

# Online Appendix

## Cooperation over finite horizons: a theory and experiments

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This appendix has three sections: 1) an example of the model with time average reciprocation, 2) proofs of the theorems stated in the main text, and 3) statements and proofs of the results described in Section 4.3 of the main text.

### 1 Example: time average reciprocation

To provide intuition for the basic features of strategic interaction between selfish and reciprocal players, we present an example to illustrate why there are decreasing contributions in equilibrium. In this specification reciprocal players reciprocate the simple time average of others' contributions:<sup>1</sup>

$$f_i^t(\cdot) = \frac{1}{t(N-1)} \left( \sum_{k=1}^t \sum_{j \in N/\{i\}} x_j^k \right) \quad \forall \quad i \in R \quad \text{and} \quad t \in \{1, \dots, T\}.$$

It is convenient to define the following terms:

$$C(t) = \frac{A}{N} \sum_{k=t+1}^T c(t, k),$$

where

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<sup>1</sup>It is easy to verify that these reciprocity functions satisfy the assumptions made in Section 4.1.

$$c(t, t+1) = \frac{N-S}{S+t(N-1)}$$

$$c(t, k) = \frac{N-S}{S+(k-1)(N-1)} \left[ 1 + \sum_{l=t+1}^{k-1} \left( \frac{N-S-1}{N-S} \right) c(t, l) \right] \text{ for } k > t+1.$$

The interpretation of these terms is the following:  $C(t)$  is the marginal impact of a contribution at  $t$  on future payoffs by reciprocal players. This includes both the direct impact of a unit contribution for the future periods and the indirect effect of a unit contribution in subsequent periods on future periods. In period  $t+1$ , the impact is  $c(t, t+1)$ , while for periods  $k > t+1$  it is  $c(t, k)$ .

**Claim 1:** Suppose  $C(t) \neq 1 - \frac{A}{N} \forall t \in \{1, \dots, T\}$  and  $S \geq 1$ . Then the above game has a unique subgame perfect Nash equilibrium, which exhibits the following contribution path. If  $C(1) < 1 - \frac{A}{N}$ , then every player contributes 0 at every period. Otherwise, let  $T^*$  be the largest integer between 1 and  $T-1$  such that  $C(T^*) > 1 - \frac{A}{N}$ . Then every selfish player contributes 1 in periods 1 to  $T^*$  and 0 afterwards. Meanwhile, reciprocal players contribute 1 until period  $T^*$ , and their contributions are strictly decreasing afterwards.

**Proof of Claim 1:** We begin with the following lemma.

**Lemma 1:** Let  $t \in \{0, \dots, T-1\}$ ,  $(x_i^1)_{i \in N}, \dots, (x_i^t)_{i \in N}$  be a length- $t$  contribution history and  $(x_i^{t+1})_{i \in S}, \dots, (x_i^T)_{i \in S}$  be a sequence of contributions by the selfish players after period  $t$ . Assume A1, A4 and A5 hold. Then there is a unique sequence of contributions by the reciprocal players after period  $t$ ,  $(x_i^{t+1})_{i \in R}, \dots, (x_i^T)_{i \in R}$  such that  $x_i^k = f_i^k((x_j^1)_{j \in N/\{i\}}, \dots, (x_j^k)_{j \in N/\{i\}}) \forall i \in R$  and  $k \in \{t+1, \dots, T\}$ , and  $x_R^k = (I - M^k)^{-1} \hat{x}_R^k$ , where  $\hat{x}_i^k = \sum_{k'=1}^{k-1} \sum_{j \in N/\{i\}} f_{i,j}^{k,k'}(x_j^{k'}) + \sum_{j \in S} f_{i,j}^{k,k}(x_j^k) \forall k \in \{t+1, \dots, T\}$  and  $i \in R$ ,  $\hat{x}_R^k = (\hat{x}_i^k)_{i \in R}$ , and  $M^k$  is the  $(N-S) \times (N-S)$  matrix whose diagonal elements are 0 and its  $(m, n)$ -th element is  $\alpha_{S+m, S+n}^{k,k}$ .

