a bubble crashes: $\frac{di_{t_0}^k}{i_{t_0}^k db_{t_0}}$ is decreasing in k.

Remark 2 *When $l > 0$, we can strengthen (ii). Indeed, in this case, the equilibrium paths for $\{b_t\}_{t\geq t_0}$ and $\{r_t\}_{t\geq t_0}$ can be shown to be increasing in the size b_{t_0} of the initial bubble. In this case, the relative variation $\left(i_t^k\right)^{-1} di_t^k/db_{t_0}$ of investment i_t^k at date $t+1$ with respect to the initial bubble b_{t_0} is decreasing in k. The reasons why a similar result cannot be proven when $l < 0$ is that in this case, it is not true anymore in general that the equilibrium paths for $\{b_t\}_{t\geq t_0}$ and $\{r_t\}_{t\geq t_0}$ are increasing in the size b_{t_0} of the initial bubble.*

IV.2 The pledgeability-value trade-off

A key theme in corporate finance is that firms can boost pledgeable income (ρ_0) at the cost of a sacrifice in value (ρ_1).[10] For example, they can pledge more collateral, creating moral hazard, monitoring costs and reduced flexibility; they may enlist a private monitor (venture capitalist, large shareholder, bank); or they can abandon private equity for a public listing, at the cost of transparency obligations, reduced incentives and so forth (more generally, some dimensions of corporate governance can be seen as enhancing pledgeability ρ_0, to the detriment of value ρ_1). The trade-off between pledgeable income and value can be formalized by a decreasing function $\rho_1 = H(\rho_0)$. Let ρ_1^t and ρ_0^t denote the values chosen by generation t, the utility of generation t entrepreneurs is

$$U_t = (\rho_1^t - \rho_0^t) i_t = \frac{\rho_1^t - \rho_0^t}{1 - \frac{\rho_0^t}{1+r_{t+1}}}[b_t + l + \rho_0^{t-1} i_{t-1}]$$

Thus, when liquidity is scarce (the interest rate is low), firms will sacrifice value in order to boost pledgeable income.

To avoid re-analyzing the complete path, let us assume that these choices between pledgeability and value are made "at the margin" and so the paths described in Propositions 1 and 2 are approximations of the realized paths with endogenous pledgeability/value decisions. The trade-off between pledgeability and value is resolved by maximizing the following

[10] See, e.g., Tirole (2006) for an overview.

objective, in every period t :

$$\frac{H\left(\rho_0^t\right)-\rho_0^t}{1-\frac{\rho_0^t}{1+r_{t+1}}}.$$

Proposition 7 *A bubble, by increasing interest rates, reduces the benefits of creating pledgeable income and is therefore conducive to lower ρ_0^t and higher ρ_1^t.*

IV.3 Stochastic bubbles (1): bubbly liquidity premium

As in Weil (1987), we can allow the bubble to burst stochastically. Suppose that each period the bubble bursts with probability $1-\lambda$. An asset's liquidity service depends on what the asset delivers when cash is particularly valuable to firms. Building on this idea, we now argue that, even in this risk neutral, constant-returns-to-scale (CRS) environment, a stochastic bubble trades at a liquidity discount or equivalently a risk premium relative to rents.

Let i_t and r_{t+1} (respectively, i_t^- and r_{t+1}^-) denote the investment levels and interest rates when the bubble has lasted until period $t+1$ and continues (respectively, bursts). The asset supply equations are given by

$$i_t=\frac{b_t+l+\rho_0 i_{t-1}}{1-\frac{\rho_0}{1+r_{t+1}}} \qquad \text{and} \qquad i_t^-=\frac{l+\rho_0 i_{t-1}}{1-\frac{\rho_0}{1+r_{t+1}^-}}.$$

Similarly, the asset demand equations are given by

$$i_t=\frac{A\left(1+r_{t+1}\right)-l-(1+r_{t+1})b_t}{\rho_0} \qquad \text{and} \qquad i_t^-=\frac{A\left(1+r_{t+1}^-\right)-l}{\rho_0}.$$

Since i_t^- and r_{t+1}^- are determined by the same set of equations as i_t and r_{t+1} but with $b_t=0$, it is clear that $r_{t+1}>r_{t+1}^-$: the burst of the bubble depresses the interest rate. The immediate response of investment i_t to a bubble crash is determined by Lemmas 1 and 2: when $l>0$, the burst of the bubble depresses investment so that $i_t>i_t^-$, and the opposite occurs when $l<0$.

At date t, generation-t entrepreneurs can hold safe assets (rents, claims on previous investments' income) or risky ones (stochastic bubble). Letting $\widetilde{r}_t$ denote the return on the bubble when it does not burst: $b_t=\left(1+\tilde{r}_t\right)b_{t-1}$ if the bubble doesn't burst and $b_t=0$

otherwise. The arbitrage equation between bubbles and rents is

$$\lambda\frac{1+r_t}{1-\frac{\rho_0}{1+r_{t+1}}}+(1-\lambda)\frac{1+r_t}{1-\frac{\rho_0}{1+r_{t+1}^-}}=\lambda\frac{1+\widetilde{r}_t}{1-\frac{\rho_0}{1+r_{t+1}}} \tag{7}$$

This in turn implies that $1+\widetilde{r}_t > (1+r_t)/\lambda$. Despite risk neutrality and CRS, stochastic bubbles trade at a positive discount $(1+r_t)^{-1}-\lambda^{-1}(1+\widetilde{r}_t)^{-1}$. Alternatively, they command positive net excess returns $\lambda(1+\widetilde{r}_t)-(1+r_t)$. The intuition is straightforward. Bubbles deliver no return when internal wealth is the most valuable: when liquidity is scarce, interest rates are low and internal funds can be levered a lot.

