

Lumpy Capacity Investment and Disinvestment Dynamics

– Online Appendix –

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1 Model

State-to-State Transitions Firm 1 cannot further decrease (increase) its capacity when its previously installed capacity is already at the minimum (maximum). Thus, if $i = 0$, we set

$$\Pr(i'|0, e_1, w_1) = \begin{cases} (1 - \delta)e_1 & \text{if } i' = 1, \\ \delta e_1 + 1 - e_1 & \text{if } i' = 0; \end{cases}$$

if $i = 1$, we set

$$\Pr(i'|1, e_1, w_1) = \begin{cases} (1 - \delta)e_1 & \text{if } i' = 2, \\ \delta e_1 + (1 - \delta)(1 - e_1 - w_1) & \text{if } i' = 1, \\ \delta(1 - e_1 - w_1) + w_1 & \text{if } i' = 0; \end{cases}$$

and, if $i = M - 1$, we set

$$\Pr(i'|M - 1, e_1, w_1) = \begin{cases} e_1 + (1 - \delta)(1 - e_1 - w_1) & \text{if } i' = M - 1, \\ \delta(1 - e_1 - w_1) + (1 - \delta)w_1 & \text{if } i' = M - 2, \\ \delta w_1 & \text{if } i' = M - 3. \end{cases}$$

Bellman Equation The private shock θ_1 is a random variable drawn from a *Beta*(3, 3) distribution. Its distribution function is

$$F(\theta_1) = \begin{cases} 0 & \text{if } \theta_1 < -1, \\ \frac{1}{2} + \frac{15}{16}\theta_1 - \frac{5}{8}\theta_1^3 + \frac{3}{16}\theta_1^5 & \text{if } -1 \leq \theta_1 < 1, \\ 1 & \text{if } \theta_1 \geq 1. \end{cases}$$

The closed-form expressions for the integrals in the Bellman equation thus are

$$\begin{aligned} \int \theta_1 \mathbf{1}[\theta_1 \leq \underline{\theta}_1] dF(\theta_1) &= \begin{cases} -\frac{5}{32} + \frac{15}{32}\underline{\theta}_1^2 - \frac{15}{32}\underline{\theta}_1^4 + \frac{5}{32}\underline{\theta}_1^6 & \text{if } -1 \leq \underline{\theta}_1 < 1, \\ 0 & \text{otherwise,} \end{cases} \\ \int \theta_1 \mathbf{1}[\theta_1 \geq \bar{\theta}_1] dF(\theta_1) &= \begin{cases} \frac{5}{32} - \frac{15}{32}\bar{\theta}_1^2 + \frac{15}{32}\bar{\theta}_1^4 - \frac{5}{32}\bar{\theta}_1^6 & \text{if } -1 \leq \bar{\theta}_1 < 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

2 Homotopy Method

Computing a MPE of a dynamic stochastic game amounts to solving a large system of equations

$$H(x, \lambda) = 0,$$

where x are the endogenous variables, i.e., values and policies, and our notation makes explicit that the equilibrium conditions H depend on the model parameters λ . In our setting, the endogenous variables are

$$x = (p, \underline{\theta}, \bar{\theta}, V),$$

where $p = (p(i, j))$, $\underline{\theta} = (\underline{\theta}(i, j))$, $\bar{\theta} = (\bar{\theta}(i, j))$, and $V = (V(i, j))$ are $M \times M$ matrices.¹

The homotopy method allows us to explore the equilibrium correspondence

$$H^{-1} = \{(x, \lambda) | H(x, \lambda) = 0\}.$$

in a systematic fashion. The homotopy method is a type of path-following method. Starting from a single MPE that has already been computed for a given parameterization of the model, the homotopy algorithm traces out an entire path of MPE by varying one or more parameters in λ .

Whenever we can find a path in H^{-1} and multiple MPE are the result of this path folding back on itself, then the homotopy method is guaranteed to identify them. We note at the outset that it is not assured that any given path computes all possible MPE for a given parameterization of the model. We refer the reader to Besanko, Doraszelski, Kryukov & Satterthwaite (2010) for further discussion and to Borkovsky, Doraszelski & Kryukov (2008) for a step-by-step guide to solving dynamic stochastic games using the homotopy method.

In the remainder of this section we describe the $4M^2$ equilibrium conditions $H = (H^p, H^\theta, H^{\bar{\theta}}, H^V)$ that determines the $4M^2$ endogenous variables $x = (p, \underline{\theta}, \bar{\theta}, V)$. The first set of equilibrium conditions H^p are the first-order conditions that determine the Nash equilibrium in prices in the product market. We proceed on a case-by-case basis.