**Proof of Lemma 1:** Assume that  $k \in \{t+1, \dots, T\}$  is such that after any length- $k$  history  $(x_i^1)_{i \in N}, \dots, (x_i^k)_{i \in N}$  and any sequence of period  $k+1$  to period  $T$  contributions  $(x_i^{k+1})_{i \in S}, \dots, (x_i^T)_{i \in S}$  by the selfish players there is a unique sequence of period  $k+1$  to period  $T$  contributions  $(x_i^{k+1})_{i \in S}, \dots, (x_i^T)_{i \in S}$  by the reciprocal players such that

$$x_i^l = f_i^l((x_j^1)_{j \in N/\{i\}}, \dots, (x_j^l)_{j \in N/\{i\}})$$

$\forall i \in R$  and  $l \in \{k+1, \dots, T\}$ . Note that the above trivially holds for  $k = T$ .

Consider now any length length- $(k-1)$  history  $(x_i^1)_{i \in N}, \dots, (x_i^{k-1})_{i \in N}$  and any sequence of period  $k$  to period  $T$  contributions  $(x_i^k)_{i \in S}, \dots, (x_i^T)_{i \in S}$  by the selfish players. By definition, if

for some  $(y_i^k)_{i \in R}$  it holds that

$$y_i^k = f_i^k((x_j^1)_{j \in N/\{i\}}, \dots, (x_j^{k-1})_{j \in N/\{i\}}, ((x_j^k)_{j \in S}, (y_j^k)_{j \in R}))$$

$\forall i \in R$  then  $x_i^k = \hat{x}_R^k + M^k x_i^k$ . Assumptions A4, A5 and  $S > 1$  imply that  $\sum_{j \in R/\{i\}} \alpha_{i,j}^{k,k} < 1$

$\forall i \in R$ . Then by a well-known theorem (see Takayama (1985, p. 381))  $I - M^k$  is invertible and therefore the solution to  $x_i^k = \hat{x}_R^k + M^k x_i^k$  is unique and satisfies  $x_R^k = (I - M^k)^{-1} \hat{x}_R$ . The claim then follows by induction.  $\diamond$

First note that  $\frac{A}{N} < 1$  implies that in any subgame perfect Nash equilibrium all selfish players contribute 0 after any  $(T - 1)$ -length history  $(x_i^1)_{i \in N}, \dots, (x_i^{T-1})_{i \in N}$ . Furthermore, in any subgame perfect Nash equilibrium  $x_i^T = E_i f_i^T((x_i^1)_{i \in N}, \dots, (x_i^{T-1})_{i \in N}, (x_j^T)_{j \in N/\{i\}}) \forall i \in R$ , where the expectation is taken with respect to player  $i$ 's beliefs concerning  $(x_j^T)_{j \in N/\{i\}}$  after history  $(x_i^1)_{i \in N}, \dots, (x_i^{T-1})_{i \in N}$ . Lemma 1 then implies that there is a unique continuation strategy profile  $(x_j^T)_{j \in N}$  after  $(x_i^1)_{i \in N}, \dots, (x_i^{T-1})_{i \in N}$  in subgame perfect Nash equilibrium, and in this continuation profile

$$\begin{aligned} x_i^T &= \frac{N - S}{S + (T - 1)(N - 1)} \sum_{t'=1}^{T-1} \sum_{j \in S} x_j^{t'} + \\ &\sum_{t'=1}^{T-1} \sum_{j \in R/\{i\}} \frac{NT - T}{S - N + 2T + NT + ST + NST - T^2 - N^2T + N^2T^2 + 1} x_j^{t'} + \\ &\sum_{t'=1}^{T-1} \frac{N - 1 - S}{S - N + 2T + NT + ST + NST - T^2 - N^2T + N^2T^2 + 1} x_i^{t'} \end{aligned}$$

$\forall i \in R$ . Note that only the first term depends on selfish players' contributions.