A steady state along the bubbly path is given by $1+\widetilde{r}^{**}=1$ and

$$1+r^{**}=\left(1+\frac{1-\lambda}{\lambda}\frac{1-\frac{\rho_0}{1+r^{**}}}{1-\frac{\rho_0}{1+r^{**-}}}\right)^{-1}, \qquad \frac{l+\rho_0\frac{A+l\frac{r^{**}}{1+r^{**}}}{1-\rho_0}}{1-\frac{\rho_0}{1+r^{**-}}}=\frac{A\left(1+r^{**-}\right)-l}{\rho_0}$$

$$i^{**}=\frac{A+l\frac{r^{**}}{1+r^{**}}}{1-\rho_0} \quad \text{and} \quad b^{**}=\frac{A+l\frac{r^{**}}{1+r^{**}}}{1-\rho_0}\left(1-\frac{\rho_0}{1+r^{**}}-\rho_0\right)-l.$$

The condition for the existence of a bubble becomes[11]

$$\frac{A+l\frac{r^{**}}{1+r^{**}}}{1-\rho_0}\left(1-\frac{\rho_0}{1+r^{**}}-\rho_0\right)-l>0. \tag{B”}$$

Proposition 8 *Suppose that (B”) holds, then:*
(i) bubbles trade at a liquidity premium;
(ii) the bursting of the bubble decreases interest rates: firms scramble for collateral, which becomes more valuable;
(ii) in steady state along the bubbly path—before the bubble bursts—as the probability of bursting $(1-\lambda)$ *increases, the interest rate* r^{**} *and the bubble* b^{**} *decrease; investment* i^{**} *decreases if* $l>0$*;*
(iii) in steady state along the bubbly path, interest rates are high ($r^{**}>r^*$)*; investment is high* ($i^{**}>i^*$) *if* $l>0$ *and low* ($i^{**}<i^*$) *if* $l<0$*;*
(iv) when the bubble bursts, investment immediately decreases if $l>0$ *and increases if* $l<0$*;*

[11] Unfortunately, this condition determines only implicitly the parameter region that leads to the possibility of bubbles. It features an endogenous objects, r^{**}. This complication arises for the following reason. Bubbles now present a risk premium and thus a positive net excess return: $\lambda(1+\tilde{r}^{**})-(1+r^{**})>0$. In a bubbly steady state, zero bubble growth pins down the expected return on bubbles: $1+\tilde{r}^{**}=1$, but the risk-free rate r^{**} has to be determined jointly with r^{**-}as solutions to a non-linear system.

then, the economy gradually converges to the bubbleless steady state.

For example, when $l > 0$, steady state investment i^{**}, bubble size b^{**} and interest rate r^{**} are all decreasing in the probability that the bubble crashes $1 - \lambda$. A more stable bubble provides more liquidity and is more conducive to investment. This in turn boosts the demand for liquidity and makes for a larger bubble.

The dynamics are more complicated. Indeed, an extra state variable is required to describe the economy. The state space is now given by the triple (i_{t-1}, b_t, r_t). The reason past interest rates r_t have to be kept track of is that the arbitrage equation (7) involves both the interest rate at date t and at date $t+1$. As a consequence, simple two-dimensional phase diagrams cannot be used anymore. A full characterization of the stability properties of the different steady states and their basin of attraction is rather involved and outside the scope of this paper.

Remark 3 *Imagine that there is a stochastic bubble in an economy with heterogeneity in pledgeable income as in Section IV.1. Equation (7) has the implication that firms with higher ρ_0^k will demand a higher expected return to hold the bubble. Therefore, in equilibrium, the bubble will be held by the firms with the lowest ρ_0^k. When $l > 0$, we showed in Proposition 6 that for a given portfolio, these firms are also the firms whose investment decreases the most in percentage terms when the bubble crashes. The result is an amplification mechanism whereby the equilibrium allocation of the bubble across firms magnifies the impact of a crash. In equilibrium, firms with a high ρ_0^k do not hold the bubble so that their net worth is insulated from the bubble crash. Moreover, they can increase their leverage more than firms with a low ρ_0^k in response to the drop in interest rates that follows the crash of the bubble. Conversely, firms with a low ρ_0^k hold the bubble so that their net worth is impaired when the bubble crashes. In addition, their leverage increases less than firms with a high ρ_0^k in response to the drop in interest rates that follows the crash of the bubble.* [12]

IV.4 Stochastic bubbles (2): endogenous crashes

Following up on section (IV.3), we modify the environment in the following way. Suppose that A follows a two-state Markov process $A \in \{A_H, A_L\}$ with $A_H > A_L$. Initially $A = A_H$.

[12]When $l < 0$, the crash of the bubble tends to increase aggregate investment. As in the case $l < 0$, the equilibrium allocation of the bubble across firms amplifies the impact of the crash.

With probability $1-\lambda>0$ per period, A transitions to A_L which is an absorbing state.

We assume that (B") is verified for an exogenous bursting probability of $(1-\lambda)$ in an economy with a deterministic and constant A equal to A_H. Similarly, we assume that for any $\lambda\in[0,1]$, (B") is violated for an exogenous bursting probability of $(1-\lambda)$ in an economy with a deterministic and constant A equal to A_L. Hence if the economy is already in the state of low net worth $A=A_L$, the demand for liquidity is low, the interest rate is high and bubbles cannot exist. Bubbles however, can exist as long as net worth is high: $A=A_H$.

