Computable Markov-Perfect Industry Dynamics – Online Appendix –

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In the main paper we establish that a computationally tractable symmetric equilibrium in pure strategies always exists. Given that an equilibrium exists, an important question is whether or not it is unique. To our knowledge, all applications of Ericson & Pakes's (1995) framework have found a single equilibrium. Pakes (2000) summarizes his experience as follows:

> ... we have experimented quite a bit with the core version of the algorithm, and we never found two sets of equilibrium policies for a given set of primitives (we frequently run the algorithm several times using different initial conditions or different orderings of points looking for other equilibria that might exist). We should emphasize here that the core version, and indeed most other versions that have been used, all use quite simple functional forms for the primitives of the problem, and multiplicity of equilibrium may well be more likely when more complicated functional forms are used. Of course, most applied work suffices with quite simple functional forms. (pp. 18–19)

Over the past few years, the industrial organization literature has increasingly acknowledged the potential for multiplicity in Ericson & Pakes's (1995) and related frameworks. Pesendorfer & Schmidt-Dengler (2008), for example, provide an entry model with one symmetric and four asymmetric equilibria.

Below we make the stronger point that multiplicity may be an issue in Ericson & Pakes's (1995) framework even if—as is always done in applications—symmetric restrictions are imposed. To this end, we provide three examples of multiple symmetric equilibria. Our examples in turn highlight firms' investment decisions, their entry/exit decisions, and product market competition as possible sources of multiplicity. Throughout we use the "quite simple

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parameter	M	$\bar{q}_1$	$\bar{q}_2$	...	$\bar{q}_{10}$	v	m	α	δ	β
value	10	0	5	...	45	1	40	2.375	0.03	$\frac{20}{21}$

Table 1: Parameter values.

functional forms" alluded to by Pakes (2000). Using more complicated functional forms in models of learning-by-doing and capacity accumulation with lumpy investment, Besanko, Doraszelski, Kryukov & Satterthwaite (2007) and Besanko, Doraszelski, Lu & Satterthwaite (2008) find much larger numbers of symmetric equilibria for some parameterizations and discuss the economics underlying this multiplicity.

Example: Investment decisions. We build on the game of capacity accumulation from section 5. There are $N = 2$ firms with $M \geq 3$ "active" states. In state ω_n firm n's capacity is $\bar{q}_{\omega_n}$. Transitions are limited to immediately adjacent states and are independent across firms. The transition probabilities at an interior state $\omega \in \{2, \ldots, M-1\}^2$ are given in Table 3 in the main paper.[1] Because the transition function $P(\cdot)$ is UIC admissible, it is guaranteed that multiplicity is not due to a violation of assumption 3.

Products are undifferentiated and firms compete in prices subject to capacity constraints. There are m identical consumers with unit demand and reservation price v. The equilibrium of this Bertrand-Edgeworth product market game is characterized in Chapter 2 of Ghemawat (1997). Let $\pi(\omega_1, \omega_2)$ denote firm 1's current profit in state $\omega = (\omega_1, \omega_2)$. Symmetry implies that firm 2's current profit in state ω is $\pi(\omega_2, \omega_1)$. Table 1 gives the parameter values.

Figure 1 illustrates the value and policy functions of two symmetric equilibria. In both investment activity is greatest in states on or near the diagonal of the state space. That is, firms with equal or similar capacities are engaged in a preemption race to become the industry leader. The difference in investment activity is greatest in state $(5,5)$ where both firms invest 1.90 in the first equilibrium compared to 1.03 in the second one. Investment activity also differs considerably in states $(1,6)$ and $(6,1)$: in the first (second) equilibrium the smaller firm invests 2.24 (3.92) and the larger firm invests 1.57 (1.46). That is, in the second equilibrium, the laggard invests heavily in a bid to catch up with the leader, and the leader to some extent accommodates the laggard. Note that multiplicity in this example rests on the dynamic nature of the game. Because product market competition takes place before investment decisions are carried out, a firm has no incentive to invest if $\beta = 0$. Hence, multiple equilibria cannot possibly arise in the static version of the game.