- Case 1: $i \geq 1, j \geq 1$.

$$H^p(i, j) \equiv q(i, j) - \frac{b}{1 - \gamma^2}(p(i, j) - c(i, j)),$$

¹Recall that we restrict attention to symmetric MPE.

where

$$\begin{aligned} q(i, j) &= \frac{1}{1 - \gamma^2} (a(1 - \gamma) - bp(i, j) + \gamma bp(j, i)), \\ c(i, j) &= \left(\frac{q(i, j)}{\bar{q}_i} \right)^\nu. \end{aligned}$$

- Case 2: $i \geq 1, j = 0$.

$$H^p(i, j) \equiv q(i, j) - b(p(i, j) - c(i, j)),$$

where

$$\begin{aligned} q(i, j) &= a - bp(i, j), \\ c(i, j) &= \left(\frac{q(i, j)}{\bar{q}_i} \right)^\nu. \end{aligned}$$

- Case 3: $i = 0, j \geq 1$.

$$H^p(i, j) \equiv p(i, j) - \frac{1}{b} (a - \gamma q(j, i)),$$

where

$$q(j, i) = a - bp(j, i).$$

- Case 4: $i = 0, j = 0$.

$$H^p(i, j) \equiv p(i, j) - \frac{a}{b}.$$

The second and third set of equilibrium conditions H^θ and $H^{\bar{\theta}}$ are the cutoffs for investment/disinvestment.

$$\begin{aligned} H^\theta(i, j) &\equiv \underline{\theta} - \min \left\{ \frac{-\phi_e - \phi_w + \beta(W^+(i, j) - W^-(i, j))}{\epsilon_e + \epsilon_w}, \frac{-\phi_e + \beta(W^+(i, j) - W^0(i, j))}{\epsilon_e} \right\}, \\ H^{\bar{\theta}}(i, j) &\equiv \bar{\theta} - \max \left\{ \frac{-\phi_w + \beta(W^0(i, j) - W^-(i, j))}{\epsilon_w}, \frac{-\phi_e - \phi_w + \beta(W^+(i, j) - W^-(i, j))}{\epsilon_e + \epsilon_w} \right\}. \end{aligned}$$

The min/max operators may lead to discontinuity in the first derivatives of the above equations. In general, the homotopy method requires the equations to be smooth in the sense of continuous differentiability. However, according to Borkovsky et al. (2008), because the homotopy algorithm takes discrete steps along the path, it may succeed in tracing out a path even if this smoothness requirement is violated. Our computational experiments suggest that this is indeed the case.

The fourth set of equilibrium conditions H^V are the Bellman equations.

$$\begin{aligned}
H^V(i, j) &\equiv -V(i, j) + \pi(i, j) - e(i, j)\phi_e - \epsilon_e \int \theta \mathbf{1}\{\theta \leq \underline{\theta}(i, j)\} dF(\theta) + e(i, j)\beta W^+(i, j) \\
&\quad + (1 - e(i, j) - w(i, j))\beta W^0(i, j) \\
&\quad + w(i, j)\phi_w + \epsilon_w \int \theta \mathbf{1}\{\theta \geq \bar{\theta}(i, j)\} dF(\theta) + w(i, j)\beta W^-(i, j),
\end{aligned}$$

where

$$\begin{aligned}
\pi(i, j) &= p(i, j)q(i, j) - C(i, j), \\
C(i, j) &= \frac{1}{1 + \nu} \left(\frac{q(i, j)}{\bar{q}_i} \right)^\nu q(i, j), \\
e(i, j) &= F(\underline{\theta}(i, j)), \\
w(i, j) &= 1 - F(\bar{\theta}(i, j))
\end{aligned}$$

and $q(i, j)$ is defined on a case-by-case basis as above.

3 Capacity Cartel

The cartel solves a single-agent dynamic programming problem. Let $V(i, j, \theta_1, \theta_2)$ denote the expected net present value of the cartel's cash flow if the state is (i, j) and the cartel has drawn θ_1 and θ_2 for the cost of capacity addition/withdrawal in plants 1 and 2, respectively. In what follows, we first characterize the value function $V(i, j, \theta_1, \theta_2)$ under the presumption that the cartel behaves optimally. In a second step, we derive the optimal investment/disinvestment decisions of the cartel, $e_1(i, j, \theta_1, \theta_2)$, $w_1(i, j, \theta_1, \theta_2)$, $e_2(i, j, \theta_1, \theta_2)$, $w_2(i, j, \theta_1, \theta_2)$.