Suppose that $A_t=A_H$ and consider the economy entering period $t+1$ with state variables given by (b_t, i_{t-1}, r_t). Then if $A_{t+1}=A_H$, (b_{t+1}, i_t, r_{t+1}) are given by the same equations as in Section IV.3 with $A=A_H$. On the other hand if $A_{t+1}=A_L$, then the bubble bursts: $b_t=0$ and i_t is given by the same equations as above with $A=A_L$ and $b_t=0$. The economy then evolves as in Section II.2.

Proposition 9 *Assume that (B") holds with $A=A_H$. Consider an economy where in the initial period the economy is in steady state along the bubbly path and $A_t=A_H$. In the first period where $A_t=A_L$, the bubble bursts. Then, the economy converges to the bubbleless steady state corresponding to $A=A_L$.*

This environment makes clear that when $l>0$, bad shocks to corporate balance sheets can potentially have an amplified negative effect on investment over and above that described in the literature emphasizing the importance of corporate net worth—for example Bernanke-Gertler (1989) and Kiyotaki-Moore (1997)—by triggering liquidity dry-ups in the form of bubble bursts.

V Discussion

V.1 Elastic savings

Entrepreneurs' preferences were chosen such that they save their entire endowment when young, and invest all the resulting wealth when middle aged. This section relaxes this assumption and allows more flexibility in the entrepreneurs' savings/dissavings choices. It turns out that the results obtained in this paper carry over, with one caveat. While it is still true that bubbles boost investment when outside liquidity l exceeds some threshold, this threshold need not be equal to zero. We now show that the threshold may indeed be positive or negative.

Example: log **preferences.** Suppose that entrepreneurs have utility from consumption vector (c^y, c^m, c^0) in the three periods of their life:

$$U = \log c^y + \beta \log c^m + \beta^2 \log c^0.$$

Let e denote their endowment when young. The rates of return on generation-t's savings when young is $1 + r_t$; the rate of return on their investment when credit-constrained at $t+1$ is

$$1 + \hat{r}_{t+1} = \frac{\rho_1 - \rho_0}{1 - \frac{\rho_0}{1+r_{t+1}}}.$$

Optimal saving behavior implies that they save

$$A \equiv \frac{\beta + \beta^2}{1 + \beta + \beta^2} e$$

when young. When middle aged, they contemplate consumptions:

$$c^m = (1 + r_t)A - \left(1 - \frac{\rho_0}{1 + r_{t+1}}\right) i_t \quad \text{and} \quad c^0 = (1 + \hat{r}_{t+1}) \left(1 - \frac{\rho_0}{1 + r_{t+1}}\right) i_t.$$

The asset supply and asset demand equations become respectively

$$i_t^s(i_{t-1}, r_{t+1}) = \frac{\beta}{1 + \beta} \frac{b_t + l + \rho_0 i_{t-1}}{1 - \frac{\rho_0}{1+r_{t+1}}}$$

and

$$i^d(r_{t+1}) \equiv \frac{A\,(1 + r_{t+1}) - l - b_t\,(1 + r_{t+1})}{\rho_0}.$$

The condition for a bubble to exist becomes

$$\frac{1 - \rho_0 \left(1 + \frac{\beta}{1+\beta}\right)}{1 - \rho_0} > \frac{l}{A}. \qquad \text{(B''')}$$

The analysis of the steady states and the dynamics of the economy is very similar to Section III. It is omitted for conciseness. Instead,we focus on one important object that behaves differently in this setup: the conditions under which bubbles increase investment. We find it most illustrative to focus on steady states and characterize the threshold for outside liquidity above which investment is higher at the bubbly steady state than at the bubbleless steady state.

Proposition 10 *Suppose that preferences are* log *and that (B"') holds. Then there exists a threshold $l_0 > 0$ such that for $l > l_0$, investment at the bubbly steady state is higher than at the bubbleless steady state $i^{**} > i^*$ and for $l < l_0$, investment at the bubbly steady state is lower than at the bubbleless steady state $i^{**} < i^*$.*

Some intuition can be given along the following lines. Let us compare log preferences with the preferences used in Section III for any given level of outside liquidity l. Since now only a fraction $\beta/(1+\beta)$ of middle-aged wealth is spent on investment by entrepreneurs, the liquidity effect of bubbles through the impact of increased interest rates on internal funds is mitigated. On the other hand, the competition effect through the impact of increased interest rates on leverage is unaffected.

Example: linear-log **preferences.** We now sketch a specification of preferences for which even for $l = 0$, investment at the bubbly steady state is higher than at the bubbleless steady state. We take

$$U = c^y + \tilde{\beta}\left[c^m(1+\beta)\right]^{\frac{1}{1+\beta}}\left[c^o(1+\beta)/\beta\right]^{\frac{\beta}{1+\beta}}$$

and refer to these preferences as linear-log. Linear-log preferences lead to the same asset supply and asset demand equations as above, with the difference that now $A = A(r_t, r_{t+1})$ is endogenous, with

$$A(r_t, r_{t+1}) = \begin{cases} e \text{ if } (1+r_t)\left(\frac{\rho_1-\rho_0}{1-\frac{\rho_0}{1+r_{t+1}}}\right)^{\frac{\beta}{1+\beta}} > \frac{1}{\tilde{\beta}}, \\ 0 \text{ if } (1+r_t)\left(\frac{\rho_1-\rho_0}{1-\frac{\rho_0}{1+r_{t+1}}}\right)^{\frac{\beta}{1+\beta}} < \frac{1}{\tilde{\beta}}. \end{cases} \tag{8}$$

The difference with the log preferences of the previous example is that now the intertemporal elasticity of substitution between c^y and a composite period that aggregates c^m and c^0 is equal to infinity. This feature reinforces the liquidity effect: bubbles, by increasing the interest rate, can increase savings when young.