Let  $t \in \{1, \dots, T - 1\}$  and assume that for every  $k \in \{t, \dots, T - 1\}$  and for every length- $k$  history  $(x_i^1)_{i \in N}, \dots, (x_i^k)_{i \in N}$  all subgame perfect Nash equilibria specify the same continuation profile, which is history-independent for selfish players and satisfies:

$$\begin{aligned} x_i^{k+1} &= \frac{N - S}{S + k(N - 1)} \sum_{t'=1}^k \sum_{j \in S} x_j^{t'} + \\ &\sum_{t'=1}^k \sum_{j \in R/\{i\}} \frac{(N - 1)(k + 1)}{S - N + 1 + (k + 1)(2 + N + S + NS - N^2) + (k + 1)^2(N^2 - 1)} x_j^{t'} + \end{aligned}$$

$$\sum_{t'=1}^k \frac{N-1-S}{S-N+1+(k+1)(2+N+S+NS-N^2)+(k+1)^2(N^2-1)} x_i^{t'}$$

for every  $i \in R$ . Then a marginal contribution by any  $i \in S$  at  $t$  has zero impact on contributions of any  $j \in S$ , and its marginal impact on the total future contributions of any  $j \in R$  is  $C(t)$ . By assumption  $C(t) \neq 1 - \frac{A}{N}$ . Then independently of history, in any subgame perfect Nash equilibrium all selfish players contribute 1 at  $t$  if  $C(t) > 1 - \frac{A}{N}$  and 0 if  $C(t) < 1 - \frac{A}{N}$ . Note that the starting assumption implies that in any subgame perfect Nash equilibrium, after any length- $t$  history all reciprocal players get a per period payoff of 0 in periods  $t+1, \dots, T$ . Then after any length- $(t-1)$  history  $(x_i^1)_{i \in N}, \dots, (x_i^{t-1})_{i \in N}$ , it has to hold that  $x_i^t = E_i f_i^t((x_i^1)_{i \in N}, \dots, (x_i^{t-1})_{i \in N}, (x_j^t)_{j \in N/\{i\}}) \forall i \in R$ , where the expectation is taken with respect to player  $i$ 's beliefs concerning  $(x_j^t)_{j \in N/\{i\}}$  after history  $(x_i^1)_{i \in N}, \dots, (x_i^{t-1})_{i \in N}$ . Then Lemma 1, together with the starting assumption, implies that for any length- $(t-1)$  history, there is a unique continuation profile in subgame perfect Nash equilibrium, which is history-independent for selfish players and satisfies:

$$x_i^k = \frac{N-S}{S+(k-1)(N-1)} \sum_{t'=1}^{k-1} \sum_{j \in S} x_j^{t'} + \sum_{t'=1}^{k-1} \sum_{j \in R/\{i\}} \frac{(N-1)k}{S-N+1+k(2+N+S+NS-N^2)+k^2(N^2-1)} x_j^{t'} + \sum_{t'=1}^{k-1} \frac{N-1-S}{S-N+1+k(2+N+S+NS-N^2)+k^2(N^2-1)} x_i^{t'}$$

for every  $k \in \{t, \dots, T\}$  and  $i \in R$ . The claim then follows by induction.  $\diamond$

It is easy to see that the property  $C(t) \neq 1 - \frac{A}{N} \forall t \in \{1, \dots, T\}$  in the claim holds generically for  $A$ , for every  $N$  and  $S$ .

## 2 Proofs of Theorems in Main Text

**Theorem 1:** If  $S \geq 1$  and A1-A5 hold, then for generic  $A$ , the public good contribution game has a unique subgame perfect Nash equilibrium, and this equilibrium exhibits a weakly decreasing pattern of contributions. If reciprocity functions are strictly concave in selfish players' contributions, then the above statement holds for all  $A$ .