Because the investment/disinvestment decisions for a given plant are mutually exclusive, we must have $e_1(i, j, \theta_1, \theta_2)w_1(i, j, \theta_1, \theta_2) = 0$ and $e_2(i, j, \theta_1, \theta_2)w_2(i, j, \theta_1, \theta_2) = 0$. This leaves us $3 \times 3 = 9$ mutually exclusive decision pairs. We denote their corresponding probabilities by $\sigma^{a_1 a_2}$, where $a_1, a_2 \in \{+, 0, -\}$ denote the investment/disinvestment decisions for the two plants and $+$ indicates investment, 0 indicates inaction, and $-$ indicates disinvestment. For example,

$$\sigma^{++}(i, j) = \iint e_1(i, j, \theta_1, \theta_2)e_2(i, j, \theta_1, \theta_2)dF(\theta_1)dF(\theta_2).$$

The expressions for the other 8 probabilities are omitted for brevity.

The value function $V(i, j, \theta_1, \theta_2)$ is defined recursively as the solution to the Bellman equation

$$\begin{aligned}
V_1(i, j, \theta_1, \theta_2) = & \max_{\substack{e_1, e_2, w_1, w_2 \in \{0,1\}, \\ e_1 w_1 = 0, e_2 w_2 = 0,}} \Pi(i, j) + e_1 e_2 \{-\eta_{e,1} - \eta_{e,2} + \beta W^{++}(i, j)\} \\
& + (1 - e_1 - w_1) e_2 \{-\eta_{e,2} + \beta W^{0+}(i, j)\} \\
& + w_1 e_2 \{\eta_{w,1} + \beta W^{-+}(i, j)\} \\
& + e_1 (1 - e_2 - w_2) \{-\eta_{e,1} + \beta W^{+0}(i, j)\} \\
& + (1 - e_1 - w_1) (1 - e_2 - w_2) \beta W^{00}(i, j) \\
& + w_1 (1 - e_2 - w_2) \{\eta_{w,1} + \beta W^{-0}(i, j)\} \\
& + e_1 w_2 \{-\eta_{e,1} + \eta_{w,2} + \beta W^{+-}(i, j)\} \\
& + (1 - e_1 - w_1) w_2 \{\eta_{w,2} + \beta W^{0-}(i, j)\} \\
& + w_1 w_2 \{\eta_{w,1} + \eta_{w,2} + \beta W^{--}(i, j)\}, \tag{1}
\end{aligned}$$

where the continuation values $W^{a_1 a_2}(i, j)$ are

$$W^{a_1 a_2}(i, j) = \sum_{i'=0}^{M-1} \sum_{j'=0}^{M-1} V(i', j') \Pr(i', j' | i, j, \sigma^{a_1 a_2}(i, j)).$$

The state-to-state transitions conditional on $\sigma^{++}(i, j)$ are defined as follows: If $i, j \in \{0, \dots, M-2\}$, then

$$\Pr(i', j' | i, j, \sigma^{++}(i, j)) = \begin{cases} (1 - \delta)^2 \sigma^{++}(i, j) & \text{if } i' = i + 1, j' = j + 1, \\ \delta(1 - \delta) \sigma^{++}(i, j) & \text{if } i' = i + 1, j' = j, \\ \delta(1 - \delta) \sigma^{++}(i, j) & \text{if } i' = i, j' = j + 1, \\ \delta^2 \sigma^{++}(i, j) & \text{if } i' = i, j' = j. \end{cases}$$

The appropriate modifications at the boundary of the state space are obvious. The expressions for the state-to-state transition conditional on the other 8 probabilities are omitted for brevity.

Turning to the optimal investment/disinvestment decisions, we have $e_1(i, j, \theta_1, \theta_2) = 1$ and $e_2(i, j, \theta_1, \theta_2) = 1$ if and only if

$$\begin{aligned}
-\eta_{e,1} - \eta_{e,2} + \beta W^{++}(i, j) \geq & \max \left\{ -\eta_{e,2} + \beta W^{0+}(i, j), \eta_{w,1} + \beta W^{-+}(i, j), \right. \\
& -\eta_{e,1} + \beta W^{+0}(i, j), \beta W^{00}(i, j), \eta_{w,1} + \beta W^{-0}(i, j), \\
& \left. -\eta_{e,1} + \eta_{w,2} + \beta W^{+-}(i, j), \eta_{w,2} + \beta W^{0-}(i, j), \eta_{w,1} + \eta_{w,2} + \beta W^{--}(i, j) \right\}.
\end{aligned}$$