The condition for a bubble to exist is:

$$\frac{1-\rho_0\left(1+\frac{\beta}{1+\beta}\right)}{1-\rho_0} > \frac{l}{e}. \tag{B$^{\text{iv}}$}$$

This condition is equivalent to $r^* < 0$, where r^* is the interest rate at the bubbleless steady state. It is verified for $l = 0$ if and only if $\rho_0\left(1 + \beta/\left(1+\beta\right)\right) < 1$.

We illustrate our point by deriving conditions for which investment i^{**} at the bubbly steady state is strictly positive but investment i^* at the bubbleless steady state is zero. Assume that (B$^{\text{iv}}$) is verified. It is possible to find $\tilde{\beta} > 0$ such that $A\left(r^*, r^*\right) = 0$ and $A\left(r^{**}, r^{**}\right) = e$ if and only if

$$\frac{1}{1+r^*}\left(\frac{1-\frac{\rho_0}{1+r^*}}{1-\rho_0}\right)^{\frac{\beta}{1+\beta}} > 1. \tag{9}$$

Since we have assumed that (B$^{\text{iv}}$) holds or equivalently that $r^* < 0$, condition (9) is always verified for β close enough to 0.

Proposition 11 *Suppose that preferences are linear*-log *and that (B^{iv}) holds. If condition (9) is verified, then there exists $\tilde{\beta} > 0$ such that investment at the bubbly steady state is higher than investment at the bubbleless steady state $i^{**} > i^* = 0$. This condition is always verified when β is small enough.*

V.2 Interest-rate sensitive outside liquidity

In this section, we provide some other microfoundations for outside liquidity. In these foundations l decreases with the interest rate. We show how to extend some of our most important results to this more general case.

Securitization. Suppose that consumers have some endowment of goods w—labor income—in their youth. They use that labor income to build a house, which has total value $y_1 j_t$ at period $t+1$, where j_t is the home investment realized in period t. However, suppose only a fraction $y_0 j_t < y_1 j_t$ can be collateralized. Consumers can invest up to $w/\left[1 - y_0/\left(1+r_t\right)\right]$ in housing. Consumers thus create $l_t = l\left(r_t\right)$ additional stores of values for the corporate sector where $l\left(r_t\right) \equiv y_0 w/\left[1 - y_0/\left(1+r_t\right)\right]$. An increase in securitization—in the form of mortgage backed securities for example—can be formalized as an increase in

y_0 towards y_1 and materializes as an increase in l_t. In this microfoundation, the amount of rents $l(r_t)$ is endogenous as it decreases with the interest rate.

Consumers as borrowers. We will also analyze a less extreme case where consumers have concave preferences and hence an elastic borrowing margin. They live for two periods and have preferences given by

$$u(c^y) + \beta u(c^o)$$

where c^y and c^o denote respectively consumption when young and old. They earn income w^y when young and w^o when old. To simplify the analysis, we focus on the case of log preferences where $u(c) = \log(c)$. In this case, consumers of generation t, facing interest rate r_t, consume

$$c_t^y = \frac{1}{1+\beta}\left(w^y + \frac{w^o}{1+r_t}\right) \text{ and } c_t^o = \frac{\beta}{1+\beta}\left((1+r_t)w^y + w^o\right).$$

The supply of rents from the consumers' sector is therefore

$$l_t = l(r_t) \equiv w^o - \frac{\beta}{1+\beta}\left((1+r_t)w^y + w^o\right)$$

where $l(r_t)$ is decreasing with r_t.

Unconstrained firms. Suppose that there also exists a competitive fringe of firms operating a concave production function $f(k_t)$. These firms are owned by consumers who only consume when young. Consumers then sell the firms to investors for a price $f(k_t)/(1+r_t)-k_t$ where k_t is the equilibrium investment level. In equilibrium, it will be the case that $f'(k_t) = 1 + r_t$ so that $k_t = k(r_t)$ where k is decreasing in r_t. This creates a net positive supply of stores of value $l_t = l(r_t) \equiv f(k(r_t))$.

Generalizing the analysis to interest-sensitive outside liquidity. Let us now analyze a framework that encompasses all the examples mentioned above. There is a net supply of stores of value owned by another sector of the economy (which we call the consumers' sector) in the amount $l_t = l(r_t)$, where l is decreasing in r_t.

The supply and demand equations for stores of values are now given by

$$i_t = \frac{l_t + b_t + \rho_0 i_{t-1}}{1 - \frac{\rho_0}{1+r_{t+1}}} \text{ and } i_t = \frac{A(1+r_{t+1}) - l_{t+1} - (1+r_{t+1})b_t}{\rho_0}.$$

The dynamics of this economy are harder to analyze because the endogeneity of l_t imposes to keep track of an additional state variable in addition to i_{t-1} and b_t: the past level of the interest rate r_t. However, the state-steady analysis remains very tractable. In the bubbly steady state

$$i^{**} = \frac{A}{1-\rho_0}, \; b^{**} = A\frac{1-2\rho_0}{1-\rho_0} - l(0), \text{ and } r^{**} = 0.$$

Note that investment in the bubbly steady state is independent of the function $l(r_t)$. There is perfect crowding out between bubbles and rents: $b^{**} + l(0)$ is independent of the function $l(r_t)$ away from zero interest rates.