**Proof of Theorem 1:** The same arguments as in the proof of Claim 1 establish that in every subgame perfect Nash equilibrium at period  $T$ , all selfish players contribute 0 after any  $(T-1)$ -

length history, and that the continuation strategy of reciprocal players after any  $(T - 1)$ -length history  $(x_i^1)_{i \in N}, \dots, (x_i^{T-1})_{i \in N}$  is uniquely determined in subgame perfect Nash equilibrium and satisfies  $x_i^T = f_i^T((x_j^1)_{j \in N}, \dots, (x_j^{T-1})_{j \in N}, ((0)_{j \in S}, (x_j^T)_{j \in R/\{i\}}) \forall i \in R$ .

Let  $t \in \{1, \dots, T - 1\}$  and assume that for almost every  $A$  (if  $f_{i,j}^{k,l}$  is strictly concave for every  $i \in R, j \in S$  and  $k, l \in \{1, \dots, T\}$  then for every  $A$ ), the following hold: for every  $k \in \{t, \dots, T - 1\}$  and for every length- $k$  history  $(x_i^1)_{i \in N}, \dots, (x_i^k)_{i \in N}$ , all subgame perfect Nash equilibria specify the same continuation profile, which is history-independent for selfish players and satisfies that  $x_i^l = f_i^l((x_j^1)_{j \in N/\{i\}}, \dots, (x_j^l)_{j \in N/\{i\}}) \forall i \in R$  and  $l \in \{k+1, \dots, T\}$ . Consider now an arbitrary  $(t - 1)$ -length history  $(x_i^1)_{i \in N}, \dots, (x_i^{t-1})_{i \in N}$ . Note that any period- $t$  contribution by a selfish player cannot influence period- $t$  contributions by other players. Furthermore, by the starting assumption, selfish players' contributions from  $t + 1$  on are generically uniquely pinned down in subgame perfect Nash equilibrium. Therefore a period- $t$  contribution by a selfish player can only influence reciprocal players' contributions from  $t + 1$  on. Let  $M^{k,k'}$  be the  $(N - S) \times (N - S)$  matrix whose diagonal components are 0 and whose  $(m, n)$ -th component is  $\alpha_{S+m, S+n}^{k,k'}$ . Then Lemma 1, together with A1, implies that contribution  $x_i^t$  by  $i \in S$  induces a total of  $\underline{1}(I - M^{t+1})^{-1}(f_{j,i}^{t+1,t}(x_i^t))_{j \in R}$  contributions at  $t+1$ , a total of  $(I - M^{t+1})^{-1}[M^{t+2,t+1}((I - M^{t+1})^{-1}(f_{j,i}^{t+1,t}(x_i^t))_{j \in R} + (f_{j,i}^{t+2,t}(x_i^t))_{j \in R})]$  contributions at  $t+2$ , and in general for  $l \in \{t+1, \dots, T\}$  a total which is equal to a linear nonnegative combination of  $(f_{j,i}^{t+1,t}(x_i^t))_{j \in R}, \dots, (f_{j,i}^{l,t}(x_i^t))_{j \in R}$  at  $l$ . The aggregate impact, and therefore the aggregate change in the flow future payoffs of  $i$ ,  $\Delta_i^{t+}(x_i^t)$  is given by a linear nonnegative combination of  $(f_{j,i}^{t+1,t}(x_i^t))_{j \in R}, \dots, (f_{j,i}^{T,t}(x_i^t))_{j \in R}$ . Since  $f_{j,i}^{l,t}$  is concave, differentiable, and increasing for every  $j \in R$  and  $l \in \{t+1, \dots, T\}$ ,  $\Delta_i^{t+}$  is also concave, differentiable, and increasing. Furthermore, if  $f_{j,i}^{l,t}$  is strictly concave for every  $j \in R$  then  $\Delta_i^{t+}$  is strictly concave. Meanwhile, the net change in the flow payoff of  $i$  at  $t$  is  $(\frac{A}{N} - 1)x_i^t$ , a linear and decreasing function. Since  $\Delta_i^{t+}$  is concave and increasing, there can only be a countable set of parameter values for  $A$  such that  $\frac{d\Delta_i^{t+}}{dx_i^t} = 1 - \frac{A}{N}$  has multiple solutions in  $[0, 1]$ , and if  $\Delta_i^{t+}$  is strictly concave then there are no parameter values like that. Therefore, for almost all values of  $A$  (for any  $A$  if  $f_{j,i}^{l,t}$  is strictly concave for every  $j \in R$  and  $l \in \{t+1, \dots, T\}$ ), the contribution of  $i$  is uniquely pinned down after  $(x_j^1)_{j \in N}, \dots, (x_i^{t-1})_{j \in N}$ . Since this holds for any  $i \in S$  and length- $(t - 1)$  history  $(x_j^1)_{j \in N}, \dots, (x_i^{t-1})_{j \in N}$ , the induction assumption implies that for almost all values of  $A$  (for any value of  $A$  if reciprocity functions towards selfish players are strictly concave), the continuation strategy after  $t - 1$  is uniquely pinned down for all selfish players in subgame perfect Nash equilibrium.