Let Ω^{++} denote the two-dimensional region of θ_1 and θ_2 that satisfies the above condition:

$$\begin{aligned}
\Omega^{++} &= \left\{ (\theta_1, \theta_2) : \theta_1, \theta_2 \in [-1, 1] \right. \\
\text{and } \theta_1 &\leq \min \left\{ \frac{\beta(W^{++}(i, j) - W^{0+}(i, j)) - \phi_e}{\epsilon_e}, \frac{\beta(W^{++}(i, j) - W^{-+}(i, j)) - \phi_e - \phi_w}{\epsilon_e + \epsilon_w} \right\} \\
\text{and } \theta_2 &\leq \min \left\{ \frac{\beta(W^{++}(i, j) - W^{00}(i, j)) - \phi_e}{\epsilon_e}, \frac{\beta(W^{++}(i, j) - W^{+-}(i, j)) - \phi_e - \phi_w}{\epsilon_e + \epsilon_w} \right\} \\
\text{and } \theta_1 + \theta_2 &\leq \min \left\{ \frac{\beta(W^{++}(i, j) - W^{00}(i, j)) - 2\phi_e}{\epsilon_e}, \frac{\beta(W^{++}(i, j) - W^{--}(i, j)) - 2(\phi_e + \phi_w)}{\epsilon_e} \right\} \\
\text{and } \theta_1 + \frac{\epsilon_e}{\epsilon_e + \epsilon_w} \theta_2 &\leq \frac{\beta(W^{++}(i, j) - W^{-0}(i, j)) - 2\phi_e - \phi_w}{\epsilon_e} \\
\text{and } \frac{\epsilon_e}{\epsilon_e + \epsilon_w} \theta_1 + \theta_2 &\leq \frac{\beta(W^{++}(i, j) - W^{0-}(i, j)) - 2\phi_e - \phi_w}{\epsilon_e} \left. \right\}.
\end{aligned}$$

The corresponding probability is obtained by integrating over Ω^{++} :

$$\sigma^{++}(i, j) = \int \int_{\Omega^{++}} dF(\theta_1) dF(\theta_2). \quad (2)$$

The derivations for the other 8 decision pairs and their corresponding probabilities are omitted for brevity.

It remains to determine the expected value function. Substituting the optimal investment/disinvestment decisions for the two plants into the Bellman equation and integrating out both sides with respect

to θ_1 and θ_2 gives

$$\begin{aligned}
V(i, j) = & \Pi(i, j) + \sigma^{++}(i, j) (-2\phi_e + \beta W^{++}(i, j)) - \epsilon_e \int \int_{\Omega^{++}} (\theta_1 + \theta_2) dF(\theta_1) dF(\theta_2) \\
& + \sigma^{0+}(i, j) (-\phi_e + \beta W^{0+}(i, j)) - \epsilon_e \int \int_{\Omega^{0+}} \theta_2 dF(\theta_1) dF(\theta_2) \\
& + \sigma^{-+}(i, j) (\phi_w - \phi_e + \beta W^{-+}(i, j)) + \epsilon_w \int \int_{\Omega^{-+}} \theta_1 dF(\theta_1) dF(\theta_2) \\
& \quad - \epsilon_e \int \int_{\Omega^{-+}} \theta_2 dF(\theta_1) dF(\theta_2) \\
& + \sigma^{+0}(i, j) (-\phi_e + \beta W^{+0}(i, j)) - \epsilon_e \int \int_{\Omega^{+0}} \theta_1 dF(\theta_1) dF(\theta_2) \\
& + \sigma^{00}(i, j) \beta W^{00}(i, j) \\
& + \sigma^{-0}(i, j) (\phi_w + \beta W^{-0}(i, j)) + \epsilon_w \int \int_{\Omega^{-0}} \theta_1 dF(\theta_1) dF(\theta_2) \\
& + \sigma^{+-}(i, j) (-\phi_e + \phi_w + \beta W^{+-}(i, j)) - \epsilon_e \int \int_{\Omega^{+-}} \theta_1 dF(\theta_1) dF(\theta_2) \\
& \quad + \epsilon_w \int \int_{\Omega^{+-}} \theta_2 dF(\theta_1) dF(\theta_2) \\
& + \sigma^{0-}(i, j) (\phi_w + \beta W^{0-}(i, j)) + \epsilon_w \int \int_{\Omega^{0-}} \theta_2 dF(\theta_1) dF(\theta_2) \\
& + \sigma^{--}(i, j) (2\phi_w + \beta W^{--}(i, j)) + \epsilon_w \int \int_{\Omega^{--}} (\theta_1 + \theta_2) dF(\theta_1) dF(\theta_2). \tag{3}
\end{aligned}$$

By checking Blackwell's sufficient conditions, it is easy to show that equation (1) is contraction mapping. Hence, there exists a unique solution to the cartel's dynamic programming problem. We compute the solution using value function iteration. The double integrals in equations (2) and (3) are computed using Matlab's native functions for numerical integration.