The computations were performed using a Matlab 5.3 implementation of the Pakes & McGuire (1994) algorithm. The first equilibrium was computed using a Gauss-Jacobi

[1] Edge states must be treated specially. If $\omega_n = 1$, then the probability of moving up to state $\omega_n' = 2$ (remaining in state $\omega_n' = 1$) is $(1-\delta)p_n$ (δp_n); if $\omega_n = M$, then the probability of dropping down to state $\omega_n' = M-1$ (remaining in state $\omega_n' = M$) is $\delta(1-p_n)$ $(1-\delta(1-p_n))$.

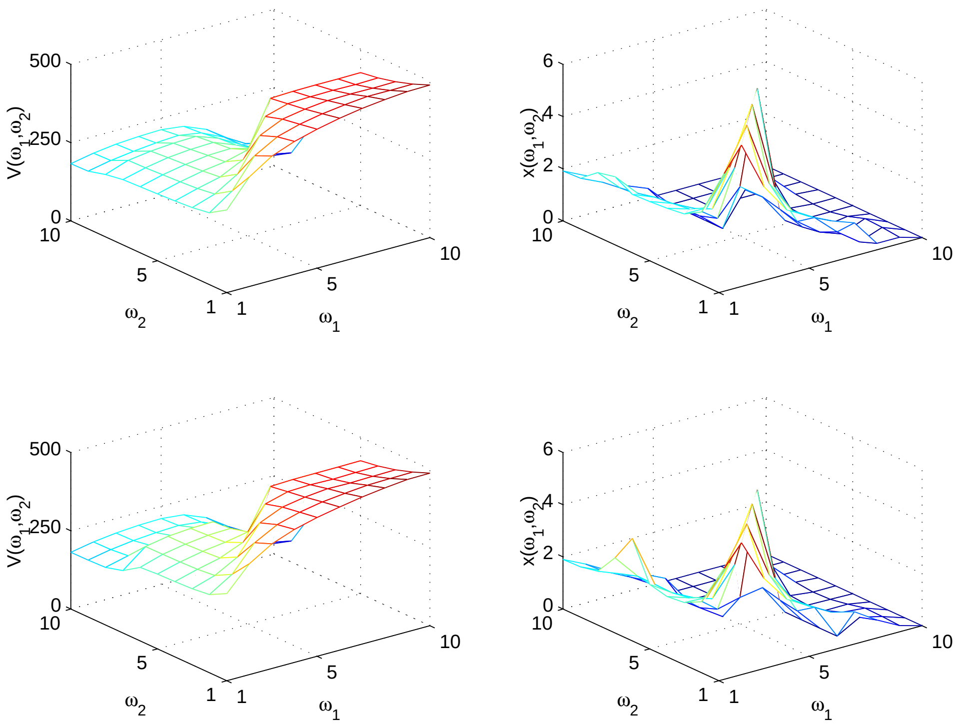

Figure 1: Two equilibria.

scheme to update the value and policy functions, the second using a Gauss-Seidel scheme (see e.g. Judd 1998). This is worth pointing out because many applications of Ericson & Pakes's (1995) framework have searched for multiple equilibria by selecting a single algorithm and varying the starting values. This approach, however, failed to identify the different equilibria in our example, and its use may thus lead one to falsely conclude that multiplicity is not an issue.

Example: Entry/exit decisions. In the above example nonuniqueness results solely from firms' investment decisions in a model without entry and exit. We next show that nonuniqueness may also result from firms' exit decisions. We slightly extend our example with random scrap values/setup costs from section 3. In particular, suppose that each firm can now be in one of two "active" states (i.e., $M = 2$) and that the current profit in states $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$ is the same. Suppose finally that a firm cannot transit between its active states (in the above example this corresponds to a situation with ineffective investment ($\alpha = 0$) and zero depreciation ($\delta = 0$)).