Note that the induction assumption implies that in any subgame perfect Nash equilibrium, after any length- $t$  history, all reciprocal players get a flow payoff of 0 in periods  $t + 1, \dots, T$ . Therefore, after any length- $(t - 1)$  history  $(x_j^1)_{j \in N}, \dots, (x_j^{t-1})_{j \in N}$  and any subgame perfect Nash equilibrium  $s$ , it holds that the action specified by  $s_i$  after  $(x_j^1)_{j \in N}, \dots, (x_j^{t-1})_{j \in N}$  is

$$E_i f_i^T((x_j^1)_{j \in N/\{i\}}, \dots, (x_j^T)_{j \in N/\{i\}})$$

$\forall i \in R$ , where the expectation is taken with respect to player  $i$ 's beliefs concerning  $(x_j^T)_{j \in N/\{i}}$  after history  $(x_i^1)_{i \in N}, \dots, (x_i^{T-1})_{i \in N}$ . Lemma 1 then implies that after  $(x_j^1)_{j \in N}, \dots, (x_j^{t-1})_{j \in N}$  the period  $t$  contribution of  $i$  is uniquely pinned down in subgame perfect Nash equilibrium, and

$$x_i^t = f_i^T((x_j^1)_{j \in N/\{i}}, \dots, (x_j^T)_{j \in N/\{i}}).$$

This concludes that for almost every  $A$  (if  $f_{i,j}^{k,l}$  is strictly concave for every  $i \in R, j \in S$  and  $k, l \in \{1, \dots, T\}$  then for every  $A$ ), the following hold: for every  $k \in \{t-1, \dots, T-1\}$  and for every length- $k$  history  $(x_i^1)_{i \in N}, \dots, (x_i^k)_{i \in N}$ , all subgame perfect Nash equilibria specify the same continuation profile, which is history-independent for selfish players and satisfies:

$$x_i^l = f_i^l((x_j^1)_{j \in N/\{i}}, \dots, (x_j^l)_{j \in N/\{i}})$$

$\forall i \in R$  and  $l \in \{k+1, \dots, T\}$ . By induction then, for almost every  $A$  (if  $f_{i,j}^{k,l}$  is strictly concave for every  $i \in R, j \in S$  and  $k, l \in \{1, \dots, T\}$  then for every  $A$ ), there is a unique subgame perfect Nash equilibrium, which is history-independent for selfish players and satisfies:

$$x_i^l = f_i^l((x_j^1)_{j \in N/\{i}}, \dots, (x_j^l)_{j \in N/\{i}})$$

$\forall i \in R$  and  $l \in \{1, \dots, T\}$ .