The condition for the bubbly steady state becomes

$$A\frac{1-2\rho_0}{1-\rho_0} - l(0) > 0. \tag{B$^{\text{v}}$}$$

We continue to denote investment and the interest rate that prevail at the bubbleless steady state by i^* and r^*.

Proposition 12 *Suppose that (B^v) holds. Then the interest rate at the bubbleless steady state is negative: $r^* < 0$.*
(i) If $l(r^*) > 0$, investment in the bubbly steady state i^{**} is higher than investment i^* in the bubbleless steady state.
(ii) If $l(r^*) < 0$ and $-\frac{1-\rho_0}{\rho_0}l(r^*) < \frac{A}{1-\rho_0}$, investment in the bubbly steady state i^{**} is lower than investment i^* in the bubbleless steady state.

Remark 4 *In the model with unconstrained firms, we have $l(r^*) = f(k^*) > 0$. Proposition 12 then shows that $i^{**} > i^*$. However, note that the steady state investment level for unconstrained firms in a bubbly steady state is lower than in the non-bubbly steady state: $k^{**} < k^*$. This is the standard crowding-out effect of bubbles on investment emphasized in* Tirole (1985)*. Therefore, bubbles crowd in the investment of constrained firms ($i^{**} > i^*$) but crowd out the investment of unconstrained firms ($k^{**} < k^*$).*

VI Some speculations on recent events and policy

VI.1 A (very coarse) narrative of the subprime crisis through the lens of the model

In this section, we attempt to use our model to shed some light on the ongoing subprime crisis. The idea is not, far from it, to provide a full account of the many complex and intricate aspects of this extraordinary episode. Rather, we want to use the model to highlight qualitatively some forces that we believe were at work.

A "global savings glut" (Bernanke 2005) was a salient characteristic of the world economy prior to the crisis. For example Caballero, Farhi and Gourinchas (2008a, 2008b) elaborate on the view that countries with high savings and underdeveloped financial markets generated a growing demand for U.S. assets, contributing to low real interest rates and persistent U.S. current account deficits.

A very stylized way to capture the impact of such "global imbalances" is as follows. In the context of the model, imagine that there is a foreign demand for domestic stores of value, which can be represented as a negative contribution $(-l^f)$ to outside liquidity. The net supply of stores of value is then $l - l^f$, which we assume remains positive. The higher l^f, the lower the real interest rate—for a given value of i_{t-1}, or at the bubbleless steady state—the levered the economy, and the more likely are the conditions for the existence of bubbles to be verified.

A consequence is that the macroeconomic environment leading to the crisis was prone to the emergence of (stochastic) bubbles. In retrospect, a possible interpretation is that this bubble was largely located in housing. As the bubble grew, the economy boomed—firms with less collateral and better growth options expanded. Along the way, firms (banks) sacrificed collateral for value, increasing their sensitivity to a potential bubble crash.

At some point, the bubble crashed, perhaps for exogenous reasons, perhaps triggered by the damages to banks' balance sheets from the first subprime defaults or the realization that future similar defaults were to come. The result was a large-scale liquidity dry-up. Poorly collateralized firms (banks) suffered and had to undergo significant downsizing. The real interest rate had to drop in order to eliminate the resulting excess demand for liquidity.

VI.2 Some speculations on policy

In this section, we sketch some implications of the model for macroeconomic policies such as capital adequacy requirements, aggregate liquidity policies, and financial development policies.

In our framework, bubbles occur when aggregate liquidity is scarce and interest rates are low. Under these conditions, they alleviate this aggregate liquidity shortage. One problem, of course, is that bubbles can crash.

However, in our model, the welfare properties of bubbles are ambiguous. Typically, bubbles do not lead to Pareto improvements. For example, the holders of rents l will in general lose from the emergence of a bubble, since it will increase interest rates and lower the price at which they can sell them. Similarly, equilibria with bubble crashes are usually not Pareto dominated by equilibria with no bubble crash.

For this reason, we do not approach public interventions from an optimal policy perspective, and adopt a more modest, positive approach, instead.

Capital adequacy requirements (CARs). A key insight is that the investment multiplier increases when the bubble crashes. Thus, if we follow the "representation hypothesis", according to which prudential regulators act on behalf of small depositors (insurees, future pensioners) and monitor the financial structure of retail institutions[13], the CARs should be higher during asset-price driven booms and smaller during busts.[14] We thus provide some support to the view that regulation should react to asset prices, and be somewhat countercyclical. Countercyclical CARs has long been frowned upon within the regulators' community, but has gained more visibility with the Spanish "dynamic provisioning" policy during the housing bubble and with the recent statements by the Committee of European Banking Supervisors[15], the US Treasury[16] and the G20 countries[17].

Aggregate liquidity policies. Authorities can to some extent regulate the possibility and the size of bubbles by controlling the supply of outside liquidity l—for example by adjusting the level of public debt, or by tightening or relaxing securitization standards (how

[13] See Dewatripont-Tirole (1994) for arguments in favor of this view.

[14] Note that this analysis is inconsistent neither with the "great deleveraging" that followed the recent crash, nor with Adrian and Shin (2008)'s evidence on the pro-cyclicity of leverage in investment banks. Under the cycle-neutral Basel rules, the sizeable losses incurred by banks forces them to much deleveraging.

[15] CEBS (2009).

[16] US Treasury (2009).

[17] In London, April 2009.

much a consumer is allowed to borrow against his house). Consider for example the effects of a lax aggregate liquidity policy (the effects are the exact opposite for a tight aggregate liquidity policy). Authorities can lower the interest rate by making the stores of value scarcer, and therefore more expensive. A temporary decrease in l implies that the bubble must grow at a slower pace (as it grows at the rate of interest). Because the steady state bubble remains the same, the bubble must increase. A permanent decrease in l lowers the interest rate and makes bubbles larger or more likely to emerge.