Again there are two symmetric equilibria. One equilibrium has both firms play the cutoff exit strategies from section 3 in states $(1,1)$, $(1,2)$, $(2,1)$, and $(2,2)$ and remain in the industry with high probability ($\xi(1,1) = \xi(1,2) = \xi(2,1) = \xi(2,2) = 0.8549$). The top panels of Table 2 summarize this equilibrium for $\epsilon = 1$. Another equilibrium is illustrated in the bottom panels of Table 2. In states $(1,1)$ and $(2,2)$ both firms continue

$V(\omega_1,\omega_2)$	$\omega_2=1$	$\omega_2=2$	$\omega_2=3$
$\omega_1=1$	15.7309	15.7309	21
$\omega_1=2$	15.7309	15.7309	21

$\xi(\omega_1,\omega_2)$	$\omega_2=1$	$\omega_2=2$	$\omega_2=3$
$\omega_1=1$	0.8549	0.8549	1
$\omega_1=2$	0.8549	0.8549	1

$V(\omega_1,\omega_2)$	$\omega_2=1$	$\omega_2=2$	$\omega_2=3$
$\omega_1=1$	15.7309	15.0238	21
$\omega_1=2$	19.8279	15.7309	21

$\xi(\omega_1,\omega_2)$	$\omega_2=1$	$\omega_2=2$	$\omega_2=3$
$\omega_1=1$	0.8549	0.1542	1
$\omega_1=2$	1	0.8549	1

Table 2: Two more equilibria.

to play the cutoff exit strategies from section 3 and remain in the industry with high probability ($\xi(1,1)=\xi(2,2)=0.8549$). In state $(1,2)$ firm 1 stays with low probability ($\xi(1,2)=0.1542$) and firm 2 stays for sure ($\xi(2,1)=1$) whereas in state $(2,1)$ firm 1 stays for sure ($\xi(2,1)=1$) and firm 2 stays with low probability ($\xi(1,2)=0.1542$). Note that this equilibrium is symmetric by construction.

The two equilibria differ starkly from each other: In the first equilibrium, a duopolistic industry may over time turn into either a monopolistic or an empty industry. In the second equilibrium, if the industry starts in states $(1,2)$ or $(2,1)$, then it always ends up as a monopoly.

Pakes & McGuire (1994) have previously conjectured that nonuniqueness may result from firms' exit decisions, especially in "situations in which one firm out of two (or several firms out of many) must exit, but either exiting would generate an equilibrium" (p. 570). To resolve this issue they suggest to assume that firms make exit decisions sequentially in a given order. The drawback is that ordering firms renders symmetry inapplicable (since we must now distinguish between firms with the same state according to their order), thereby increasing the size of the state space and the computational burden. Further, Pakes & McGuire's (1994) suggestion may fail to rule out all but one equilibrium. To see this, consider our example with deterministic scrap values/setup costs from section 3. Suppose that there is just one "active" state (i.e., $M=1$) and that, in each period, firm 1 makes its exit decision before firm 2. Hence, firm 1 takes into account how its exit decision affects that of firm 2. It is easy to see that there are two equilibria. One equilibrium has firm 1 stay in the industry in state $(1,1)$ and firm 2 exit whereas another equilibrium has firm 1 exit and firm 2 stay. Intuitively, there is little reason to believe that sequential exit decisions alleviate the multiplicity problem unless this assumption enables one to apply a backwards induction argument. But this is not the case in a model with an infinite horizon such as the Ericson & Pakes (1995) model.

Example: Product market competition. In the main paper treat the profit function $\pi_n(\cdot)$ as a primitive. Instead we could have gone back to demand and cost fundamentals and explicitly modeled competition in the product market. To the extent that this game admits more than one equilibrium $\pi_n(\cdot)$ fails to be determined uniquely, thereby making product market competition yet another source of multiplicity.

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