A4 implies that  $\frac{d\Delta_i^{t+}(x)}{dx} \geq \frac{d\Delta_i^{(t+1)+}(x)}{dx}$  for every  $i \in S, x \in [0, 1]$  and  $t \in \{1, \dots, T-1\}$ ; therefore, for generic parameter values, the contributions of selfish players are weakly decreasing in subgame perfect Nash equilibrium. A3 then implies that along the equilibrium path, the contributions of all players are weakly decreasing.  $\diamond$

**Theorem 2:** Suppose assumptions A1-A5 hold and the game has a unique subgame perfect Nash equilibrium  $s$ . Then if  $s_i^t > 0$  for some  $i \in S$ , then  $s_j^k > 0 \forall j \in R$  and  $k \in \{1, \dots, t\}$ .

**Proof of Theorem 2:** By Theorem 1,  $s_i^t > 0$  for some  $i \in S$  implies  $s_i^k > 0 \forall k \in \{1, \dots, t\}$ . A4 then implies the claim.  $\diamond$

### 3 Statements and Proofs of Additional Results

#### *Private Return from Contributing*

**Proposition 1:** Assume  $S \geq 1$ . Let  $A$  and  $\widehat{A}$  be such that that the games in which the return of private investment are  $A$  and  $\widehat{A}$  have unique subgame perfect Nash equilibria  $s$  and  $\widehat{s}$ . Then  $A < \widehat{A}$  implies  $s_i^t \leq \widehat{s}_i^t \forall t \in \{1, \dots, T\}$  and  $i \in N$ .

**Proof of Proposition 1:** We first begin with the following Lemma:

**Lemma 2:** Consider two contribution games  $\mathcal{G}$  and  $\widehat{\mathcal{G}}$  with individual contributions revealed after rounds, in which the players are the same:  $N = \widehat{N}$ ,  $S = \widehat{S}$ , and  $f_i^t = \widehat{f}_i^t \forall i \in R$  and  $t \in \{1, \dots, \min(T, \widehat{T})\}$ . Suppose  $\mathcal{G}$  and  $\widehat{\mathcal{G}}$  have unique subgame perfect Nash equilibria  $s$  and  $\widehat{s}$ . If  $s_i^t \leq \widehat{s}_i^t \forall i \in S$ , then  $s_i^t \leq \widehat{s}_i^t \forall i \in N$ .

**Proof of Lemma 2:** Let  $T^* = \min(T, \widehat{T})$ . Let  $\widehat{y}_i^k = \sum_{k'=1}^{k-1} \sum_{j \in N/\{i\}} f_{i,j}^{k,k'}(\widehat{s}_j^{k'}) + \sum_{j \in S} f_{i,j}^{k,k}(\widehat{s}_j^1)$ , and let  $y_i^k = \sum_{k'=1}^{k-1} \sum_{j \in N/\{i\}} f_{i,j}^{k,k'}(s_j^{k'}) + \sum_{j \in S} f_{i,j}^{k,k}(s_j^1) \forall i \in R$  and  $k \in \{1, \dots, T^*\}$ . Suppose that for some  $k \in \{1, \dots, T^*\}$ , it holds that  $\widehat{s}_i^t \geq s_i^t \forall i \in N$  and  $t \in \{1, \dots, k-1\}$ , and  $\widehat{s}_i^k \geq s_i^k \forall i \in S$ . Note that this holds for  $k=1$ . Lemma 1 implies that  $\widehat{s}_R^k = (I - M^k)^{-1} \widehat{y}_R^k$  and  $s_R^k = (I - M^k)^{-1} y_R^k$ , where  $\widehat{y}_R^k = (\widehat{y}_i^k)_{i \in R}$ ,  $y_R^k = (y_i^k)_{i \in R}$ ,  $\widehat{s}_R^k = (\widehat{s}_i^k)_{i \in R}$ ,  $s_R^k = (s_i^k)_{i \in R}$ , and  $M^k$  is the  $(N-S) \times (N-S)$  matrix whose diagonal elements are 0 and its  $(m, n)$ -th element is  $\alpha_{S+m, S+n}^{k,k}$ . Since  $\sum_{j \in S} f_{i,j}^{k,k}(s_j^k)$  is increasing in  $s_j^k \forall i \in R$  and  $j \in S$ , and  $(I - M^k)^{-1} y_R^k$  is increasing in  $y_R^k$  in the relevant nonnegative range, the above establishes that  $\widehat{s}_i^k \geq s_i^k \forall i \in R$ . Then  $k < T$  implies  $\widehat{s}_i^t \geq s_i^t \forall i \in N$  and  $t \in \{1, \dots, k\}$ , and  $\widehat{s}_i^{k+1} \geq s_i^{k+1} \forall i \in S$ . The claim then follows by induction.  $\diamond$