Our model offers a possible interpretation of the "Rubinomics" claims according to which a reduction in U.S. public debt in the 1990s lowered interest rates and boosted investment: a reduction in public debt made possible the joint emergence of the dot com bubble and of an investment boom through a liquidity effect. This is an (non-mutually exclusive) alternative to a more standard narrative which holds that lower public debt reduced interest rates and boosted investment through a cost of capital channel.

Financial development. "Financial development" in our model is captured in two related ways: governance (ρ_0) and existing stock of traded securities ($\rho_0 i_{t-1}$), thereby reflecting the empirical literature. In practice, a country's quality of governance relates to its legal system's investor protection or to the existence of intermediaries with financial and human capital that can monitor industry and thereby enhance its access to capital. In our theory, governance increases funding liquidity and reduces the demand for stores of value, making bubbles smaller and less likely ceteris paribus. Similarly, a larger inherited stock of traded securities increases the amount of stores of value and therefore also makes bubbles less likely.

VII Conclusion

This paper has made several contributions. First, we have studied the interplay between inside and outside liquidity. Outside liquidity helps firms address the asynchronicity between their access to and need for cash—the liquidity effect—but also compete for savings with productive investment—the competition effect. The liquidity effect dominates when outside liquidity is abundant.

Second, we have shown that bubbles are more likely to exist and can be larger when inside and outside liquidity are scarce. Dynamic efficiency is consistent with the existence of bubbles, provided that the rate of return on internal funds exceeds that on borrowed ones, i.e. provided that capital markets are imperfect.

Third, bubbles are a form of outside liquidity. They are more likely to crowd the financially-constrained corporate sector's investment in (out), the more (less) abundant the outside liquidity. The bursting of a bubble has a negative effect on firms' financial net worth, and further reduces liquidity. Conversely, permanent net worth losses by firms make it harder to sustain a bubble, and so financial disturbances amplify real ones.

Even in a risk neutral, constant-returns-to-scale environment, a stochastic bubble trades at a liquidity discount relative to rents since it pays little or zero in states where internal funds can be levered the most.

Finally, bubbles impact other corporate decisions as well. In particular, firms are predicted to sacrifice value for pledgeability in periods of scarce liquidity, as when a bubble has burst.

Our analysis brings support to the idea that pricking bubbles may be hazardous. But it also suggests when this will be particularly so, namely when outside liquidity is abundant.

Our imperfect-capital-markets analysis implies a divergence between the rates of returns on internal and borrowed funds, and therefore that rates of return on borrowed funds below the rate of growth of the economy are consistent with Abel et al. (1989)'s finding that the productive sector may disgorge at least as much as it invests. The outflow measure in Abel et al. aggregates a variety of firms with wildly different governance structures and therefore pledgeable income (publicly traded firms, family and private equity, startups). Using the theoretical analysis to build a modified version of Abel et al.'s clever test of potential existence of bubbles would be of much interest as well.

This paper points to a number of promising research avenues. Three themes seem to us particularly interesting. First, we have only briefly touched in Section IV.3 on the question of who should hold the bubble. There we saw that a stochastic bubble is held in equilibrium by the firms with the lowest pledgeable income. It would be interesting to explore this question further. For example, one could allow less levered agents such as consumers to hold the bubble. Incorporating a meaningful trade-off between size (leverage) and liquidity as in Holmström and Tirole (1998) would allow a better analysis of the role of lax capital adequacy requirements. Second, our model is too stylized to analyze the specific role of monetary policy, or the interaction between prudential regulation and interest rate policy; developing the model along this dimension would be interesting. Third, it would be interesting to introduce the possibility of bailouts. In Farhi and Tirole (2009), we investigate this question and point out that systemic bailouts (for example through lax interest rate policy in response to a crisis) result in strategic complementarities in liquidity and portfolio choices. The financial institutions' incentive to correlate their positions on the overvalued assets would have interesting implications for macro-dynamics and public debt.

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Appendix

A.1 Proofs for Section II

Proof of Lemmas 1 and 2. We can solve for investment i_t as a function of i_{t-1} and b_t:

$$i_t = \frac{A + \rho_0 i_{t-1} + [1 - \frac{1}{\rho_0}]l + \sqrt{\left\{A + \rho_0 i_{t-1} + [1 - \frac{1}{\rho_0}]l\right\}^2 + 4\frac{l}{\rho_0}[b_t + l + \rho_0 i_{t-1}]}}{2}$$

From this expression, it is clear that i_t is increasing in i_{t-1} and b_t if $l > 0$ and decreases with b_t if $l < 0$.

Proof of Proposition 1. Part (i) can be proved as follows. Let us now provide a condition for (5) in terms of the primitives of the model. The steady state is given by

$$1 + r^* = \frac{A\rho_0 + l + \sqrt{[A\rho_0 + l]^2 - 4Al(1 - \rho_0)\rho_0}}{2A(1 - \rho_0)}$$

and

$$i^* = \frac{A(1 + r^*) - l}{\rho_0}.$$

Hence (5) is verified if and only if

$$\frac{A\rho_0 + l + \sqrt{[A\rho_0 + l]^2 - 4Al(1 - \rho_0)\rho_0}}{4A\rho_0(1 - \rho_0)} \geq 1.$$

A.2 Proofs for Section III

Dynamics in section III.2. The $i_t = i_{t-1}$ schedule is given by

$$b_t^i(i_{t-1}) = i_{t-1}^2 \frac{\rho_0}{l}(1 - \rho_0) - \frac{\rho_0 i_{t-1}}{l}\left[A + \left(2 - \frac{1}{\rho_0}\right)l\right] - l.$$