4 Results

Investment Sunkness We set $\gamma = 0.99$ to model almost homogenous goods and contrast the case of partially sunk investment ($\phi_w = 24$) with the case of fully sunk investment ($\phi_w = -1000$). Table 1 lists the investment/disinvestment probabilities $e(i, j)$ (upper panel) and $w(i, j)$ (lower panel) for the case of partially sunk investment. Table 2 does the same for the case of fully sunk investment.

$e(i, j)$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$i = 0$	0.986	0.580	0.550	0.496	0.456	0.420	0.366	0.292	0.215	0.152
$i = 1$	1	0.841	0.015	0	0	0	0	0	0	0
$i = 2$	0.362	0.408	0.774	0	0	0	0	0	0	0
$i = 3$	0	0	0.227	0.764	0	0	0	0	0	0
$i = 4$	0	0	0	0	0.721	0	0	0	0	0
$i = 5$	0	0	0	0	0	0.626	0.316	0	0	0
$i = 6$	0	0	0	0	0	0.685	0.680	0	0	0
$i = 7$	0	0	0	0	0	0	0	0.561	0.342	0
$i = 8$	0	0	0	0	0	0	0	0.605	0.625	0
$i = 9$	0	0	0	0	0	0	0	0	0	0
$w(i, j)$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$i = 0$	0	0	0	0	0	0	0	0	0	0
$i = 1$	0	0	0	0	0	0	0	0	0	0
$i = 2$	0	0	0	0.035	0.319	0.312	0.272	0.285	0.316	0.338
$i = 3$	0	0	0	0.030	0.596	0.603	0.635	0.614	0.583	0.560
$i = 4$	0.244	0.475	0	0	0.046	0.950	0.967	0.839	0.773	0.728
$i = 5$	0.666	0.904	0.860	0.816	0.041	0.054	0.684	0.998	0.907	0.854
$i = 6$	0.899	1	1	0.997	0.960	0.315	0.087	1	1	0.945
$i = 7$	0.974	1	1	1	1	1	0	0.062	0.658	1
$i = 8$	0.985	1	1	1	1	1	1	0.395	0.106	1
$i = 9$	0.977	1	1	1	1	1	1	1	0	0.068

Table 1: Investment/disinvestment probabilities $e(i, j)$ and $w(i, j)$. $\gamma = 0.99$, $\phi_w = 24$, and $\delta = 0$.

$e(i, j)$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$i = 0$	0.975	0.624	0.591	0.557	0.378	0.488	0.028	0	0	0
$i = 1$	1	0.799	0.270	0.201	0.040	0	0	0	0	0
$i = 2$	0.491	0.787	0.427	0	0	0	0	0	0	0
$i = 3$	0	0	0	0	0	0	0	0	0	0
$i = 4$	0	0	0	0	0	0	0	0	0	0
$i = 5$	0.230	0	0	0	0	0	0	0	0	0
$i = 6$	0.547	0	0	0	0	0	0	0	0	0
$i = 7$	0	0	0	0	0	0	0	0	0	0
$i = 8$	0	0	0	0	0	0	0	0	0	0
$i = 9$	0	0	0	0	0	0	0	0	0	0

$w(i, j)$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$	$j = 8$	$j = 9$
$i = 0$	0	0	0	0	0	0	0	0	0	0
$i = 1$	0	0	0	0	0	0	0	0	0	0
$i = 2$	0	0	0	0	0	0	0	0	0	0
$i = 3$	0	0	0	0	0	0	0	0	0	0
$i = 4$	0	0	0	0	0	0	0	0	0	0
$i = 5$	0	0	0	0	0	0	0	0	0	0
$i = 6$	0	0	0	0	0	0	0	0	0	0
$i = 7$	0	0	0	0	0	0	0	0	0	0
$i = 8$	0	0	0	0	0	0	0	0	0	0
$i = 9$	0	0	0	0	0	0	0	0	0	0

Table 2: Investment/disinvestment probabilities $e(i, j)$ and $w(i, j)$. $\gamma = 0.99$, $\phi_w = -1000$, and $\delta = 0$.

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