The proof of Theorem 1 establishes that the equilibrium contribution of any player  $i \in S$  in any period  $t \in \{1, \dots, T\}$  is given by  $x_i^t = \arg \max_{x \in [0,1]} ((\frac{A}{N} - 1)x + \Delta_i^{t+}(x))$  if the return to contributing is  $A$ , and it is given by  $x_i^t = \arg \max_{x \in [0,1]} ((\frac{\widehat{A}}{N} - 1)x + \Delta_i^{t+}(x))$  if the return to contributing is  $\widehat{A}$ , where  $\Delta_i^{t+}$  is a term that increases in  $x$ .  $\widehat{A} > A$  then implies that  $\widehat{s}_i^t \geq s_i^t \forall i \in S$  and  $t \in \{1, \dots, T\}$ , where  $\widehat{s}$  is the unique subgame perfect Nash equilibria of the game in which the return to contributing is  $\widehat{A}$  and  $s$  is the unique subgame perfect Nash equilibria of the game in which the return to contributing is  $A$ . The claim then follows from Lemma 2.  $\diamond$

### Number of Periods

**Proposition 2:** Assume  $S \geq 1$ . Let  $A$  be such that the games with  $T$  and  $\widehat{T}$  number of periods have unique subgame perfect Nash equilibria  $s$  and  $\widehat{s}$ . Then  $T < \widehat{T}$ ,  $f_i^t = \widehat{f}_i^t$  implies  $s_i^t \leq \widehat{s}_i^t \forall t \in \{1, \dots, T\}$  such that  $t \leq \min(T, \widehat{T})$  and  $i \in N$ .

**Proof of Proposition 2:** Theorem 1 establishes that the equilibrium contribution of any player  $i \in S$  in any period  $t \in \{1, \dots, T\}$  is given by  $x_i^t = \arg \max_{x \in [0,1]} ((\frac{A}{N} - 1)x + \Delta_i^{t+}(x))$ , where  $\Delta_i^{t+}$  is a term that increases in  $x$  and increases in  $T$ . This implies  $\widehat{s}_i^t \geq s_i^t \forall i \in S$  and  $t \in \{1, \dots, T\}$ . The claim then follows from Lemma 2.  $\diamond$

### One-shot Games

**Proposition 3:** If  $T = 1$  and  $S \geq 1$ , then the game has a unique Nash equilibrium, which involves all players contributing 0 to the public good.

**Proof of Proposition 3:** For any  $i \in S$ , contributing 0 is a dominant strategy, therefore  $s_i^1 = 0$  for any Nash equilibrium  $s$ . By Lemma 1, there is a unique vector of contributions  $x_{S+1}, \dots, x_N$  by the reciprocal players such that  $f_i(0, \dots, 0, x_{S+1}, \dots, x_N) = x_i \forall i \in R$ . Since  $f_i(0, \dots, 0) = 0 \forall i \in R$ , the unique Nash equilibrium of the game is then  $s_i^1 = 0 \forall i \in N$ .  $\diamond$