The $b_t = b_{t+1}$ schedule is given by

$$b_t^b = -\rho_0^2 i_{t-1} + (1 - \rho_0)A - l.$$

We have

$$b_t^i(0) = -l$$
$$b_t^b(0) = (1-\rho_0)\,A - l > 0 \text{ from the bubble existence condition.}$$

$$\frac{db_t^i}{di_{t-1}} = 2i_{t-1}\frac{\rho_0}{l}\,(1-\rho_0) - \frac{\rho_0}{l}\left[A + \left(2-\frac{1}{\rho_0}\right)l\right]$$

So

$$\frac{db_t^i}{di_{t-1}}|_{i_{t-1}=0} = -\frac{\rho_0}{l}\left[A + \left(2-\frac{1}{\rho_0}\right)l\right],$$

which can be positive or negative.

Proof of Proposition 4. Define $\tilde{l} \equiv -l$. The $\tilde{b}^i$ and $\tilde{b}^b$ schedules are now

$$b_t = -\,(1-\rho_0)\,\rho_0\frac{1}{\tilde{l}}i_{t-1}^2 + \frac{\rho_0 i_{t-1}}{\tilde{l}}\left[A + (1-2\rho_0)\,\frac{\tilde{l}}{\rho_0}\right] + \tilde{l}$$

and

$$b_t = -\rho_0^2 i_{t-1} + (1-\rho_0)\,A + \tilde{l}$$

Investment is given by the following equation

$$i_t = \frac{\frac{1}{\rho_0}\,(1-\rho_0)\,\tilde{l} + \rho_0 i_{t-1} + A + \sqrt{[\frac{1}{\rho_0}\,(1-\rho_0)\,\tilde{l} + \rho_0 i_{t-1} + A]^2 - 4\frac{1}{\rho_0}\tilde{l}\left[b_t + \rho_0 i_{t-1} - \tilde{l}\right]}}{2}$$

Note that we have

$$\tilde{b}^b(i_{t-1}) \geq \tilde{b}^i(i_{t-1})$$

if and only if $i_{t-1} \notin [i_2, i_1]$ where

$$i_1 = \frac{A}{1-\rho_0}$$
$$i_2 = \frac{1-\rho_0}{\rho_0}\tilde{l}$$

and (6) guarantees that $i_2 < i_1$. A phase diagram analysis shows that the bubbly steady state is saddle path stable, and the results in the proposition follow.

A.3 Bifurcation analysis: the relationship with Woodford (1990).

In this appendix, we clarify the relationship of our model with that of Woodford (1990). We show that the mechanism by which bubbles affect investment is quite different in our model and in Woodford's model.

Our model is closest to Woodford's in a particular case of ours, when $\rho_0 = 0$ and $l = 0$. We by no means imply that this particular case is exactly the model he analyzed. There are important differences. Notably in his model, preferences are not linear, production functions are concave, and agents are infinitely lived. In particular, the bifurcations that we emphasize below in our version of Woodford's model would not occur in his model. Nonetheless, these results are useful to emphasize how different the mechanisms by which bubbles affect investment are in our model and in his.

When $\rho_0 = 0$ and $l = 0$, in the absence of a bubble, there are no stores of value. As a consequence, in the bubbleless steady state (there is no transitional dynamics), investment i^{W*} is equal to 0: (i) the returns to investment are entirely non-pledgeable so that no outside funds can be raised in the intermediate period; (ii) there is no way for the entrepreneurs to transfer funds from the early period to the intermediate period. The implicit interest rate is $1 + r^{W*} = 0$. By constrast, in the bubbly steady state, $i^{W**} = A$ and the interest rate is $1 + r^{W**} = 1$. The bubble size b^{W**} is commensurate to the amount of wealth that entrepreneurs have in the early period of their life: $b^{W**} = A$. The bubble provides stores of value and allows the entrepreneurs to transfer their funds to the intermediate period of their life where they can invest. The bubble simply allows different generations to realize some trades that were impossible in its absence: they now have an asset that they can pass on to each other.

Consider now our model when $\rho_0 > 0$. We maintain the assumption that $l = 0$. It turns out that in this case, there are two bubbleless steady states. The first one features zero investment $i_1^* = 0$ and a zero gross interest interest rate $1 + r_1^* = 0$, exactly as in the bubbleless steady state of our version of the Woodford model. The second one features positive investment $i_2^* = \frac{A}{1-\rho_0}$ and a positive gross interest rate $1 + r_2^* = \frac{\rho_0}{1-\rho_0}$. The first steady state is unstable and the second one is stable. This can be understood as follows: in the first steady state, the price of stores of value is essentially infinite so that any perturbation that introduces some positive amount of investment for generation t will generate a lot of stores of value for generation $t + 1$. This in turn will allow generation $t + 1$ to invest and will put the economy on a path converging to the second steady state. If $1 + r_2^* < 1$, there is also a bubbly steady state where investment is given by $i^{**} = \frac{A}{1-\rho_0}$ and $1 + r^{**} = 1$. Note that we have $i^{**} = i_2^* > i_1^*$. Hence, if we ignore the first unstable bubbleless steady state, bubbles have no effect on investment: in contrast with both Woodford's model and our version of it, there is neither crowding in nor crowding out.

Recall that our intuition is that the effect of bubbles on investment can be understood as follows: bubbles raise the interest rate and therefore crowd investment in if outside liquidity is positive. In considering the case $l = 0$, we have essentially gotten rid of the consumers sector. That in this context, bubbles have no effect on investment is therefore consistent with our intuition.[18]

Note also the contrast with the case that corresponds to our version of Woodford's model ($\rho_0 = 0$ and $l = 0$). There, the second bubbleless steady state disappears and the first bubbleless steady state becomes stable. To use the language of dynamic systems, there is a bifurcation at $\rho_0 = 0$. As we take ρ_0 to 0, the first bubbleless steady state remains unchanged and the interest rate $1 + r_2^*$ of the second bubbleless steady state converges to $1 + r_1^* = 0$. But the corresponding investment level i_2^* converges to $A > i_1^* = 0$. This discountinuity at $\rho_0 = 0$ is behind the bifurcation result. Both in Woodford's model and in our version of it, bubbles systematically crowd investment in, even when $l = 0$. When $\rho_0 > 0$ this direct effect disappears and bubbles have no effect on investment when $l = 0$.

When $l > 0$, the first unstable bubbleless steady state disappears but the second stable bubbleless steady state is contiunous at $l = 0$. This gives yet another justification for the emphasis on the second bubbleless steady state when $l = 0$. In light of this discussion, it should become clear that the mechanism by which bubbles affect investment is quite different in our model and in Woodford's model. In Woodford's model, bubbles crowd investment in even when $l = 0$. This effect is absent in our model. In our model, bubbles affect investment by raising interest rates and transfering sellers of stores of values to purchasers of stores of value. Only when outside liquidity is positive do bubbles crowd investment in.

A.4 Proofs for Section V.1

Let $\alpha = \beta / (1 + \beta)$. Investment in the bubbly steady state is given by

$$i^{**} = \frac{\alpha A}{1 - \rho_0}$$

and investment in the bubbleless steady state is given by

$$i^* = \frac{\rho_0 A - l(1 - 2\alpha\rho_0) + \sqrt{[\rho_0 A - l(1 - 2\alpha\rho_0)]^2 + 4\alpha l^2 \rho_0 (1 - \alpha\rho_0)}}{2\rho_0 (1 - \alpha\rho_0)}$$

[18]For crowding in to occur in Woodford's model, in addition, there needs to be sufficient intertemporal substituability in consumption.

and the interest rate is

$$1+r^{*}=\frac{\rho_0 A+l+\sqrt{\left[\rho_0 A-l\left(1-2\alpha\rho_0\right)\right]^2+4\alpha l^2\rho_0\left(1-\alpha\rho_0\right)}}{2A\left(1-\alpha\rho_0\right)}.$$

It can be verified that (B"') is equivalent to $r^{*}<0$.

After a few manipulations, it can be verified that $i^{**}>i^{*}$ if and only if

$$l^2-\frac{1-2\alpha\rho_0}{1-\rho_0}Al-\frac{\rho_0}{1-\rho_0}\left[\alpha\frac{1-\alpha\rho_0}{1-\rho_0}-1\right]A^2<0.$$

Together with condition (B"'), this is equivalent to $l\in[l_0,l_1]$ where

$$\begin{aligned} l_0 &= A\frac{\frac{1-2\alpha\rho_0}{1-\rho_0}-\sqrt{\left(\frac{1-2\alpha\rho_0}{1-\rho_0}\right)^2-4\rho_0\left(1-\alpha\right)\frac{1-\rho_0(1+\alpha)}{1-\rho_0}}}{2}\\ l_1 &= A\frac{\frac{1-2\alpha\rho_0}{1-\rho_0}+\sqrt{\left(\frac{1-2\alpha\rho_0}{1-\rho_0}\right)^2-4\rho_0\left(1-\alpha\right)\frac{1-\rho_0(1+\alpha)}{1-\rho_0}}}{2}. \end{aligned}$$

It can be shown that as long as (B"') holds, $l_0>0$ and $l<l_1$.

A.5 Proofs for Section V.2

Proof of Proposition 12. The proof proceeds as follows. Consider the economy where rents supplied by consumers are fixed and equal to $\tilde{l}=l(r^{*})$. The corresponding variables for this economy are denoted with a tilde. We can use the phase diagram derived in Sections III.2 and III.3 to analyze this economy. At the intersection of $\tilde{b}_t^i$ and $\tilde{b}_t^b$, investment is equal to $\tilde{\imath}^{**}=\frac{A}{1-\rho_0}$, the bubble is equal to $\tilde{b}^{**}=A-\frac{1-2\rho_0}{1-\rho_0}-\tilde{l}$ (possibly negative) and the interest rate is equal to 0. Note that this economy is such that, $i^{*}=\tilde{\imath}^{*}$, $r^{*}=\tilde{r}^{*}$ and $i^{**}=\tilde{\imath}^{**}$. Hence $i^{**}>i^{*}$ if and only if $\tilde{\imath}^{**}>\tilde{\imath}^{*}$.

Let us first prove by contradiction that (B$^{\text{v}}$) implies that $r^{*}<0$. Assume that $r^{*}\geq 0$. Then $\tilde{l}<l\left(n\right)$ so that $\tilde{b}^{**}>0$. But this is possible only if $\tilde{r}^{*}<0$ i.e. $r^{*}<0$. This is a contradiction. Hence $r^{*}<0$.

Let us now move on to prove (i) and (ii) in Proposition 12. Suppose first that $\tilde{l}>0$. We know from the phase diagram analysis in Section III.2 and the fact that $r^{*}<0$ that $\tilde{\imath}^{**}>\tilde{\imath}^{*}$. We therefore conclude that $i^{**}>i^{*}$. The proof of (ii) proceeds exactly along the same lines.